Algebraic Continuation of Multiple Zeta Values by Renormalization

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- This approach gives a better understanding of the renormalization process in physics,
- and allows the method of renormalization to be applied to mathematical problems.
- Key in the Connes-Kreimer framework is the Algebraic Birkhoff Decomposition which splits a Feynman integral into a purely convergent component and a purely divergent component. This splitting is compatible with the hierarchy structure of the Feynman integrals.

- By regularization (deformation), the Feynman integral is deformed into an expression in the ring $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ of Laurent series.

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with $\begin{cases} \phi_{\text{FG},-} : \mathcal{H}_{\text{FG}} \to \mathbb{C}[\varepsilon^{-1}] \\ \phi_{\text{FG},+} : \mathcal{F}_{\text{FG}} \to \mathbb{C}[[\varepsilon]] \end{cases}$

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- Then the renormalized values of the Feynman integrals are given by $\phi_{\text{FG},+}\big|_{\varepsilon=0}$.

- **2. Rota-Baxter algebras:**
  Fix λ in a base ring **k**. A Rota-Baxter operator or a Baxter operator of weight λ on a **k**-algebra R is a linear map $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, y \in R.$$

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$$F(x) := P[f](x) = \int_0^x f(t)\,dt, \quad G(x) := P[g](x) = \int_0^x g(t)\,dt.$$

  Then the integration by parts formula states

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- that is,
$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x).$$

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$$P[f](x)P[g](x) = \Big(\sum_{n\geq 1} f(x+n)\Big)\Big(\sum_{m\geq 1} g(x+m)\Big)$$

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$$= \Big(\sum_{n>m\geq 1} + \sum_{m>n\geq 1} + \sum_{m=n\geq 1}\Big) f(x+n)g(x+m)$$

$$= \sum_{m\geq 1}\Big(\sum_{k\geq 1} f(x+\underbrace{k+m}_{=n})\Big)g(x+m) + \sum_{n\geq 1}\Big(\sum_{k\geq 1} g(x+\underbrace{k+n}_{=m})\Big)f(x+n)$$

$$+ \sum_{n\geq 1} f(x+n)g(x+n)$$

$$= P(P(f)g)(x) + P(fP(g))(x) + P(fg)(x).$$

- QFT dimensional regularization: Let $R = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n \varepsilon^n,\ k \geq 0$. Define

$$P(\sum_{n=-k}^{\infty} a_n \varepsilon^n) = \sum_{n=-k}^{-1} a_n \varepsilon^n.$$

  Then P is a Rota-Baxter operator of weight -1.

  Others: Partial sums, scalar product, Hochschild homology ring, classical and associative Yang-Baxter equations, dendriform algebras, rooted trees, divided powers, ...

- **3. Algebraic Birkhoff decomposition.**
  Let $\mathcal{H}$ be a connected filtered Hopf algebra, that is, $\mathcal{H}$ has a decreasing filtration $\mathcal{H}_n \subset \mathcal{H}, n \geq 0$ that is compatible with the Hopf algebra structure of $\mathcal{H}$ and $\mathcal{H}_0 = \mathbb{C}$.
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- **Theorem** (Algebraic Birkhoff decomposition) Any algebra homomorphism $\phi : \mathcal{H} \to R$ has a unique decomposition into algebra homomorphisms

$$\phi = \phi_-^{-1} \star \phi_+, \quad \begin{cases} \phi_- : \mathcal{H} \to \mathbb{C} + P(R) \text{ (counter term)} \\ \phi_+ : \mathcal{H} \to \mathbb{C} + (\mathrm{id} - P)(R) \text{ (renormalization)} \end{cases}$$

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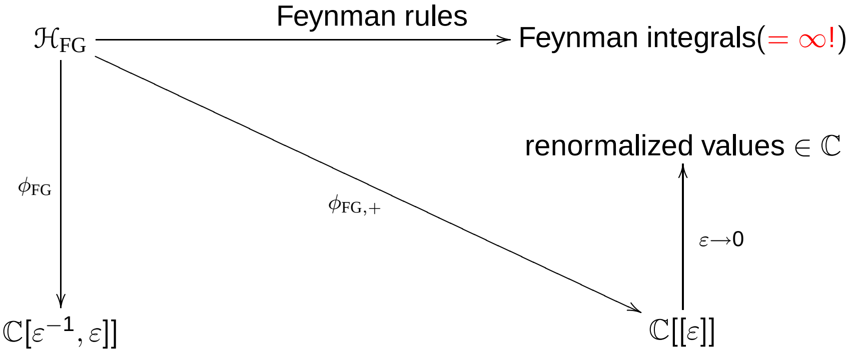

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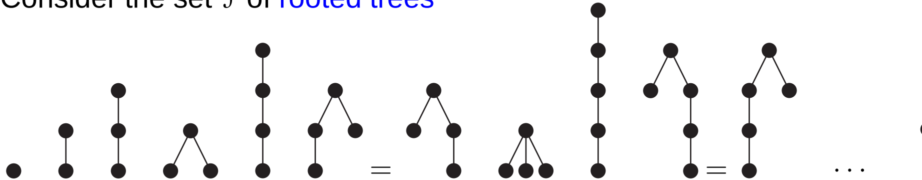

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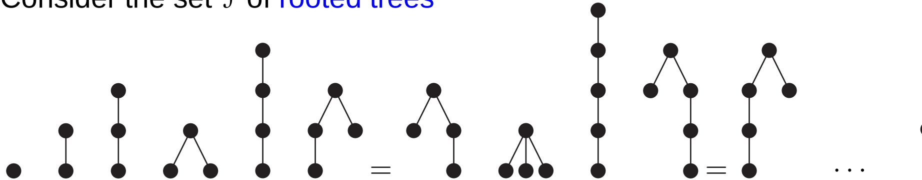

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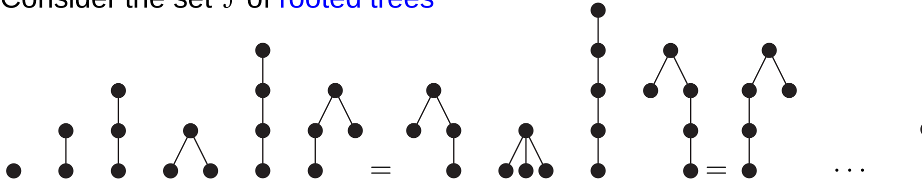
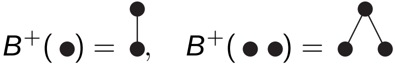


- The set $\mathcal{F}$ of rooted forests is the free semigroup (with concatenation product) generated by $\mathcal{T}$.
- $\mathcal{F}$ is equipped with the grafting operator $B^+ : \mathcal{F} \to \mathcal{F}$ where $B^+(F)$ is the tree by adding a new root to the trees in F:



- A commutative operated semigroup is a pair (U, α) where U is a commutative semigroup and $\alpha : U \to U$ is a map. A morphism from (U, α) to (V, β) is a semigroup homomorphism $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.

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- Both $\mathcal{G}$ and $\mathcal{F}$ are examples of operated semigroups, as is the set $\mathcal{G}_{reg}$: $\int_0^\infty \frac{x^{-\varepsilon}dx}{x+c}$, $\int_0^\infty \left(\int_0^\infty \frac{x_1^{-\varepsilon}dx_1}{x_1+x}\right) \frac{x^{-\varepsilon}dx}{x+c}$,
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- A free commutative operated semigroup on a set X is defined by the usual universal property.
- Theorem. The operated semigroup $\mathcal{F}$, with the concatenation product and the grafting operator B^+, is the free commutative operated semigroup on $\bullet$.
- Corollary. There are unique surjective operated semigroup morphisms $\phi : \mathcal{F} \to \mathcal{G}$ such that $\phi(\bullet) = \int_0^\infty \frac{dx}{x+c}$

$$\phi_{reg} : \mathcal{F} \to \mathcal{G}_{reg} \text{ such that } \phi(\bullet) = \int_0^\infty \frac{x^{-\varepsilon}dx}{x+c}.$$

- $\mathcal{H} := \mathbb{C} \bigoplus \mathbb{C}\mathcal{F}$ has a coproduct

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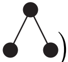
- For $F \in \mathcal{F}$, the renormalized value of $\phi(F)$ is defined to be $\phi_{reg,+}(F)\big|_{\varepsilon=0}$.
- Example. $\phi(\bullet) = \displaystyle\int_0^\infty \frac{dx}{x+c} \mapsto -\ln c$,

$\phi($  $) = \displaystyle\int_0^\infty \left(\int_0^\infty \frac{dx_1}{x_1+x}\right) \frac{dx}{x+c} \mapsto \frac{1}{2}(\ln c)^2 + \frac{\pi^2}{4}.$

- **5. MZVs with positive arguments**

  Multiple zeta values are defined to be the evaluation of the multiple complex variable function

$$\zeta(\vec{s}) = \zeta(s_1, \cdots, s_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

  at positive integers $s_1, \cdots, s_k$ with $s_1 > 1$. It has been studied by many authors (Euler, Hoffman, Zagier, ...) from different viewpoints.

- Recall,

$$P(f)(x) := \sum_{n \geq 1} f(x+n)$$

is a Rota-Baxter operator of weight 1:

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- Taking $f = 1/x^{s_1}$ and $g = 1/x^{s_2}$, we have
$$\zeta(s_1)\zeta(s_2) = P(\frac{1}{x^{s_1}})(0)P(\frac{1}{x^{s_2}})(0) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$
$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4).$$

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$$P(f)(x) := \sum_{n\geq 1} f(x+n)$$

is a Rota-Baxter operator of weight 1:

$$P(f)P(g) = P(P(f)g)(x) + P(fP(g))(x) + P(fg)(x).$$

- Taking $f = 1/x^{s_1}$ and $g = 1/x^{s_2}$, we have

$$\zeta(s_1)\zeta(s_2) = P(\frac{1}{x^{s_1}})(0)P(\frac{1}{x^{s_2}})(0) = \zeta(s_1,s_2) + \zeta(s_2,s_1) + \zeta(s_1+s_2).$$

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4).$$

- More generally, for $s \geq 1$, define $f_s(x) = 1/x^s$. Then

$$\zeta(s_1,s_2) = P(f_{s_2}P(f_{s_1}))(0).$$

$$\zeta(s_1,\cdots,s_k) = P(f_{s_k}P(f_{s_{k-1}}P(\cdots f_{s_2}P(f_{s_1})\cdots)))(0).$$

Then $\zeta(s_1,\cdots,s_k)$ satisfies the quasi-shuffle relation:

$$\vec{s} * \vec{t} = \sum_{\vec{w}} \vec{w} \Rightarrow \zeta(\vec{s})\zeta(\vec{t}) = \sum_{\vec{w}} \zeta(\vec{w}).$$

- **Quasi-shuffle product:** Hoffman (2000)
  Let A be a $\mathbf{k}$-module. Let $Ш^+(A) = T(A)$ be the tensor algebra on A. Suppose A is a $\mathbf{k}$-algebra. Define another products on $Ш^+(A)$. Define $1 \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$.
  Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Recursively define

  $$(a_1 \otimes \mathfrak{a}') * (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes \mathfrak{b}'))) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') * \mathfrak{b}') + (a_1 b_1) \otimes (\mathfrak{a}' * \mathfrak{b}'),$$

  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity.
  Example.
  $a_1 * (b_1 \otimes b_2) = a_1 \otimes (1 * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes (1 * b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes (a_1 b_2) + (a_1 b_1) \otimes b_2.$

- **Mixable shuffle product:** Guo-Keigher (2000) on Rota-Baxter algebras, Goncharov (2002) on motivic shuffle relations and Hazewinckle on overlapping shuffle products.

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- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.

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- **Example:**

$$
\begin{aligned}
& a_1 \diamond (b_1 \otimes b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\
& + a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad \text{(merged shuffles)}.
\end{aligned}
$$

- A **free commutative Rota-Baxter algebra over another commutative algebra** A is a commutative Rota-Baxter algebra $Ш(A)$ with an algebra homomorphism $j_A : A \to Ш(A)$ such that for any commutative Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\;j_A\;} & Ш(A) \\ & \searrow^{f} & \downarrow{\bar{f}} \\ & & R \end{array}$$

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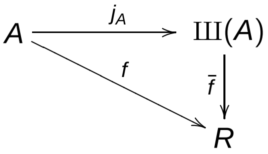

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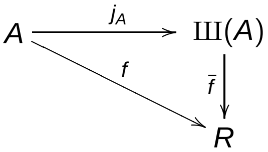

- When $A = \mathbf{k}[X]$, we have the free commutative Rota-Baxter algebra over X.
- Recall $(\text{Ш}^+(A), \diamond)$ is a commutative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is a commutative A-algebra.
  Theorem (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over A.

- So MZVs with positive arguments satisfy the quasi-shuffle relation.

$$\vec{s} \star \vec{t} = \sum_{\vec{w}} \vec{w} \;\Rightarrow\; \zeta(\vec{s})\zeta(\vec{t}) = \sum_{w} \zeta(\vec{w}).$$

$$\zeta(s_1)\zeta(t_1,t_2) = \zeta(s_1,t_1,t_2) + \zeta(t_1,s_1,t_2) + \zeta(t_1,t_2,s_1) + \zeta(s_1+t_1,t_2) + \zeta(t_1,s_1+t_2).$$

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- Integration representation of MZVs and shuffle relation: For one variable zeta function, this is called the Leibniz formula:

$$\zeta(s) = \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{s-2}} \int_0^{x_{s-1}} \frac{dx_s}{1-x_s} \frac{dx_{s-1}}{x_{s-1}} \cdots \frac{dx_2}{x_2} \frac{dx_1}{x_1}$$

For multiple variables,

$$\zeta(s_1,\cdots,s_k) = \int_0^1 \cdots \int \frac{dx}{1-x} \underbrace{\frac{dx}{x} \cdots \frac{dx}{x}}_{s_1-1 \text{ terms}} \cdots \frac{dx}{1-x} \underbrace{\frac{dx}{x} \cdots \frac{dx}{x}}_{s_k-1 \text{ terms}}$$

Thus the product of MZVs also satisfies the shuffle relation:

$$\zeta(\vec{s})\zeta(\vec{t}) = \zeta(\vec{s} ш \vec{t}).$$

$$\zeta(2)\zeta(2) = 4\zeta(3,1) + 2\zeta(2,2).$$

- Combining
$$\zeta(2)\zeta(2) = 4\zeta(3,1) + 2\zeta(2,2).$$
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- Conjecturally, all algebraic relations among MZVs are given by the double shuffle relation and their degenerated cases.
- Question: What about MZVs with nonpositive arguments?

- **6. Non-positive MZVs by analytic continuation**
  One variable case: $\zeta(z)$ has analytic continuation to the whole complex plane with a simple pole at $z = 1$. So $\zeta(k), k \leq 0$, are defined. Further,
$$\zeta(k) = (-1)^k \frac{B_{-k+1}}{-k+1}$$

- **Multiple variable case:** In two variable case considered by Atkinson (1949) and in general by Goncharov, Zhao, Arakawa-Kaneko, and Akiyama-Egami-Tanigawa (around 2000).

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- Theorem. $\zeta(s_1, \cdots, s_k)$ can be meromorphically continued to $\mathbb{C}^k$ with singularities on the subvarieties

$$\begin{aligned} s_1 &= 1; \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \cdots; \text{ and} \\ \sum_{i=1}^{j} s_i &\in \mathbb{Z}_{\leq j}, \ (3 \leq j \leq k). \end{aligned}$$

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- Behavior near the singularities: Simple poles. For example, near $(0, 0)$

$$\zeta(s_1, s_2) = \frac{5s_1 + 4s_2}{12(s_1 + s_2)} + R_2(s_1, s_2)$$

where $R_2(s_1, s_2)$ is an entire function near $(0, 0)$ with $R_2(0, 0) = 0$.

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- Trouble: Most special values of $\zeta(s_1, \cdots, s_k)$, $k > 1$, at nonpositive integers are not defined.

- MZV**s as directional limits** (Akiyama, Egami and Tanigawa):

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- The definitions do not agree with each other. For $\zeta(r_1,r_2)$ at $(r_1,r_2)=(0,0)$, the proposed values are $5/12, 1/3, 3/8$ respectively. In fact, by letting (r_1,r_2) approach to $(0,0)$ along different paths, one can get any values, as well as the infinity, to be the limit.

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- The values do not satisfy either the shuffle of quasi-shuffle relation.
- Renormalization approach:

  • in one variable case, use the generating function of Bernoulli numbers;

  • in multiple variable case, use the renormalization procedure in QFT, in the Hopf algebra framework of Connes-Kreimer.

► 7. Non-positive MZVs by renormalization.

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$$\frac{e^{\varepsilon}}{1-e^{\varepsilon}} = -\frac{1}{\varepsilon}\frac{-\varepsilon}{e^{-\varepsilon}-1} = -\frac{1}{\varepsilon} + \sum_{k\geq 0} \zeta(-k)\frac{\varepsilon^k}{k!}$$

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$$Z(s;\varepsilon):=\sum_{n\geq 1}\frac{e^{n\varepsilon}}{n^s}=\frac{d^{-s}}{d\varepsilon}\left(\frac{e^{\varepsilon}}{1-e^{\varepsilon}}\right)=\frac{d^{-s}}{d\varepsilon}\left(-\frac{1}{\varepsilon}+\sum_{k\geq 0}\zeta(-k)\frac{\varepsilon^k}{k!}\right).$$

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- Define

$$\tilde{P}\left(\sum_{n\geq N} a_n\varepsilon^n\right) = \sum_{n\geq 0} a_n\varepsilon^n$$

Then for the "renormalized value", we have (Euler)

$$\tilde{P}Z(s,\varepsilon)\big|_{\varepsilon=0} = \zeta(s).$$

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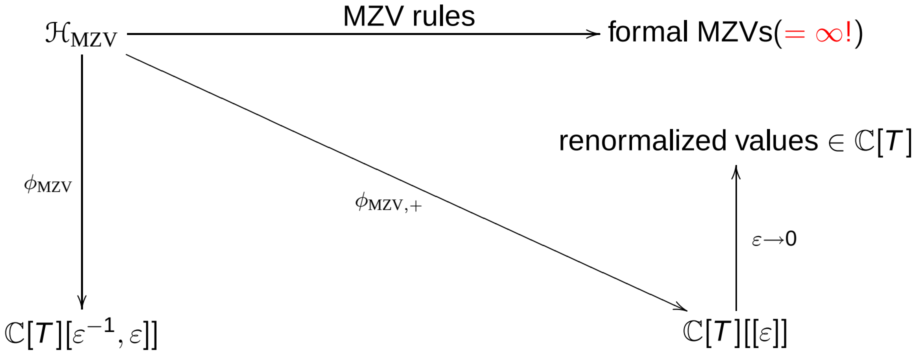
$$\phi_{\text{MZV}} = \phi_{\text{MZV},-} \star \phi_{\text{MZV},+}.$$



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- **Regularized MZVs.** For $\vec{s} \in \mathbb{Z}^k$, define the formal MZV

$$\zeta(\vec{s}) = \sum_{n_1 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

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- For $(\vec{s}, \vec{r}) \in \mathbb{Z}^k \times \mathbb{R}^k_{>0}$, define regularized MZV

$$Z\left(\left[{\vec{s} \atop \vec{r}}\right]; \varepsilon\right) = \sum_{n_1 > \cdots > n_k > 0} \frac{e^{n_1 r_1 \varepsilon} \cdots e^{n_k r_k \varepsilon}}{n_1^{s_1} \cdots n_k^{s_k}},$$

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  convergent for $Re(\varepsilon) < 0$.
- Key properties: 1. $Z\left(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon\right)$ multiply according to the quasi-shuffle product: $Z\left(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon\right) Z\left(\left[\begin{smallmatrix}\vec{s}\,'\\ \vec{r}\,'\end{smallmatrix}\right]; \varepsilon\right) = Z\left(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right] \star \left[\begin{smallmatrix}\vec{s}\,'\\ \vec{r}\,'\end{smallmatrix}\right]; \varepsilon\right)$.

  2. $Z\left(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon\right)$ has Laurent series expansion in $\mathbb{C}[T][\varepsilon^{-1}, \varepsilon]]$.

  3. When $\vec{s} \leq 0$, $Z\left(\left[\begin{smallmatrix}\vec{s}\\ \vec{r}\end{smallmatrix}\right]; \varepsilon\right)$ has Laurent series expansion in $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$.

- **The Hopf algebra of general MZVs:** Take the commutative semigroup

$$\mathfrak{M} = \{f_{\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]} \mid (s,r) \in \mathbb{Z} \times \mathbb{R}_{>0}\}$$

with the multiplication $f_{\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]} f_{\left[\begin{smallmatrix} s' \\ r' \end{smallmatrix}\right]} = f_{\left[\begin{smallmatrix} s+s' \\ r+r' \end{smallmatrix}\right]}$. Then

$$\mathcal{H}_{\mathrm{MZV}} := Ш^{+}(\mathfrak{M}) = \sum_{n \geq 0} \mathbb{C}\, \mathfrak{M}^n,$$

with the quasi-shuffle (=mixable shuffle) product and deconcatenation coproduct, is a filtered connected Hopf algebra.

- **The Hopf algebra of general MZVs:** Take the commutative semigroup

$$\mathfrak{M} = \{ f_{\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]} \mid (s, r) \in \mathbb{Z} \times \mathbb{R}_{>0} \}$$

with the multiplication $f_{\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]} f_{\left[\begin{smallmatrix} s' \\ r' \end{smallmatrix}\right]} = f_{\left[\begin{smallmatrix} s+s' \\ r+r' \end{smallmatrix}\right]}$. Then

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- Deconcatenation coproduct: $\Delta(a_1, \cdots, a_m) = (a_1, \cdots, a_m) \bigotimes 1 + (a_1, \cdots, a_{m-1}) \bigotimes a_m + \cdots + 1 \bigotimes (a_1, \cdots, a_m)$

- **Laurent series of regularized MZVs:**
  Let $\mathcal{R}_{\mathrm{MZV}} = \mathbb{C}[T][[\varepsilon]][\varepsilon^{-1}]$ and $R_{\mathrm{MZV}} := \mathbb{C}[[\varepsilon]][\varepsilon^{-1}]$. Let $\mathcal{P}$ and P be the projections to the pole parts. Then both $(\mathcal{R}, \mathcal{P})$ and (R, P) are Rota-Baxter algebras.

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- **Regularized MZV rule ϕ_{MZV}:** For $(\vec{s}, \vec{r}) \in \mathbb{Z}^k \times \mathbb{R}^k_{>0}$, define

$$\phi_{\text{MZV}}(f_{[\vec{s} \atop \vec{r}]}) := Z([{\vec{s} \atop \vec{r}}]; \varepsilon) = \sum_{n_1 > \cdots > n_k > 0} \frac{e^{n_1 r_1 \varepsilon} \cdots e^{n_k r_k \varepsilon}}{n_1^{s_1} \cdots n_k^{s_k}}$$

The map $\phi_{\text{MZV}} := Z : \mathcal{H}_{\text{MZV}} \to \mathcal{R}_{\text{MZV}}, f_{[\vec{s} \atop \vec{r}]} \mapsto Z([{\vec{s} \atop \vec{r}}]; \varepsilon)$ is an algebra homomorphism.

- **9. Renormalization of regularized MZVs:**
  Let $\mathcal{R}_+ = \mathbb{C}[T][[\varepsilon]], \mathcal{R}_- = \mathbb{C}[T][\varepsilon^{-1}]$.
  Algebraic Birkhoff for $Z = \phi_{\mathrm{MZV}}$: There is a unique decomposition

$$Z = Z_-^{-1} \star Z_+$$

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- Define the directional renormalized MZVs

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- Proof: $\zeta : \mathcal{H}_{\rm MZV} \to \mathbb{C}$ is the composition of algebra homomorphisms $Z_+ : \mathcal{H}_{\rm MZV} \to \mathcal{R}_+$ and $eva_{\varepsilon=0} : \mathcal{R}_+ \to \mathbb{C}[T]$, so is still an algebra homomorphism. Since product in $\mathcal{H}_{\rm MZV}$ satisfies quasi-shuffle relation, so does their images.

- **Renormalized MZVs.** For $\vec{s} \in \mathbb{Z}^k_{>0} \cup \mathbb{Z}^k_{\leq 0}$, define

$$\bar{\zeta}(\vec{s}) = \zeta\left(\left[\begin{smallmatrix}\vec{s}\\ -\vec{s}\end{smallmatrix}\right]\right) = \lim_{\delta \to 0^+} \zeta\left(\left[\begin{smallmatrix}\vec{s}\\ |\vec{s}| + \delta\end{smallmatrix}\right]\right),$$

where, for $\vec{s} = (s_1, \cdots, s_k)$ and $\delta \in \mathbb{R}_{>0}$, we denote $|\vec{s}| = (|s_1|, \cdots, |s_k|)$ and $|\vec{s}| + \delta = (|s_1| + \delta, \cdots, |s_k| + \delta)$. These $\bar{\zeta}(\vec{s})$ are called the renormalized multiple zeta values of the multiple zeta function $\zeta(u_1, \cdots, u_k)$ at $\vec{s}$.

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- **Theorem:** (1) $\bar{\zeta}(\vec{s})$ are well-defined and satisfy the quasi-shuffle relation.
  (2) $\bar{\zeta}(\vec{s})$ agree with the usual definitions of $\zeta(\vec{s})$, defined
  - either by convergence of the sums,
  - or by the regularization of Zagier et al.
  - or by analytic continuation.

- In the following table, the element in row s_1 and column s_2 is $\bar{\zeta}(-s_1, -s_2)$, $1 \leq s_1 \leq 7,\ 1 \leq s_2 \leq 8$.

$\frac{1}{288}$	$-\frac{1}{240}$	$\frac{83}{64512}$	$\frac{1}{504}$	$-\frac{3925}{2239488}$	$-\frac{1}{480}$	$\frac{342884347}{99656663040}$
$-\frac{1}{240}$	0	$\frac{1}{504}$	$-\frac{319}{437400}$	$-\frac{1}{480}$	$\frac{2494519}{1362493440}$	$\frac{1}{264}$
$-\frac{71}{35840}$	$\frac{1}{504}$	$\frac{1}{28800}$	$-\frac{1}{480}$	$\frac{114139507}{139519328256}$	$\frac{1}{264}$	$-\frac{313042283533}{93600000000000}$
$\frac{1}{504}$	$\frac{319}{437400}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$-\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$
$\frac{32659}{15676416}$	$-\frac{1}{480}$	$-\frac{21991341}{25836912640}$	$\frac{1}{264}$	$\frac{1}{127008}$	$-\frac{691}{65520}$	$\frac{26194796926873}{5884626295848960}$
$-\frac{1}{480}$	$-\frac{2494519}{1362493440}$	$\frac{1}{264}$	$\frac{41796929201}{26873437500000}$	$-\frac{691}{65520}$	0	$\frac{1}{24}$
$\frac{-75497471}{19931332608}$	$\frac{1}{264}$	$\frac{316292283533}{93600000000000}$	$-\frac{691}{65520}$	$\frac{-36808933898915}{8238476814188544}$	$\frac{1}{24}$	$\frac{1}{115200}$
$\frac{1}{264}$	$\frac{16608667097}{2879296875000}$	$-\frac{691}{65520}$	$-\frac{4607695}{491051484}$	$\frac{1}{24}$	$\frac{63967403428993199}{3561322226607185040}$	$-\frac{3617}{16320}$

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