Some background for Manin's theorem $\mathbf{K}(\mathbb{F}_1) \sim \mathbb{S}$

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§I. Introduction

Forty years ago Quillen created modern algebraic K-theory by proposing a new foundation for the subject, in which ($i \geq 1$)

$$K_{2i}(\mathbb{F}_q) = 0, \;\; K_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1)\mathbb{Z} \; .$$

The revolutionary ideas behind this calculation, more than the calculation itself, have had continuously growing impact, leading us out of the Wilderness of Algebra into the Promised Land of Topology – or, more precisely, of homotopy theory. I'll try to justify that claim in this talk.

The techniques Quillen pioneered generalize those of classical homological algebra; in his language a **resolution** can be interpreted as a kind of homotopy-equivalent replacement with better properties. This has deep roots in earlier work of Dold, Kan, and others on non-abelian homological algebra, which Quillen first applied to Grothendieck's program for the construction of a cotangent complex for morphisms of commutative rings; but his theory of (what are now called) **model** categories[1] does not require the categories of interest to be additive.

Using Kan's theory of (semi)simplicial sets, he took seriously the idea that categories should be treated as concrete **things**, like groups or spaces. This led to the idea that (higher) K-theory should be definable for a broad class of symmetric monoidal categories, not just that of (projective) modules over a ring. This has been immensely productive, resulting in an evolutionary chain

$$\text{Kan \& Grothendieck} \to \text{Quillen} \to \text{Segal} \to \text{Waldhausen} \to \text{Lurie} \dots$$

which continues to develop and thrive.

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[1] I have not tried to define model categories here; roughly, they are nice categories with special classes of morphisms called fibrations, cofibrations, and weak equivalences, related by suitable axioms. Inverting the weak equivalence defines an associated homotopy category. Algebraists seem very comfortable in such contexts; model categories are in many ways like rigidified versions of derived categories (which they generalize).

{ This commentary was added as an afterthought. I apologize for the rather polemic tone of this talk: it's a little heavy on the rhetoric. But I wanted to emphasize that many constructions of classical algebra (eg, the theory of modular forms) are beginning to be seen to have deep homotopy-theoretic foundations. That story arguably began with Quillen's work.

I should have included the work of Boardman-Vogt and Joyal, and probably many others, in this line of descent. }

§II. Symmetric monoidal categories

These are categories $\mathcal{C}$ with a product operation $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ which is commutative and associative **up to canonical isomorphisms**. Here are some examples:

- $(k - \mathrm{Vect}_{\mathrm{fg}}), \cong, \oplus$: finite-dimensional k-vector spaces up to isomorphism;

- $(\mathrm{Proj}_{\mathrm{fg}}/R), \cong, \oplus$: finitely-generated projective R-modules up to isomorphism;

- $((\text{finite) Sets}), \cong, \coprod$: finite sets up to bijection, with disjoint union as composition;

- $((\text{finite CW) Spaces}), \cong, \vee$: finite cell complexes, under wedge sum.

{ These examples emphasize algebra at the expense of topology: there is a nice K-theory of piecewise linear bundles, for example, and recent work of Madsen-Tillmann-Weiss on the Mumford conjecture work with categories of Riemann surfaces under glueing. }

We want to generalize Grothendieck's original set-theoretic construction

$$K : (\text{commutative monoids}) \to (\text{abelian groups})$$

to this categorical context. Segal, following Grothendieck, understood that any category $\mathcal{C}$ can be presented as the simplicial set with strings

$$V_0 \to V_1 \to \cdots \to V_n$$

of composable morphisms as its collection $\mathcal{C}[n]$ of n-simplices; alternately

$$\mathcal{C}[n] := \mathrm{Func}(\mathrm{Ord}(n), \mathcal{C})$$

(where $\text{Ord}(n) = \{0 < 1 \cdots < n\}$ regarded as a category). Monotone maps between ordered sets define functors between such categories, defining face and degeneracy maps on the sets $\{\mathcal{C}[n]\}$.

{ NB, $\mathcal{C}[n]$ is not a set of isomorphism classes. This can seriously strain one's set theory, but it is generally agreed that the resulting size issues are rarely serious. }

Such a simplicial object in the category of sets has a geometrical realization (or nerve, or classifying space)

$$\mathcal{C} \mapsto |\mathcal{C}| = \coprod_{n \geq 0} \mathcal{C}[n] \times \Delta^n / (\text{face \& deg rel'ns}) ,$$

defining a **product-preserving** functor from categories to spaces.

Remarks and examples:

i) $\pi_0(|\mathcal{C}|)$ is the set of isomorphism classes of objects of $\mathcal{C}$.

ii) A group G defines a category $\mathcal{G}$ with one object and set $\mathcal{G}[1] = G$ of morphisms; its geometrical realization $|\mathcal{G}| = BG$ is a classifying space for G-bundles[2], and

$$H^*(BG) \cong H^*_{\text{alg}}(G) .$$

iii) It will important later that there is a similar geometrical realization construction for simplicial **spaces** – though this may involve working in some category of compactly generated spaces, and requires more care. For example, a topological group defines a topological category, whose realization is the classifying space for the topological group.

iv) When $\mathcal{C}$ is a symmetric monoidal category, $|\mathcal{C}|$ is **up to homotopy** a commutative monoid in the category of spaces. In the first example above, the classifying space is

$|(k - \text{Vect}_{\text{fg}})| = \coprod_{n \geq 0} B\text{Gl}_n(k)$;

in the second,

[2] At the beginning of the twentieth century, the logician Frege defined the cardinal number of a set S to the the class of all classes in bijection with S. In modern terms this is not a set, but a category, whose nerve is the classifying space for symmetric group Σ_S. This seems to confirm one's intuition that Frege's notion of cardinal number is a very subtle object

$|(\mathrm{Proj}_{\mathrm{fg}}/R)| = \coprod_{\pi_0} B\mathrm{Aut}(P)$;

the third has

$|((\text{finite}) \text{ Sets})| = \coprod_{n \geq 0} B\Sigma_n$,

while the fourth is a little too complicated to write down without a distracting digression; it involves the study of an analog of the group ring for the monoid of stable self-maps of the sphere.

§III. Commutative group objects in the homotopy category

It is important that these objects are commutative and associative only up to homotopy; they are not topological monoids in the strict sense. However, a delicate argument (which can be swept under the rug by astute constructions)[3] implies that if the isomorphisms in the underlying category are coherent in the sense of MacLane, then **the classifying space construction can be iterated**, yielding an 'infinite loop-spectrum'

$$B^k|\mathcal{C}| \cong \Omega B^{k+1}|\mathcal{C}| \cong \cdots \cong \Omega^n B^{k+n}|\mathcal{C}| \cong \cdots$$

$(k \geq 1)$. The construction

$$\mathcal{C} \mapsto \Omega B|\mathcal{C}| := \mathbf{K}(\mathcal{C}) \in \text{Spectra}$$

is the homotopy-theoretic analog of Grothendieck's functor (which replaces a set-theoretic monoid with its best approximation by an abelian group). In particular,

$$\pi_0 \mathbf{K}(\mathcal{C}) = \pi_0 \Omega B|\mathcal{C}| \cong K(\pi_0|\mathcal{C}|)$$

is the 'group completion' of the monoid of isomorphism classes of objects in $\mathcal{C}$. This generalizes to an isomorphism

$$H_*(\Omega B|\mathcal{C}|) \cong H_*(|\mathcal{C}|) \otimes_{\mathbb{Z}[\pi_0|\mathcal{C}|]} \mathbb{Z}[K_0(\mathcal{C})] \ .$$

{ Ω and B are adjoint in some sense, and $B\Omega X \sim X$. }

[3]Segal's idea is, very roughly, to replace a system of composite maps from A^m to A^n which might commute only modulo a highly nontrivial system of coherent isomorphisms, with a related system of **strictly** commutative diagrams of maps between objects $A\{m\}$ and $A\{n\}$ which are homotopy-equivalent to those of the original system. In fact this idea has a long history in homotopy theory, and I won't try to do it justice here.

We obtain a K-theory for symmetric monoidal categories which takes values in the (additive) category of infinite loop-spectra. For example,

$$\Omega B(\coprod B\mathrm{Gl}_n(k)) \cong \mathbb{Z} \times B\mathrm{Gl}_\infty^+(k) := \mathbf{K}(k) ,$$

{ NB the composition comes from 'Whitney sum' $\oplus$, corresponding to block sum of matrices. }

while

$$\Omega B(\coprod B\Sigma_n) = \lim \Omega^n S^n := Q(S^0)$$

produces the infinite loop-spectrum $\mathbb{S}$ representing stable homotopy theory:

$$\pi_i(Q(S^0)) = \lim \pi_{n+i}(S^n) = \pi_i^S(\mathrm{pt}) .$$

Digression:

Group multiplication $G \times G \to G$ is a group homomorphism iff $G = A$ is abelian. It follows that the category $\mathcal{A}$ defined by an **abelian** group has a **strictly** commutative and associative monoidal structure, so for such spaces the naive classifying space construction can be iterated, yielding the infinite loop-spectrum $B^k|\mathcal{A}| = H(A,k)$ of Eilenberg-MacLane spaces, with

$$\pi_i H(A,k) = A, \text{ if } i = k; \; = 0 \text{ otherwise.}$$

If X is a (connected, pointed) space, then the infinite symmetric product

$$\mathrm{SP}^\infty X = \bigcup_{n \geq 0} X^n/\Sigma_n$$

is the free commutative monoid generated by X; it is a generalized EM space $[\pi_* \mathrm{SP}^\infty X \cong H_*(X,\mathbb{Z})]$ by a theorem of Dold and Thom.

On the other hand

$$Q(X) = \bigcup_{n \geq 0} \Omega^n S^n X$$

is the free **homotopy**-commutative monoid generated by X.

These constructions are related through a diagram[4]

$$X^n/\Sigma_n \leftarrow \mathrm{Config}^n(\mathbb{R}^k) \times_{\Sigma_n} X^n \to \mathrm{Maps}(\mathbb{R}^k_+, X \wedge \mathbb{R}^k_+) = \Omega^n S^n X ;$$

[4]The right-hand arrow generalizes the construction which sends an n-tuple $\{z_k \in \mathbb{C}\}$ of distinct points to the polynomial $p(z) = \prod(z - z_k)$, regarded as a map from the Riemann sphere to itself.

where $\mathrm{Config}^n(\mathbb{R}^k)$ is the space of n-tuples of distinct points in $\mathbb{R}^k$: it admits a free action of the symmetric group, and becomes contractible as n goes to ∞. In (co)homology with rational coefficients the distinction between the left and right objects is lost. They differ over $\mathbb{Z}$ because the homotopy-theoretic object sees the subtleties in allowing indexing sets for compositions which wander around at will in the universe of discourse.

Abelian group objects in the homotopy category of spaces (ie, spectra) are thus richer and more subtle objects than abelian groups in the category of sets. Taking homotopy groups maps them to the category of graded abelian groups, but that loses a great deal of information (encoded in classical terms by their Postnikov invariants).

Quillen's approach to K-theory was the beginning of a great mathematical migration in algebra, from the world of set-theory to the homotopy-theoretic world of spectra. { More propaganda, but intended seriously: for example, noncommutative motives will probably require nontrivial homotopy-theoretic techniques . . . }

Waldhausen, motivated by questions from geometric topology, later seized on these ideas and reworked them further, extending them to a very general class of categories with a significant weakening of Quillen's model structures. In particular, he showed that ring objects in the category of spectra (the things that represent cohomology theories with good multiplicative structures) have a reasonable K-theory of their own. Similarly, spectra associated to categories of well-behaved functors between symmetric monoidal categories can be used to construct bivariant versions of K-theory.

The K-theory of the category of finite spaces (example *iv* in §2) is deeply related to manifold topology; it can also be described as the K-theory of a category of modules over the sphere spectrum $\mathbb{S}$ representing stable homotopy theory. Waldhausen and his school also worked out computational methods for studying such theories, based on generalizations of Hochschild and cyclic homology to the world of spectra.

IV. Finite fields

1 The most natural route to Quillen's calculations uses the étale homotopy theory of Artin and Mazur, which provides an isomorphism

$$\hat{\mathbf{K}}(\overline{\mathbb{F}}_p) \cong \hat{\mathbf{K}}^{\mathrm{top}}(\mathbb{C})_{\neg p}$$

of the profinite completion of Quillen's groups with that of the complex topological K-spectrum, away from the prime p. [Such isomorphisms have been vastly extended by Suslin, to general algebraically closed fields.] However, in his Annals paper Quillen avoided this by reinterpreting classical (and more elementary) work of Brauer on representations of finite groups.

He showed that the isomorphism above can be constructed by lifting modular characters to complex-valued ones, providing a homotopy-theoretic analog for an honest complex representation of $\mathrm{Gl}_n(\overline{\mathbb{F}}_p)$. It's a beautiful argument, but I won't try to reproduce it here.

The next step is to realize that the classifying space $|\mathbb{F}_q - \mathrm{Vect}_{\mathrm{fg}}|$ is the fixed-points of the Frobenius endomorphism

$$x \mapsto \sigma^n(x) = x^q, \; q = p^n$$

acting on the classifying space $|\overline{\mathbb{F}}_p - \mathrm{Vect}_{\mathrm{fg}}|$. More precisely, Lang's fibration

$$B\mathrm{Gl}_n(\mathbb{F}_q) \to B\mathrm{Gl}_n(\overline{\mathbb{F}}_p) \to B\mathrm{Gl}_n(\overline{\mathbb{F}}_p)$$

(the second map sends X to $X^{-1}\sigma^n(X)$, with σ applied elementwise to the matrix) leads to a fibration

$$\mathbf{K}(\mathbb{F}_q) \to \mathbf{K}(\overline{\mathbb{F}}_p) \to \mathbf{K}(\overline{\mathbb{F}}_p)$$

of spectra.

The Frobenius endomorphism can be calculated on the model of $\mathbf{K}(\overline{\mathbb{F}}_p)$ provided by topological K-theory: its action is determined on characters, or equivalently on line bundles, where it can be identified with the representation-theoretic operation ψ^q defined by Adams' (and Newton's) formula[5]

$$\frac{\lambda_t'(V)}{\lambda_t(V)} = \sum_{k \geq 0} \psi^k(V)\, t^k \; .$$

Adams showed that

$$\psi^k : K_{2i}(\mathbb{C}) \to K_{2i}(\mathbb{C})$$

is multiplication by k^i, from which it follows that

$$K_{2i-1}(\mathbb{F}_q) \cong \pi_{2i-1}(\text{fiber of } \psi^q - 1)$$

equals

$$\mathrm{coker}(q^i - 1 : \hat{\mathbb{Z}}_{\neg p} \to \hat{\mathbb{Z}}_{\neg p}) \; .$$

[5] $\psi^q(L) = L^{\otimes q}$ for line bundles L.

2 The vector space

$$k\langle S\rangle = \mathrm{Maps}(S,k)^*$$

generated by S satisfies

$$k\langle S \coprod S'\rangle \cong k\langle S\rangle \oplus k\langle S'\rangle$$

and thus defines a monoidal functor from finite sets to k-vector spaces. According to the philosophy above, this gives us a map

$$Q(S^0) \to \Omega B|k - \mathrm{Vect}_{\mathrm{fg}}| : S^0 \to \mathbf{K}(k)$$

of (commutative ring) spectra.

This map is very interesting when k is finite, but the construction makes sense for $k = \mathbb{Z}$, and the maps to $\mathbf{K}(\mathbb{F}_q)$ factor through this 'universal' example: see Quillen's letter [6 §2] to Milnor. The image of this Hurewicz homomorphism[6] is cyclic, isomorphic to

$$\langle \zeta(1-2n)\rangle \subset \mathbb{Q}/\mathbb{Z}\ ;$$

a theorem of von Staudt-Clausen (and Atiyah) identifies its $p(\neq 2)$-component with a cohomology group

$$H^1_c(\mathbb{Z}_p^\times, \mathbb{Z}_p(n))$$

of Galois type (with $\mathbb{Z}_p(n)$ a kind of p-adic Tate representation

$$u, v \mapsto u^n v : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to \mathbb{Z}_p).$$

This cyclic subgroup is the image of Whitehead's J-homomorphism

$$\mathbb{Z} \cong \pi_{4n}(B\mathrm{O}) \cong \pi_{4n-1}(\mathrm{O}) \to \pi^S_{4n-1}(\mathrm{pt})\ ,$$

induced by the inclusion

$$\mathrm{O} = \lim \mathrm{O}(n) \to \lim \Omega^{n-1} S^{n-1} = Q(S^0)\ .$$

There is thus a deep and intimate relation between the algebraic K-theory of finite fields and stable homotopy theory; but there is more to the stable homotopy ring than the elements (now said to be of chromatic level one) constructed in this way.

[6] ignoring its 2-component

3 To get some sense of what's at issue, note that a set of cardinality n defines an $\mathbb{F}_q$-vector space of cardinality q^n; this construction is monoidal, since $q^{n+m} = q^n \cdot q^m$. To go backwards, however, is more difficult. An $\mathbb{F}_q$-vector space of dimension n defines a projective space of cardinality

$$\#(\mathbb{P}_n) = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1} ;$$

this function is not monoidal, but it becomes so as $q \to 1$. It appears that, to see the rest of stable homotopy from this viewpoint, we'll have to look for some higher corrections to l'Hopital's rule.

This highlights the apparent fact that the spectrum $\mathbb{S}^\times$ defined by the symmetric monoidal category of finite pointed sets under **Cartesian product** has not been systematically studied. The functor which sends an $\mathbb{F}_q$-vector space to its underlying set, with the origin as basepoint, defines a very interesting composition

$$\mathbb{S} \to K(\mathbb{F}_q) \to \mathbb{S}^\times$$

of maps of spectra. Note that $\pi_0\mathbb{S}^\times \cong \mathbb{Q}_+^\times$ is the free abelian group generated by the primes. It is not at all clear if or how $\pi_*\mathbb{S}^\times$ might be related to the (graded) multiplicative group

$$(1 + t\pi_*^S(\text{pt})[[t]])^\times ,$$

but it seems interesting that the homomorphism $n \mapsto (1 + t)^n$ specializes, for $t = q - 1$, to the function $n \mapsto q^n$. It would be interesting to understand its relation, if any, to $\mathbf{K}(\mathbb{Z})$.

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