MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 42/81

Funktionalanalysis: C^*-Algebren

27.9. - 3.10.1981

Die jährliche Funktionalanalysistagung in Oberwolfach wurde in diesem Jahr unter das Thema "C^*-Algebren" gestellt. Unter der Leitung von A. Connes (Paris), J. Cuntz (Heidelberg, z. Zt. Marseille) und R. Nagel (Tübingen) nahmen daran über 40 Mathematiker aus Europa und Amerika teil. Ein Interessenschwerpunkt bestand in der Untersuchung der Struktur solcher C^*-Algebren, die mit dynamischen Systemen und geometrischen Objekten assoziiert werden können. Vor allem aber wurden die aktuellen Entwicklungen auf dem Gebiet der K-Theorie und allgemeinerer Homologie-Kohomologie-Theorien für nichtkommutative C^*-Algebren und ihre Beziehungen zu topologischen Fragen diskutiert. Als Höhepunkt kann ohne Zweifel der mehrstündige Vortrag von A. Connes über "Spectral sequence and homology of currents for operator algebras" bezeichnet werden. Die angenehme "Oberwolfach-Atmosphäre" tat ein übriges, den Aufenthalt für alle Teilnehmer fruchtbar und angenehm zu gestalten. Daraus resultierte der Wunsch, daß diese bisher erste Oberwolfachtagung über C^*-Algebren eine Wiederholung finden möge.

implies $e^{-tf(H)} \geq e^{-tf(K)} \geq 0$ for all $t \geq 0$ and all $H = H^* \geq 0$, $K = K^* \geq 0$ if and only if $f(0) \leq f(0+)$ and the restriction of f to $(0,+\infty)$ is C^∞ with $(-1)^n f^{(n+1)}(x) \geq 0$ for all $x > 0$, $n = 0,1,2,\ldots$.

A. CONNES:

Spectral sequence and homology of currents for operator algebras

The transversal elliptic theory for foliations requires as a preliminary a purely algebraic work of computing for a non-commutative algebra A the homology of the following complex: n-cochains and multilinear fcts. $\varphi(f^1,\ldots,f^n)$ of $f^1,\ldots,f^n \in A$ with $(f^1,f^2,\ldots,f^n,f^o) = (-1)^n \varphi(f^o,f^1,\ldots,f^n)$ and the boundary is $b\varphi(f^o,\ldots,f^{n+1}) = \varphi(f^o f^1,f^2,\ldots,f^{n+1}) - \varphi(f^o,f^1 f^2,\ldots,f^{n+1}) + \ldots + (-1)^{n+1}\varphi(f^{n+1}f^o,\ldots,f^n)$. The basic class associated to a transversally elliptic operator for A = the algebra of the foliation is given by

$$\varphi(f^o,\ldots,f^n) = \mathrm{Trace}\,(\varepsilon\, F\,[F,f^o][F,f^1]\ \ldots\ [F,f^n])\ ,\quad f^i \in A$$

where $F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$, Q is a parametrix of P and $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. An operator $S : H^n(A) \to H^{n+2}(A)$ is constructed as well as a pairing $K(A) \times H(A) \to \mathbb{C}$ where $K(A)$ is the algebraic K-theory of A, it gives the index of the operator from its associated class φ, moreover $\langle e,\varphi\rangle = \langle e,S\varphi\rangle$ so that the important group to determine is the inductive limit $H_g = \varinjlim H^n(A)$ for the map S being the tools of homological algebra the groups $H^n(A, A^*)$ of Hochschildt cohomology with coeff. in the bimodule A^* are easier to determine and the solution of the problem is obtained in two steps:

1. The construction of a map $B : H^n(A, A^*) \to H^{n-1}(A)$ and the proof of a long exact sequence

$$\ldots \to H^n(A, A^*) \xrightarrow{B} (H^{n-1}(A) \xrightarrow{S} H^{n+1}(A) \xrightarrow{I} H^{n+1}(A, A^*) \to \ldots$$

where I is the obvious map from the cohomology of the above complex to the Hochschildt cohomology.

2. The contruction of a spectral sequence with E_2 given by the cohomology of degree -1 differential $I \circ B$ on the Hochschildt group $H^n(A, A^*)$ and which converges strongly to a graded group associated to the above inductive limit.

This purely algebraic theory is then used for $A = C^\infty(V)$ one gets the de Rham homology of currents and for the pseudo torus, i.e. the algebra of the Kronecker foliation one finds that the Hochschildt cohomology depends on the deophactive nature of the rotation number while the above theory gives $H_g{}^o$ of dim 1, $H_g{}^1$ of dim 2 and $H_g{}^2$ of dim 2 as expected, but by some remarkable cancellations.

G.A. ELLIOTT:

Temperature-density state spaces of dynamical systems

If $(A, \mathbb{R}^2, \alpha)$ is an almost periodic C^*-dynamical system and if whenever a nontrivial one-parameter subgroup (read "density") has an α-invariant ground state this is unique, then, at least if A is unital and each ideal of the fixed point algebra is generated by projections, the space of α-invariant ground states must be totally disconnected.

Group actions on AF algebras; Joint with Wulf Rossmann

Product type and non product type actions onto of finite groups on UHF algebras are studied & classified; the Grothendieck gp of the fixed subalgebra is an ordered module over the representation ring of the group, and this structure with an additional datum is a complete invariant for stable conjugacy (i.e. conjugacy after tensoring with the regular representation. The product type actions yield rank 1 modules, and thus their classification is analogous to that of UHF algebras; non product type actions yield modules of large rank and thus correspond to AF algebras with $\mathbb{Z}$ replaced by the representation ring of the group as an ordered ring.

David Handelman (Ottawa)

Spectral sequence and homology of currents for operator algebras

The transversal elliptic theory for foliations requires as a preliminary a purely algebraic work, of computing for a non commutative algebra $\mathcal{A}$ the cohomology of the following complex: n-cochains are multilinear fcts $\varphi(f^0,\dots,f^n)$ of $f^0,\dots,f^n\in\mathcal{A}$ with $\varphi(f^1,f^2,\dots,f^n,f^0)=(-1)^n\varphi(f^0,f^1,\dots,f^n)$ and the boundary is $b\varphi(f^0,\dots,f^{n+1})=\varphi(f^0f^1,\dots,f^{n+1})-\varphi(f^0,f^1f^2,\dots,f^{n+1})+\dots+(-1)^{n+1}\varphi(f^{n+1}f^0,\dots,f^n)$. The basic class associated to a transversally elliptic operator, for $\mathcal{A}$ = the algebra of the foliation is given by:

$$\varphi(f^0,\dots,f^n)=\mathrm{Trace}\left(\varepsilon F[F,f^0][F,f^1]\dots[F,f^n]\right),\quad f^i\in\mathcal{A}$$

where $F=\begin{bmatrix}0&Q\\P&0\end{bmatrix}$, Q is a parametrix of P and $\varepsilon=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ – An operation $S: H^n(\mathcal{A})\to H^{n+2}(\mathcal{A})$ is constructed, as well as a pairing $K(\mathcal{A})\times H(\mathcal{A})\to\mathbb{C}$ where $K(\mathcal{A})$ is the algebraic K theory of $\mathcal{A}$, it gives the index of the operator from its associated class φ, moreover $\langle e,\varphi\rangle=\langle e,S\varphi\rangle$ so that the important group to determine is the inductive limit $H_\varphi=\varinjlim H^n(\mathcal{A})$ for the map S. Using the tools of homological algebra the groups $H^n(\mathcal{A},\mathcal{A}^*)$ of Hochschild cohomology with coeff in the bimodule $\mathcal{A}^*$ are easier to determine and the solution of the problem is obtained in two steps.

(1) the construction of a map $B: H^n(\mathcal{A},\mathcal{A}^*) \to H^{n-1}(\mathcal{A})$ and the proof of a long exact sequence:

$$\to H^n(\mathcal{A},\mathcal{A}^*) \xrightarrow{B} H^{n-1}(\mathcal{A}) \xrightarrow{S} H^{n+1}(\mathcal{A}) \xrightarrow{I} H^{n+1}(\mathcal{A},\mathcal{A}^*) \to \dots$$

where I is the obvious map from the cohomology of the above complex to the Hochschild cohomology.

(2) the construction of a spectral sequence with E_2 term given by the cohomology of the degree -1 differential IB on the Hochschild groups $H^n(\mathcal{A},\mathcal{A}^*)$, and which converges strongly to a graded group associated to the above inductive limit.

This purely algebraic theory is then used for $\mathcal{A} = C^\infty(V)$ one gets the de Rham homology of currents, and for the pseudo torus, i.e. the algebra of the Kronecker foliation, one finds that the Hochschild cohomology depends on the diophantine nature of the rotation number while the above theory gives H^0_λ of dim 1, H^1_λ of dim 2 and H^2_λ of dim 1 as expected, but from some remarkable cancellations, [illegible]

A. CONNES
(PARIS)

Hahn-Banach methods for completely bounded maps into $B(K)$.

Let $\mathfrak{X}$ be a $*$-invariant subspace for some C^* algebra $\mathcal{A}$, $\varphi: \mathfrak{X} \to B(K)$ linear and completely bounded. Then there exists a completely bounded extension $\Phi: \mathcal{A} \to B(K)$ preserving the cb-norm $\|\varphi\|_{cb}$. If $\varphi = \varphi^*$, it may be decomposed $\varphi = \varphi_+ - \varphi_-$ as the difference of two completely positive maps $\varphi_\pm$ such that φ_+ and φ_- are disjoint in the set of all completely positive maps and the norm condition $\|\varphi\|_{cb} = \|\varphi_+ + \varphi_-\|$ holds. If $\mathfrak{X} = \mathcal{A}$ is a W^*-algebra and φ is σ-w^*-continuous, $\varphi_\pm$ are normal. This implies a Stinespring representation for φ. The same conclusions hold for module homomorphisms.

G. Wittstock