Cyclic Cohomology, Quantum group Symmetries and the Local Index Formula for $SU_q(2)$

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Abstract

We analyse the NC-space underlying the quantum group $SU_q(2)$ from the spectral point of view which is the basis of noncommutative geometry, and show how the general theory developped in our joint work with H. Moscovici applies to the specific spectral triple defined by Chakraborty and Pal. This provides the pseudo-differential calculus, the Wodzciki-type residue, and the local cyclic cocycle giving the index formula. The cochain whose coboundary is the difference between the original Chern character and the local one is given by the remainders in the rational approximation of the logarithmic derivative of the Dedekind eta function. This specific example allows to illustrate the general notion of locality in NCG. The formulas computing the residue are "local". Locality by stripping all the expressions from irrelevant details makes them computable. The key feature of this spectral triple is its equivariance, i.e. the $SU_q(2)$-symmetry. We shall explain how this leads naturally to the general concept of invariant cyclic cohomology in the framework of quantum group symmetries.

1 Introduction

In noncommutative geometry a geometric space is described from a spectral point of view, as a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of a $*$-algebra $\mathcal{A}$ represented in a Hilbert space $\mathcal{H}$ together with an unbounded selfadjoint operator D, with compact resolvent, which interacts with the algebra in a bounded fashion. This spectral data embodies both the metric and the differential structure of the geometric space.
An essential ingredient of the general theory is the Chern character in K-homology which together with cyclic cohomology and the spectral sequence relating it to Hochschild cohomology, were defined in 1981 (cf. [4],[5],[6]). The essence of the theory is to allow for computations of differential geometric nature in the non-commutative framework.
While basic examples such as the non-commutative tori were analysed as early as 1980 (cf. [7]), the case of the underlying NC-spaces to quantum groups has been left aside till recently, mainly because of the "drop of dimension" which occurs when the deformation parameter q affects non-classical values $q \neq 1$. Thus for instance the Hochschild dimension of $SU_q(2)$ drops from the classical value $d = 3$ to $d = 1$ and these NC-spaces seem at first rather esoteric.
A very interesting spectral triple for $SU_q(2)$, $q \neq 1$, has been proposed in [3]. Thus the algebra $\mathcal{A}$ is the algebra of functions on $SU_q(2)$ and the representation in $\mathcal{H}$ is the coregular representation of $SU_q(2)$. The operator D is very simple, and is invariant under the action of the quantum group $SU_q(2)$. (The Anzats proposed in a remark at the end of [8] provides the right formula for $|D|$ but not for the sign of D as pointed out in [17]).
Our purpose in this paper is to show that the general theory developped by Henri Moscovici and the author (cf.[9]) applies perfectly to the above spectral triple.
The power of the general theory comes from general theorems such as the local computation of the analogue of Pontrjagin classes: *i.e.* of the components of the cyclic cocycle which is the Chern character of the K-homology class of D and which make sense in general. This result allows, using the infinitesimal calculus, to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$. The notion of locality which is straightforward for classical spaces is more elaborate in the non-commutative situation and relies essentially on the non-commutative integral which is the Dixmier trace in the simplest case and the analogue of the Wodzicki residue in general. Its validity requires the discreteness of the dimension spectrum, a subset of $\mathbb{C}$ which is an elaboration of the classical notion of dimension. At an intuitive level this subset is the set of "dimensions", possibly complex, in which the NC-space underlying the spectral triple manifests itself non-trivially. At the technical level it is the set of singularities of functions,

$$\zeta_b(z) = \mathrm{Trace}\,(b|D|^{-z}) \qquad \mathrm{Re}\, z > p\ ,\ \ b \in \mathcal{B}\,. \tag{1}$$

where $b \in \mathcal{B}$ varies in a suitable algebra canonically associated to the triple and allowing to develop the pseudo-differential calculus.
Our first result is that in the above case of $SU_q(2)$, the dimension spectrum is simple and equal to $\{1, 2, 3\} \subset \mathbb{C}$. Simplicity of the dimension spectrum means that the singularities of the functions (1) are at most simple poles. It then follows from the general results of [9] that the equality,

$$⨍ P = \mathrm{Res}_{z=0}\ \mathrm{Trace}(P|D|^{-z}) \tag{2}$$

defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.
Our second result is the explicit computation of this functional in the above case of $SU_q(2)$. In doing so we shall also determine the analogue of the cosphere bundle in that example and find an interesting space S_q^*. This space is endowed with a one parameter group γ_t of automorphisms playing the role of the geodesic flow, and is intimately related to the product $D^2_{q+} \times D^2_{q-}$, of two NC-two-disks, while the coproduct gives its relation to $SU_q(2)$. The formulas computing the residue will be "local" and very simple, locality by stripping all the expressions from irrelevant details makes them computable.
Our third result which is really the main point of the paper, is the explicit formula for the local index cocycle, which owing to the metric dimension 3 is a priori given by the following cocycle,

$$\varphi_1(a^0, a^1) = ⨍ a^0[D, a^1]|D|^{-1} - \frac{1}{4} ⨍ a^0 \nabla([D, a^1])\,|D|^{-3} + \frac{1}{8} ⨍ a^0 \nabla^2([D, a^1])\,|D|^{-5} \tag{3}$$

and,

$$\varphi_3(a^0, a^1, a^2, a^3) = \frac{1}{12} ⨍ a^0[D, a^1][D, a^2][D, a^3]\,|D|^{-3}\,, \tag{4}$$

where $\nabla(T) = [D^2, T] \quad \forall T$ operator in $\mathcal{H}$. We shall begin by working out the degenerate case $q = 0$ with a luxury of details, mainly to show that the numerical coefficients involved in the above formula are in fact unique in order to get a (non-trivial) cocycle. The coboundary involved in the formula (theorem 3) will then be conceptually explained (in the section "η-Cochain") and the specific values $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$ of the Riemann Zeta function will account for the numerical coefficients encountered in the coboundary.
We shall then move on to the general case $q \in]0, 1[$ and construct the pseudo-differential calculus on $\mathrm{SU}_q(2)$ following the general theory of [9]. We shall determine the algebra of complete symbols by computing the quotient by smoothing operators. This will give the cosphere bundle S_q^* of $\mathrm{SU}_q(2)$ already mentionned above. The analogue of the geodesic flow will give a one-parameter group of automorphisms γ_t of $C^\infty(S_q^*)$. We shall also construct the restriction morphism r to the product of two non-commutative 2-disks,

$$r : C^\infty(S_q^*) \to C^\infty(D^2_{q+} \times D^2_{q-}) \tag{5}$$

We shall then show that the dimension spectrum of $\mathrm{SU}_q(2)$ in the above spectral sense is $\{1, 2, 3\}$ and compute the residues in terms of the symbol $\rho(b) \in C^\infty(S_q^*)$ of the operator b of order 0. If one lets $\rho(b)^0$ be the component of degree 0 for the geodesic flow γ_t, the formulas for the residues are,

$$⨍ b\,|D|^{-3} = (\tau_1 \otimes \tau_1)(r\rho(b)^0)$$

$$⨍ b\,|D|^{-2} = (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r\rho(b)^0)$$

$$⨍ b\,|D|^{-1} = (\tau_0 \otimes \tau_0)(r\rho(b)^0)\,,$$

where r is the above restriction map to $D^2_{q+} \times D^2_{q-}$. The algebras $C^\infty(D^2_{q\pm})$ are Toeplitz algebras and as such are extensions of the form,

$$0 \longrightarrow \mathcal{S} \longrightarrow C^\infty(D^2_{q\pm}) \stackrel{\sigma}{\longrightarrow} C^\infty(S^1) \longrightarrow 0 \tag{6}$$

where the ideal $\mathcal{S}$ is isomorphic to the algebra of matrices of rapid decay. The functional τ_1 is the trace obtained by integrating $\sigma(x)$ on S^1, while τ_0 is a regularized form of the trace on the ideal $\mathcal{S}$. Due to the need of regularization, τ_0 is not a trace but its Hochschild coboundary (which measures the failure of the trace property) is easily computed in terms of the canonical morphism σ.
A similar long exact sequence, and pair of functionals τ_j make sense for $\mathcal{A} = C^\infty(SU_q(2))$. They are invariant under the one parameter group of automorphisms generated by the derivation ∂, which rotates the canonical generators in opposite ways. Using this derivation together with the second derivative of $\sigma(x)$ to define the differential we then show how to construct a one dimensional cycle (in the sense of ([5])) whose character is extremely simple to compute. This shows how to bypass the shortage of traces on $\mathcal{A} = C^\infty(SU_q(2))$ to obtain a significant calculus.
Our main result (theorem 5) is that the local formula for the Chern character of the above spectral triple gives exactly the above cycle, thus completing the original computation. Another quite remarkable point is that the cochain whose coboundary is the difference between the original Chern character and the local one is given by the remainders in the rational approximation of the logarithmic derivative of the Dedekind eta function. The computation of this non-local cochain is very involved.
One fundamental property of the above spectral triple is its equivariance ([3]) under the action of the quantum group $SU_q(2)$. In the last section we shall use this example to obtain and explain in general a new concept of quantum group invariance in cyclic cohomology.

2 Operator theoretic Local Index Formula

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The Fredholm index of the operator D determines (in the odd case) an additive map $K_1(\mathcal{A}) \longrightarrow \mathbb{Z}$ given by the equality

$$\varphi([u]) = \text{Index}\,(PuP), u \in GL_1(\mathcal{A}) \tag{7}$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}\,(D)$.
This map is computed by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, \ldots, a^n) = \text{Trace}\,(a^0[F, a^1] \ldots [F, a^n]) \qquad \forall\, a^j \in \mathcal{A} \tag{8}$$

where $F = \text{Sign}\ D$ and we assume that the dimension p of our space is finite, which means that the characteristic values μ_k of $(D+i)^{-1}$ decay like $k^{-1/p}$, also $n \geq p$ is an odd integer. There are similar formulas involving the grading γ in the even case.
The cyclic cohomology $HC^n(\mathcal{A})$ is defined as the cohomology of the complex of cyclic cochains, i.e. those satisfying

$$\psi(a^1, \ldots, a^n, a^0) = (-1)^n\ \psi(a^0, \ldots, a^n)\ , \qquad \forall a^j \in \mathcal{A}\ , \tag{9}$$

under the coboundary operation b given by:

$$(b\psi)(a^0, \ldots, a^{n+1}) = \sum_0^n (-1)^j\ \psi(a^0, \ldots, a^j\ a^{j+1}, \ldots, a^{n+1}) \tag{10}$$
$$+(-1)^{n+1}\ \psi(a^{n+1}\ a^0, \ldots, a^n)\ , \quad \forall a^j \in \mathcal{A}\ .$$

Equivalently, $HC^n(\mathcal{A})$ can be described in terms of the second filtration of the (b, B) bicomplex of arbitrary (non cyclic) cochains on $\mathcal{A}$, where $B : C^m \to C^{m-1}$ is given by

$$(B_0\varphi)(a^0, \ldots, a^{m-1}) = \varphi(1, a^0, \ldots, a^{m-1}) - (-1)^m \varphi(a^0, \ldots, a^{m-1}, 1)$$
$$B = AB_0, \quad (A\psi)(a^0, \ldots, a^{m-1}) = \sum (-1)^{(m-1)j}\ \psi(a^j, \ldots, a^{j-1}) \tag{11}$$

To an n-dimensional cyclic cocycle ψ one associates the (b, B) cocycle $\varphi \in Z^p(F^q\ C)$, $n = p - 2q$ given by

$$(-1)^{[n/2]}\ (n!)^{-1}\ \psi = \varphi_{p,q} \tag{12}$$

where $\varphi_{p,q}$ is the only non zero component of φ.
Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $D^{-1} \in \mathcal{L}^{(p,\infty)}$, the precise normalization for its Chern character in cyclic cohomology is obtained from the following cyclic cocycle τ_n, $n \geq p$, n odd,

$$\tau_n(a^0, \ldots, a^n) = \lambda_n\ \text{Tr}'\left(a^0[F, a^1] \ldots [F, a^n]\right)\ , \qquad \forall a^j \in \mathcal{A}\ , \tag{13}$$

where $F = \mathrm{Sign} D$, $\lambda_n = \sqrt{2i}\,(-1)^{\frac{n(n-1)}{2}}\,\Gamma\left(\frac{n}{2}+1\right)$ and

$$\mathrm{Tr}'(T) = \frac{1}{2}\,\mathrm{Trace}(F(FT+TF)) \qquad (14)$$

If one wants to regard the cocycle τ_n of (13) as a cochain of the (b,B) bicomplex, one takes (12) into account and use instead of λ_n, the normalization constant $\mu_n = (-1)^{[n/2]}\,(n!)^{-1}\,\lambda_n = \sqrt{2i}\,\frac{\Gamma\left(\frac{n}{2}+1\right)}{n!}$.
It is difficult to compute the cocycle τ_n in general because the formula (13) involves the ordinary trace instead of the local trace $⨍$ and it is crucial to obtain a local form of the above cocycle.
In [13] we obtained the following general formula for the Hochschild cohomology class of τ_n in terms of the Dixmier trace :

$$\varphi_n(a^0,\ldots,a^n) = \lambda_n\ \mathrm{Tr}_\omega\left(a^0[D,a^1]\ldots[D,a^n]\,|D|^{-n}\right)\ ,\quad \forall a^j\in\mathcal{A}\ . \qquad (15)$$

The problem of finding a local formula for the *cyclic cohomology* Chern character, i.e. for the class of τ_n is solved by a general formula [9] which is expressed in terms of the (b,B) bicomplex and which we now explain.
Let us make the following regularity hypothesis on $(\mathcal{A},\mathcal{H},D)$

$$a \text{ and } [D,a] \in \cap\,\mathrm{Dom}\,\delta^k,\ \forall\, a\in\mathcal{A} \qquad (16)$$

where δ is the derivation $\delta(T) = [|D|,T]$ for any operator T.
We let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$, $\delta^k([D,a])$. The usual notion of *dimension* of a space is replaced by the *dimension spectrum* which is a subset of $\mathbb{C}$. The precise definition of the dimension spectrum is the subset $\Sigma\subset\mathbb{C}$ of singularities of the analytic functions

$$\zeta_b(z) = \mathrm{Trace}\,(b|D|^{-z}) \qquad \mathrm{Re} z > p\ ,\ b\in\mathcal{B}\,. \qquad (17)$$

Note that D may have a non-trivial kernel so that $|D|^{-s}$ is ill defined there. However the kernel of D is finite dimensional and the poles and residues of the above function are independent of the arbitrary choice of a non-zero positive value $|D| = \varepsilon$ on this kernel. The dimension spectrum of a manifold M consists of relative integers less than $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z\notin\mathbb{R}$ in the dimension spectrum.
We assume that Σ is discrete and simple, i.e. that ζ_b can be extended to $\mathbb{C}/\Sigma$ with simple poles in Σ. In fact the hypothesis only matters in a neighborhood of $\{z, \mathrm{Re}(z)\geq 0\}$.
Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple satisfying the hypothesis (16) and (17).
We shall use the following notations:

$$\nabla(a) = [D^2,a] \quad ; \quad a^{(k)} = \nabla^k(a)\ ,\quad \forall a \text{ operator in } \mathcal{H}\ .$$

The local index theorem is the following, [9]:

Theorem 1.

1. *The equality*

$$⨍ P = \mathrm{Res}_{z=0}\ \mathrm{Trace}(P|D|^{-z})$$

defines a trace on the algebra generated by $\mathcal{A}$, $[D,\mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.

2. *There is only a finite number of non–zero terms in the following formula which defines the odd components $(\varphi_n)_{n=1,3,\ldots}$ of a cocycle in the bicomplex (b,B) of $\mathcal{A}$,*

$$\varphi_n(a^0,\ldots,a^n) = \sum_k c_{n,k} ⨍ a^0[D,a^1]^{(k_1)}\ldots[D,a^n]^{(k_n)}\,|D|^{-n-2|k|} \qquad \forall\, a^j \in \mathcal{A}$$

where k is a multi-index, $|k| = k_1 + \ldots + k_n$,

$$c_{n,k} = (-1)^{|k|}\,\sqrt{2i}(k_1!\ldots k_n!)^{-1}\,((k_1{+}1)\ldots(k_1{+}k_2{+}\ldots{+}k_n{+}n))^{-1}\,\Gamma\left(|k| + \frac{n}{2}\right).$$

3. *The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.*

For the normalization of the pairing between HC^* and $K(\mathcal{A})$ see [13]. In the even case, i.e. when $\mathcal{H}$ is $\mathbb{Z}/2$ graded by γ,

$$\gamma = \gamma^*,\quad \gamma^2 = 1,\quad \gamma a = a\gamma \quad \forall\, a \in \mathcal{A},\ \gamma D = -D\gamma,$$

there is an analogous formula for a cocycle (φ_n), n even, which gives the Fredholm index of D with coefficients in K_0. However, φ_0 is not expressed in terms of the residue $⨍$ because the character can be non-trivial for a finite dimensional $\mathcal{H}$, in which case all residues vanish.
To give some concreteness to this general result we shall undertake the computation in an example, that of the quantum group $\mathrm{SU}_q(2)$. Its original interest is that it lies rather far from ordinary manifolds and is thus a good test case for the general theory.

3 Dimension Spectrum of $\mathrm{SU}_q(2)$: Case $q = 0$.

Let q be a real number $0 \le q < 1$. We start with the presentation of the algebra of coordinates on the quantum group $\mathrm{SU}_q(2)$ in the form,

$$\alpha^*\alpha+\beta^*\beta = 1\,,\ \alpha\alpha^*+q^2\beta\beta^* = 1\,,\ \alpha\beta = q\beta\alpha\,,\ \alpha\beta^* = q\beta^*\alpha\,,\ \beta\beta^* = \beta^*\beta\,. \quad (18)$$

Let us recall the notations for the standard representation of that algebra. One lets $\mathcal{H}$ be the Hilbert space with orthonormal basis $e_{ij}^{(n)}$ where $n \in \frac{1}{2}\mathbb{N}$ varies among half-integers while $i, j \in \{-n, -n+1, \ldots, n\}$.
Thus the first elements are,

$$e_{00}^{(0)}\ ,\ e_{ij}^{(1/2)}\ ,\ i,j \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \ldots$$

The following formulas define a unitary representation in $\mathcal{H}$,

$$\alpha e_{ij}^{(n)} = a_+(n,i,j)\, e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + a_-(n,i,j)\, e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})} \quad (19)$$

$$\beta e_{ij}^{(n)} = b_+(n,i,j)\, e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + b_-(n,i,j)\, e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}$$

where the explicit form of $a_\pm$ and $b_\pm$ is,

$$a_+(n,i,j) = q^{2n+i+j+1}\,\frac{(1-q^{2n-2j+2})^{1/2}(1-q^{2n-2i+2})^{1/2}}{(1-q^{4n+2})^{1/2}(1-q^{4n+4})^{1/2}} \quad (20)$$

$$a_-(n,i,j) = \frac{(1-q^{2n+2j})^{1/2}(1-q^{2n+2i})^{1/2}}{(1-q^{4n})^{1/2}(1-q^{4n+2})^{1/2}}$$

and

$$b_+(n,i,j) = -q^{n+j}\,\frac{(1-q^{2n-2j+2})^{1/2}(1-q^{2n+2i+2})^{1/2}}{(1-q^{4n+2})^{1/2}(1-q^{4n+4})^{1/2}} \quad (21)$$

$$b_-(n,i,j) = q^{n+i}\,\frac{(1-q^{2n+2j})^{1/2}(1-q^{2n-2i})^{1/2}}{(1-q^{4n})^{1/2}(1-q^{4n+2})^{1/2}}\,.$$

Note that a_- does vanish if $i = -n$ or $j = -n$, which gives meaning to $a_-(n,i,j)\, e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}$ for these values while $i - \frac{1}{2} \notin \left[-\left(n-\frac{1}{2}\right), n-\frac{1}{2}\right]$ or $j - \frac{1}{2} \notin \left[-\left(n-\frac{1}{2}\right), n-\frac{1}{2}\right]$. Similarly b_- vanishes for $j = -n$ or $i = n$.

Let now as in ([3]), D be the diagonal operator in $\mathcal{H}$ given by,

$$D(e_{ij}^{(n)}) = (2\,\delta_0(n-i) - 1)\, 2n\ e_{ij}^{(n)} \quad (22)$$

where $\delta_0(k) = 0$ if $k \neq 0$ and $\delta_0(0) = 1$. It follows from [3] that the triple

$$(\mathcal{A}, \mathcal{H}, D) \quad (23)$$

is a spectral triple.

In order to simplify we start the discussion with the case $q = 0$. We then have the simpler formulas,

$$\begin{aligned} & a_+(n,i,j) = 0 \\ & a_-(n,i,j) = 0 \text{ if } i = -n \text{ or } j = -n \\ & a_-(n,i,j) = 1 \text{ if } i \neq -n \text{ and } j \neq -n \end{aligned} \tag{24}$$

$$\begin{aligned} & b_+(n,i,j) = 0 \text{ if } j \neq -n \\ & b_+(n,i,j) = -1 \text{ if } j = -n \\ & b_-(n,i,j) = 0 \text{ if } i \neq -n \text{ or } j = -n \\ & b_-(n,i,j) = 1 \text{ if } i = -n, j \neq -n\,. \end{aligned} \tag{25}$$

Thus for $q = 0$ the operators α and β in $\mathcal{H}$ are given by,

$$\alpha e_{ij}^{(n)} = e_{i-\frac{1}{2},j-\frac{1}{2}}^{\left(n-\frac{1}{2}\right)} \qquad \text{if} \quad i > -n, j > -n \tag{26}$$

and $\alpha e_{ij}^{(n)} = 0$ if $i = -n$ or $j = -n$.

$$\beta e_{ij}^{(n)} = 0 \qquad \text{if} \quad i \neq -n \quad \text{and} \quad j \neq -n \tag{27}$$

$$\beta e_{-n,j}^{(n)} = e_{-\left(n-\frac{1}{2}\right),j-\frac{1}{2}}^{\left(n-\frac{1}{2}\right)} \qquad \text{if} \quad j \neq -n \tag{28}$$

and

$$\beta e_{i,-n}^{(n)} = -e_{i+\frac{1}{2},-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)}\,. \tag{29}$$

By construction $\beta\beta^* = \beta^*\beta$ is the projection e on the subset $\{i = -n \text{ or } j = -n\}$ of the basis.

Also α is a partial isometry with initial support $1 - e$ and final support $1 = \alpha\alpha^*$. The basic relations between α and β are,

$$\alpha^*\alpha + \beta^*\beta = 1\,,\ \alpha\alpha^* = 1\,,\ \alpha\beta = \alpha\beta^* = 0\,,\ \beta\beta^* = \beta^*\beta\,. \tag{30}$$

For $f \in C^\infty(S^1)$, $f = \sum \widehat{f}_n\, e^{in\theta}$, we let

$$f(\beta) = \sum_{n>0} \widehat{f}_n\, \beta^n + \sum_{n<0} \widehat{f}_n\, \beta^{*(-n)} + \widehat{f}_0\, e \tag{31}$$

and the map $f \to f(\beta)$ gives a (degenerate) representation of $C^\infty(S^1)$ in $\mathcal{H}$.

Now let $\mathcal{A}$ be the linear space of sums,

$$a = \sum_{k,\ell\geq 0} \alpha^{*k} f_{k\ell}(\beta)\, \alpha^\ell + \sum_{\ell\geq 0} \lambda_\ell\, \alpha^\ell + \sum_{k>0} \lambda'_k\, \alpha^{*k} \tag{32}$$

where λ and λ' are sequences (of complex numbers) of rapid decay and $(f_{k\ell})$ is a sequence of rapid decay with values in $C^\infty(S^1)$.

We let A be the C^* algebra in $\mathcal{H}$ generated by α and β.

Proposition 1. *The subspace $\mathcal{A} \subset A$ is a subalgebra stable under holomorphic functional calculus.*

Proof. Let σ be the linear map from $\mathcal{A}$ to $C^\infty(S^1)$ given by,

$$\sigma(a) = \sum_{\ell \geq 0} \lambda_\ell \, u^\ell + \sum_{k>0} \lambda'_k \, u^{-k} \tag{33}$$

where $u = e^{i\theta}$ is the generator of $C^\infty(S^1)$. Let $\mathcal{J} = \operatorname{Ker} \sigma$. For $a \in \mathcal{J}$ one has $a = \sum \alpha^{*k} f_{k\ell} \, \alpha^\ell$ and the equality,

$$\alpha^{*k} f_{k\ell} \, \alpha^\ell \alpha^{*k'} g_{k'\ell'} \, \alpha^{\ell'} = \delta_{\ell,k'} \, \alpha^{*k} f_{k\ell} \, g_{k'\ell'} \, \alpha^{\ell'} \tag{34}$$

shows that $\mathcal{J}$ is an algebra and is isomorphic to the topological tensor product

$$C^\infty(S^1) \otimes \mathcal{S} = C^\infty(S^1, \mathcal{S}) \tag{35}$$

where $\mathcal{S}$ is the algebra of matrices of rapid decay.

Since $\mathcal{S}$ is stable under holomorphic functional calculus (h.f.c.) in its norm closure $\mathcal{K}$ (the C^* algebra of compact operators), it follows from (35) that $\mathcal{J}$ is stable under h.f.c. in its norm closure $\overline{\mathcal{J}} \subset A$.

The equalities $\alpha f(\beta) = 0 \, \forall f \in C^\infty(S^1)$ and $\alpha\alpha^* = 1$ show that $\mathcal{J}$ is stable under left multiplication by α^* and α. It follows using (30) that $\mathcal{A}$ is an algebra, $\mathcal{J}$ a two sided ideal of $\mathcal{A}$ and that one has the exact sequence,

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} C^\infty(S^1) \longrightarrow 0\,. \tag{36}$$

By construction $\mathcal{A}$ is dense in A. Let us check that it is stable under h.f.c. in A. Let $a \in \mathcal{A}$ be such that $a^{-1} \in A$. Let us show that $a^{-1} \in \mathcal{A}$.

Let ∂_α be the derivation of $\mathcal{A}$ given by,

$$\partial_\alpha \alpha = \alpha\,, \; \partial_\alpha \beta = 0\,. \tag{37}$$

The one parameter group $\exp(it\partial_\alpha)$ of automorphisms of $\mathcal{A}$ is implemented by unitary operators in $\mathcal{H}$ (cf.(66) below) and extends to A. Moreover $\mathcal{A}$ is dense in the domain,

$$\operatorname{Dom} \partial_\alpha^j = \{x \in A\,; \; \partial_\alpha^j x \in A\} \tag{38}$$

in the graph norm.

Since $a^{-1} \in \operatorname{Dom} \partial_\alpha^j$ we can, given any $\varepsilon > 0$, find $b \in \mathcal{A}$ such that,

$$\|\partial_\alpha^j (b - a^{-1})\| < \varepsilon \qquad j = 0, 1, 2\,. \tag{39}$$

Thus, given $\varepsilon > 0$, we can find $b \in \mathcal{A}$ such that, with $x = ab$,

$$\|\partial_\alpha^j (x-1)\| < \varepsilon \qquad j = 0,1,2\,. \tag{40}$$

For ε small enough it follows that if we let $\sigma(x^{-1})^\wedge_n$ be the Fourier coefficients of $\sigma(x^{-1})$,

$$c = \sum_{n \geq 0} \sigma(x^{-1})^\wedge_n\, \alpha^n + \sum_{n<0} \sigma(x^{-1})^\wedge_n\, \alpha^{*-n} \tag{41}$$

is an element of $\mathcal{A}$, invertible in A, such that,

$$\sigma(c) = \sigma(x^{-1})\,. \tag{42}$$

(Since one controls $n^2 \sigma(x^{-1})^\wedge_n$ from $\|\partial_\alpha^j (x^{-1}-1)\|$.)

Thus $\sigma(xc) = 1$ and since xc is invertible in A (by (40), (42)) and $1 - xc \in \mathcal{J}$ the stability of $\mathcal{J}$ under h.f.c. shows that $(xc)^{-1} = y \in \mathcal{A}$. Then $abcy = 1$ and $a^{-1} = bcy \in \mathcal{A}$. $\square$

Our next result determines the dimension spectrum of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ defined above in (23),

Theorem 2. *The dimension spectrum of the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is* simple *and equal to* $\{1,2,3\} \subset \mathbb{C}$.

Thus we let $\mathcal{B}$ be the algebra generated by the

$$\delta^k(a)\,,\ \delta^k([D,a])\,,\ a \in \mathcal{A}\,,\ k \in \mathbb{N} \tag{43}$$

where δ is the unbounded derivation of $\mathcal{L}(\mathcal{H})$ given by the commutator with $|D|$,

$$\delta(T) = |D|\, T - T\, |D|\,. \tag{44}$$

(It is part of the statement that the elements in (43) are in the domain of δ^k.)

For $b \in \mathcal{B}$ we consider the function,

$$\zeta_b(s) = \mathrm{Trace}\,(b\, |D|^{-s}) \tag{45}$$

where we take care of the eigenvalue $D = 0$ by replacing $|D|$ by an arbitrary $\varepsilon > 0$ there. The statement of the theorem is that all the functions $\zeta_b(s)$ which are a priori only defined for $\mathrm{Re}\,(s) > 3$, do extend to meromorphic function on $\mathbb{C}$ and only admit *simple* poles at the 3 points $\{1,2,3\} \subset \mathbb{C}$.

To prove it we shall first describe the algebra $\mathcal{B}$. We let,

$$F = \mathrm{Sign}\, D \tag{46}$$

so that $F = 2P - 1$ where P is the orthogonal projection on the subset $\{i = n\}$ of the basis. Concerning the generator α one has,

$$\delta(\alpha) = -\alpha\,,\ \delta(\alpha^*) = \alpha^*\,,\ [F, \alpha] = 0\,. \tag{47}$$

It follows that $[D, \alpha^*] = F\delta(\alpha^*) = F\alpha^* = \alpha^* F$. Thus $\alpha\,[D, \alpha^*] = \alpha\alpha^* F = F$,

$$F = \alpha\,[D, \alpha^*]\,. \tag{48}$$

This shows that $F \in \mathcal{B}$.
Concerning the generator β one has

$$\delta(\beta) = \beta K\,,\ \ \delta(\beta^*) = -K\beta^* \tag{49}$$

where K is the multiplication operator,

$$K(e_{ij}^{(n)}) = k(n,i,j)\, e_{ij}^{(n)} \tag{50}$$

with

$$\begin{aligned} k(n,i,j) = 0 \qquad &\text{unless } i = -n \quad \text{or} \quad j = -n \\ k(n,-n,j) = -1 \qquad &\text{if } j \neq -n \\ k(n,i,-n) = 1\,. & \end{aligned} \tag{51}$$

Thus the support of K is $e = \beta^*\beta$ and,

$$K^2 = e\,. \tag{52}$$

We let $e_0 = \frac{1}{2}\,(K - \beta K \beta^*)$. It is the orthogonal projection on the subset of the basis $\{i = -n \text{ and } j = -n\}$. For each m one lets,

$$e_m = \beta^m e_0\, \beta^{*m} \tag{53}$$

and the e_m are pairwise orthogonal projections such that,

$$\sum_{m\in\mathbb{Z}} e_m = e\,. \tag{54}$$

We let $\mathcal{L}$ be the algebra of double sums with rapid decay,

$$\mathcal{L} = \left\{ \sum \lambda_{n,m}\, \beta^n e_m\,;\ \lambda \in \mathcal{S} \right\} \tag{55}$$

(where $\beta^{-\ell} = \beta^{*\ell}$ for $\ell > 0$).
One has $\delta(e_n) = 0$, $[K, \beta] = 2e_0\beta$, $Ke_m = \text{sign}(m)e_m$ and using (49),

$$\delta(\beta^n) = n\beta^n K \qquad \text{modulo } \mathcal{L}\,. \tag{56}$$

Thus $\mathcal{L}$ is invariant (globally) under δ. Also for any $f \in C^\infty(S^1)$ one has

$$[K, f(\beta)] \in \mathcal{L} \tag{57}$$

and the algebra B_0,

$$B_0 = \{f_0(\beta) + f_1(\beta)\, K + h\,;\ f_j \in C^\infty(S^1)\,,\ h \in \mathcal{L}\} \tag{58}$$

is stable by the derivation δ.

A similar result holds if we further adjoin the operator

$$F_1 = eF = Fe\,. \tag{59}$$

Indeed $e'_0 = \frac{1}{2}(F_1 - \beta F_1 \beta^*)$ is the projection on the element $\{e^{(0)}_{0,0}\}$ of the basis and the $e'_n = \beta^n e'_0 \beta^{*n}$ are pairwise orthogonal projections on the one dimensional subspaces spanned for $n > 0$ by $e^{(n/2)}_{n/2,-n/2}$ and for $n < 0$, $n = -k$, by $e^{(k/2)}_{-k/2,k/2}$.

We let,

$$\mathcal{L}' = \left\{\sum \lambda_{n,m}\, \beta^n e'_m\,;\ \lambda \in \mathcal{S}\right\}\,. \tag{60}$$

One has,

$$[F, f(\beta)] \in \mathcal{L}' \qquad \forall f \in C^\infty(S^1) \tag{61}$$

and $F_1\mathcal{L}' = \mathcal{L}' F_1 = \mathcal{L}'$.

Also $e'_n \leq e_n$ for each n so that $\mathcal{L}\mathcal{L}' \subset \mathcal{L}'$ and $\mathcal{L}'\mathcal{L} \subset \mathcal{L}'$ which shows that the sum,

$$\mathcal{L}'' = \mathcal{L} + \mathcal{L}' \tag{62}$$

is an algebra.

Thus the algebra generated in $e\mathcal{H}$ by the $\delta^k(f)$, $f \in C^\infty(S^1)$ and F_1, is contained in the algebra B_1

$$B_1 = \{f_0 + f_1 K + f_2 F_1 + h\,;\ f_j \in C^\infty(S^1)\,,\ h \in \mathcal{L}''\}\,. \tag{63}$$

Note that $F_1 K = 1 + 2(K - F_1)$ so that we do not need terms in $F_1 K$.

We then let B be the algebra of double sums,

$$B = \left\{\sum \alpha^{*k} b_{k\ell}\, \alpha^\ell + A_0 + A_1 F\right\} \tag{64}$$

where $b_{k\ell} \in B_1$ and the sequence $(b_{k\ell})$ is of rapid decay while A_0, A_1 are sums of rapid decay of the form,

$$A = \sum_{\ell \geq 0} a_\ell\, \alpha^\ell + \sum_{k>0} a_{-k}\, \alpha^{*k}\,.$$

Since F commutes with α and α^* it commutes with A_j. Thus one checks that B is an algebra, that it is stable under δ and contains both F and $\mathcal{A}$, thus it contains $\mathcal{B}$.

Let then $b \in B$ and consider the function,

$$\zeta_b(s) = \mathrm{Trace}\,(b\,|D|^{-s}) \tag{65}$$

which is well defined for $\mathrm{Re}\,(s) > 3$.

There is a natural bigrading corresponding to the degrees in α and β. It is implemented by the following action of $\mathbb{T}^2$ in $\mathcal{H}$,

$$V(u,v)\, e_{k,\ell}^{(n)} = \exp i(-u(k+\ell)+v(k-\ell))\, e_{k,\ell}^{(n)} \,. \tag{66}$$

Note that both $k+\ell$ and $k-\ell$ are integers so that one gets an action of $\mathbb{T}^2$.
The indices k,ℓ are transformed to $k-\frac{1}{2}$, $\ell-\frac{1}{2}$ by α, so that

$$V(u,v)\, \alpha(e_{k,\ell}^{(n)}) = \exp i(-u(k+\ell-1)+v(k-\ell))\, \alpha(e_{k,\ell}^{(n)})$$

and we get,

$$V(u,v)\, \alpha\, V(-u,-v) = e^{iu}\alpha \,. \tag{67}$$

The indices k,ℓ are transformed to $k+\frac{1}{2}$, $\ell-\frac{1}{2}$ by β and,

$$V(u,v)\, \beta(e_{k,\ell}^{(n)}) = \exp i(-u(k+\ell)+v(k-\ell+1))\, \beta(e_{k,\ell}^{(n)})$$

so that,

$$V(u,v)\, \beta\, V(-u,-v) = e^{iv}\beta \,. \tag{68}$$

Moreover, since V is a multiplication operator it commutes with $|D|$, D, and $F = 2P-1$.

Using the restriction of this bigrading to B (which gives bidegree $(0,0)$ for diagonal operators, $(1,0)$ for α and $(0,1)$ for β) one checks that homogeneous elements of bidegree $\neq (0,0)$ satisfy $\zeta_b(s) \equiv 0$, thus one can assume that b is of bidegree $(0,0)$.

Any $b \in B^{(0,0)}$ is of the form,

$$b = \sum \alpha^{*k}\, b_k\, \alpha^k + a_0 + a_1 F \tag{69}$$

where a_0, a_1 are *scalars* and (b_k) is a sequence of rapid decay with $b_k \in B_1^{(0,0)}$.
Elements c of $B_1^{(0,0)}$ are of the form,

$$c = \lambda_0 + \lambda_1 K + \lambda_2 F_1 + h \tag{70}$$

where λ_j are *scalars* and $h \in \mathcal{L}''^{(0,0)}$. Finally elements of $\mathcal{L}''^{(0,0)}$ are of the form,

$$h = \sum h_n e_n + \sum h'_m e'_m \tag{71}$$

where (h_n) and (h'_m) are *scalar* sequences of rapid decay.
The equality,

$$|D|^z\, \alpha^{*k} = \alpha^{*k}(|D|+k)^z \qquad z \in \mathbb{C} \tag{72}$$

is checked directly $(k \geq 0)$.

Using $\alpha^k \alpha^{*k} = 1$ it follows that with b as in (69),

$$\mathrm{Trace}\,(b\,|D|^{-s}) = \mathrm{Trace}\,((a_0 + a_1 F)\,|D|^{-s}) + \sum_{k\geq 0} \mathrm{Trace}\,(b_k(|D|+k)^{-s})\,. \quad (73)$$

Now for h as in (71) one has

$$\mathrm{Trace}\,(h\,|D|^{-s}) = \sum h_n \mathrm{Trace}\,(e_n\,|D|^{-s}) + \sum h'_m \mathrm{Trace}\,(e'_m\,|D|^{-s})\,.$$

Moreover

$$\mathrm{Trace}\,(e_n\,|D|^{-s}) = \sum_{\ell=0}^{\infty} \frac{1}{(|n|+\ell)^s} = \zeta(s) - \left(\sum_{0}^{|n|-1} \frac{1}{r^s} \right)\,.$$

But $\sum h_n \left(\sum\limits_{0}^{|n|-1} \frac{1}{r^s} \right) = \rho_1(s)$ is a holomorphic function of $s \in \mathbb{C}$ and similarly, since $\mathrm{Trace}\,(e'_m\,|D|^{-s}) = \frac{1}{|m|^s}$, the function $\rho_2(s) = \sum h'_m \mathrm{Trace}\,(e'_m\,|D|^{-s})$ is holomorphic in $s \in \mathbb{C}$. Thus modulo holomorphic functions one has,

$$\mathrm{Trace}\,(h\,|D|^{-s}) \sim \left(\sum h_n\right) \zeta(s)\,. \quad (74)$$

Next,

$$\mathrm{Trace}\,(e\,|D|^{-s}) = \sum_{0}^{\infty} \frac{2n+1}{n^s} = 2\,\zeta(s-1) + \zeta(s) + \varepsilon^{-s} \quad (75)$$

$$\mathrm{Trace}\,(K\,|D|^{-s}) = \sum_{n\in\mathbb{Z}} \sum_{\ell\geq 0} \frac{\mathrm{sign}(n)}{(|n|+\ell)^s} = \zeta(s) + \varepsilon^{-s}$$

and with $F = 2P - 1$ we also have,

$$\mathrm{Trace}\,(eP\,|D|^{-s}) = \zeta(s) + \varepsilon^{-s}. \quad (76)$$

Thus, with c as in (70) we get,

$$\zeta_c(s) = \lambda\,\zeta(s-1) + \mu\,\zeta(s) + \rho(s) \quad (77)$$

where λ, μ are scalars and ρ is a holomorphic function of $s \in \mathbb{C}$.
A similar result holds for

$$\sum_{k\geq 0} \mathrm{Trace}\,(b_k(|D|+k)^{-s})\,.$$

For instance one rewrites the double sum

$$\sum h_{n,k} \mathrm{Trace}\,(e_m(|D|+k)^{-s}) = \sum_{n,k,\ell} h_{n,k} \frac{1}{(|n|+k+\ell)^s}$$

as

$$\sum_m \left(\sum_{|n|+k\leq m} h_{n,k} \right) \frac{1}{m^s} = a\,\zeta(s) + \rho(s)$$

where $a = \sum h_{n,k}$ and ρ is holomorphic in $s \in \mathbb{C}$.
Finally

$$\mathrm{Trace}\,(P\,|D|^{-s}) = \sum_0^\infty \frac{(n+1)}{n^s} = \zeta(s-1) + \zeta(s)$$

and

$$\mathrm{Trace}\,(|D|^{-s}) = \sum_0^\infty \frac{(n+1)^2}{n^s} = \zeta(s-2) + 2\,\zeta(s-1) + \zeta(s)\,.$$

Thus we conclude that for any $b \in B$ one has

$$\zeta_b(s) = \lambda_3\,\zeta(s-2) + \lambda_2\,\zeta(s-1) + \lambda_1\,\zeta(s) + \rho(s) \tag{78}$$

where the λ_j are scalars and ρ is a holomorphic function of $s \in \mathbb{C}$, thus proving theorem 2.

4 The Local Index Formula for $SU_q(2)$, $(q = 0)$.

In this section we shall compute the local index formula for the above spectral triple. Since the dimension spectrum is simple and equal to $\{1,2,3\} \subset \mathbb{C}$ the cyclic cocycle given by the local index formula has two components φ_1 and φ_3 of degree 1 and 3 given, up to an overall multiplication by $(2i\pi)^{1/2}$, by

$$\varphi_1(a^0,a^1) = {\int\!\!\!\!\!\!-} a^0[D,a^1]|D|^{-1} - \frac{1}{4}{\int\!\!\!\!\!\!-} a^0\nabla([D,a^1])\,|D|^{-3} + \frac{1}{8}{\int\!\!\!\!\!\!-} a^0\nabla^2([D,a^1])\,|D|^{-5} \tag{79}$$

and,

$$\varphi_3(a^0,a^1,a^2,a^3) = \frac{1}{12}{\int\!\!\!\!\!\!-} a^0[D,a^1][D,a^2][D,a^3]\,|D|^{-3}\,. \tag{80}$$

With these notations the cocycle equation is,

$$b\varphi_1 + B\varphi_3 = 0\,. \tag{81}$$

The following formulas define a cyclic cocycle τ_1 on $\mathcal{A}$,

$$\tau_1(\alpha^{*k},x) = \tau_1(x,\alpha^{*k}) = \tau_1(\alpha^l,x) = \tau_1(x,\alpha^l) = 0, \tag{82}$$

for all integers k, l and any $x \in \mathcal{A}$,

$$\tau_1(\alpha^{*k} f(\beta)\alpha^{\ell}, \alpha^{*k'} g(\beta)\alpha^{\ell'}) = 0 \qquad (83)$$

unless $\ell' = k$, $k' = \ell$ and

$$\tau_1(\alpha^{*k} f(\beta)\alpha^{\ell}, \alpha^{*\ell} g(\beta)\alpha^{k}) = \frac{1}{\pi i}\int_{S^1} f\ \mathrm{d}g\ .$$

Let φ_0 be the 0-cochain given by $\varphi_0(\alpha^{*k} f(\beta)\alpha^{\ell}) = 0$ unless $k = \ell$ and,

$$\varphi_0(\alpha^{*k} f(\beta)\alpha^{k}) = \rho(k)\ \frac{1}{2\pi}\int_{S^1} f\ \mathrm{d}\theta, \qquad (84)$$

where $\rho(j) = \frac{2}{3} - j - j^2$. Finally, let φ_2 be the 2-cochain given by the pull back by σ of the cochain $\frac{-1}{24}\frac{1}{2\pi i}\int f_0 f_1' f_2''\,\mathrm{d}\theta$ on $C^\infty(S^1)$.
Our next task is to prove the following result,

Theorem 3. *The local index formula of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by the cyclic cocycle τ_1 up to the coboundary of the cochain (φ_0, φ_2).*

The precise equations are,

$$\varphi_1 = \tau_1 + b\varphi_0 + B\varphi_2\,, \qquad \varphi_3 = b\varphi_2\,. \qquad (85)$$

The proof is a computation but we shall go through it in details in order to get familiar with various ways of computing residues and manipulating "infinitesimals" in the sense of the quantized calculus. In other words our purpose is not concision but rather a leisurly account of the details.

4.1 Restriction to $C^\infty(\beta)$

Let us first concentrate on the restriction of the cocycle φ to the subalgebra $C^\infty(\beta)$ generated by β and β^*. To see the subspace of $\mathcal{H}$ responsible for the non-triviality of that cocycle we follow the action of β on the vectors,

$$\xi_{-n} = e^{(\frac{n}{2})}_{-\frac{n}{2},\frac{n}{2}} \qquad n \geq 0\,,\ n \in \mathbb{N}\,. \qquad (86)$$

and,

$$\xi_n = e^{(n/2)}_{n/2,-n/2} \qquad n \geq 0\,,\ n \in \mathbb{N} \qquad (87)$$

For $n > 0$, (28) shows that $\beta(\xi_{-n}) = \xi_{-(n-1)} = \xi_{-n+1}$, with $\beta(\xi_{-1}) = e^{(0)}_{0,0} = \xi_0$. Next, $\beta(\xi_0) = \beta(e^{(0)}_{0,0}) = -e^{(1/2)}_{(1/2),-1/2} = -\xi_1$, and for $n > 0$ (29) shows that $\beta(\xi_n) = -\xi_{n+1}$. Thus,

$$\beta(\xi_n) = -\mathrm{sign}(n)\ \xi_{n+1} \qquad (\mathrm{sign}(0) = 1) \qquad (88)$$

We let $\ell^2(\mathbb{Z}) = \mathcal{H}_0 \subset \mathcal{H}$ be the subspace of $\mathcal{H}$ spanned by the ξ_n and rewrite the above equality as,

$$\beta = -UH \text{ on } \ell^2(\mathbb{Z}) \subset \mathcal{H} \tag{89}$$

where H is the sign operator and U the shift,

$$U\xi_n = \xi_{n+1}\,. \tag{90}$$

The operator D also restricts to the subspace $\ell^2(\mathbb{Z}) = \mathcal{H}_0 \subset \mathcal{H}$ and its restriction D_0 is given by,

$$D_0\xi_n = \text{sign}(n)\,|n|\,\xi_n = n\,\xi_n \qquad \forall n\,. \tag{91}$$

The unitary $W = e^{i\frac{\pi}{2}(|D_0|-D_0)}$ commutes with D_0 and conjugates U to $-UH$,

$$W\,U\,W^* = -UH \tag{92}$$

Thus the triple $(\beta,\,\mathcal{H}_0,D_0)$ is isomorphic to,

$$\left(e^{i\theta},\ L^2(S^1),\,(-i)\,\frac{\partial}{\partial\,\theta}\right)\,. \tag{93}$$

In particular the index and cyclic cohomology pairings with the restriction to $\mathcal{H}_0$ are non trivial and we control,

$$\text{Res}_{s=1}\,\text{Trace}_{\mathcal{H}_0}\,(\beta^*[D_0,\beta]\,|D_0|^{-s})\ = 2. \tag{94}$$

This however does not suffice to get the non-triviality of the restriction of φ to $C^\infty(\beta)$ since we need to control the residues on $e\mathcal{H}$ where, as above e is the support of β. To see what happens we shall conjugate the restriction of both β and D to the orthogonal complement of $\mathcal{H}_0$ in $e\mathcal{H}$ with a very simple triple. Let us define for each $k \in \mathbb{N}$ the vectors,

$$\xi^{(k)}_{-n} = e^{\left(\frac{k+n}{2}\right)}_{-\left(\frac{k+n}{2}\right),\frac{n-k}{2}} \qquad n \geq 0 \tag{95}$$

$$\xi^{(k)}_{n} = e^{\frac{k+n}{2}}_{\frac{n-k}{2},-\left(\frac{k+n}{2}\right)} \qquad n \geq 0$$

so that $\xi^{(k)}_0 = e^{(k/2)}_{-k/2,-k/2}$.

For $n > 0$ one has

$$\beta(\xi^{(k)}_{-n}) = \beta\left(e^{\left(\frac{k+n}{2}\right)}_{-\left(\frac{k+n}{2}\right),\frac{n-k}{2}}\right) = e^{\left(\frac{k+n-1}{2}\right)}_{-\left(\frac{k+n-1}{2}\right),\frac{n-1-k}{2}} = \xi^{(k)}_{-(n-1)} = \xi^{(k)}_{-n+1}\,.$$

For $n = 0$,

$$\beta(\xi^{(k)}_0) = \beta(e^{(k/2)}_{-k/2,-k/2}) = -e^{\left(\frac{k+1}{2}\right)}_{-\frac{k}{2}+\frac{1}{2},-\left(\frac{k+1}{2}\right)} = \xi^{(k)}_1$$

and,
$$\beta(\xi_n^{(k)}) = \beta\left(e^{\left(\frac{k+n}{2}\right)}_{\frac{n-k}{2},-\left(\frac{n+k}{2}\right)}\right) = -e^{\left(\frac{k+n+1}{2}\right)}_{\frac{n+1-k}{2},-\left(\frac{n+1+k}{2}\right)} = -\xi_{n+1}^{(k)}\,.$$
Thus, as in (88) we have,
$$\beta(\xi_n^{(k)}) = -\mathrm{sign}\,(n)\,\xi_{n+1}^{(k)}\,. \tag{96}$$
Now $\beta(e_{ij}^{(m)}) = 0$ unless $i = -m$ or $j = -m$ and for any $m \in \frac{1}{2}\,\mathbb{N}$ the vectors $e_{-m,j}^{(m)}$ and $e_{i,(-m)}^{(m)}$ are of the form $\xi_n^{(k)}$. Indeed in the first case one takes $n = m+j$, $k = m-j$ which are both in $\mathbb{N}$, and $\xi_{-n}^{(k)} = e_{-m,j}^{(m)}$. In the second case $n = m+i$, $k = m-i$ are both in $\mathbb{N}$ and $\xi_n^{(k)} = e_{i,-m}^{(m)}$.

We then let $\mathcal{H}_k$ be the span of the $\xi_n^{(k)}$, $n \in \mathbb{Z}$ and
$$\mathcal{H}' = \bigoplus_{k\geq 1} \mathcal{H}_k = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}^+)\,. \tag{97}$$
The operator D restricts to $\mathcal{H}'$ and is given there by,
$$D' = |D_0| \otimes 1 + 1 \otimes N \tag{98}$$
where N is the number operator $N\varepsilon_k = k\varepsilon_k$.

Also β is $-UH \otimes 1$ and we can conjugate it as in (92) back to $U \otimes 1$.

Thus the triple $(\beta, \mathcal{H}', D')$ is isomorphic to
$$(U \otimes 1,\ \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}^+),\ |D_0| \otimes 1 + 1 \otimes N) \tag{99}$$

The metric dimension is 2 in this situation, and the contribution φ' of $\mathcal{H}'$ to the restriction of φ to $C^\infty(\beta)$ only has a one dimensional component φ_1' which involves the two terms,
$$\varphi_1'(a^0, a^1) = ⨍ a^0[D', a^1]|D'|^{-1} - \frac{1}{4} ⨍ a^0\nabla([D', a^1])\,|D'|^{-3},\ \ a^j \in C^\infty(\beta) \tag{100}$$
Since D' is positive, it is K-homologically trivial and the above cocycle must vanish identically on $C^\infty(\beta)$. As we shall see this vanishing holds because of the precise ratio $-\frac{1}{4}$ of the coefficients in the local index formula.

To see this, we need to compute the poles and residues of functions of the form $\mathrm{Trace}\,((T \otimes 1)\,|D'|^{-s})$ for operators T in $\ell^2(\mathbb{Z})$. For that purpose it is most efficient to use the well known relation between residues of zeta functions and asymptotic expansions of related theta functions. More specifically, for $\lambda > 0$ and $Re(s) > 0$, the equality,
$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda}\, t^s\, \frac{dt}{t} \tag{101}$$

gives

$$\begin{aligned}\mathrm{Trace}\,((T\otimes 1)\,|D'|^{-s}) &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Trace}\,((T\otimes 1)\,e^{-tD'})\,t^s\,\frac{dt}{t}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Trace}\,(Te^{-t|D_0|})\left(\frac{1}{e^t-1}\right)t^s\,\frac{dt}{t}\,.\end{aligned}$$

Thus if we assume that one has an expansion of the form $\mathrm{Trace}\,(Te^{-t|D_0|}) = \frac{a}{t}+b+c\,t+0(t^2)$ one gets, using

$$\frac{1}{e^t-1}=\frac{1}{t}-\frac{1}{2}+\frac{t}{12}+0(t^2) \tag{102}$$

the equality modulo holomorphic functions of s, $Re(s)>0$,

$$\mathrm{Trace}\,((T\otimes 1)\,|D'|^{-s}) \sim \frac{1}{\Gamma(s)}\int_0^1 \varphi(t)\,t^s\,\frac{dt}{t},$$

where,

$$\varphi(t)=\frac{a}{t^2}+\frac{\left(b-\frac{a}{2}\right)}{t}+\frac{a}{12}-\frac{b}{2}+c\,.$$

One has $\int_0^1 t^\alpha\,\frac{dt}{t}=\frac{1}{\alpha}$, thus one gets 2 poles $s=2$ and $s=1$, and the expansion,

$$\mathrm{Trace}\,((T\otimes 1)\,|D'|^{-s}) \sim \frac{a}{\Gamma(s)(s-2)}+\left(b-\frac{a}{2}\right)\frac{1}{\Gamma(s)(s-1)}+\cdots$$

so that,

$$\mathrm{Res}_{s=2}(\mathrm{Trace}\,((T\otimes 1)|D'|^{-s}))=a \tag{103}$$

$$\mathrm{Res}_{s=1}(\mathrm{Trace}\,((T\otimes 1)|D'|^{-s}))=b-\frac{a}{2} \tag{104}$$

Let us compute $\varphi_1'(\beta^*,\beta)$. The first term in (100) is $⨍\,\beta^*[D',\beta]|D'|^{-1}$. Thus we take $T=U^*[|D_0|,U]$. One has $[|D_0|,U]=UH$, $T=H$ and

$$\mathrm{Trace}\,(Te^{-t|D_0|})=\sum_{\mathbb{Z}}\mathrm{sign}\,(n)\,e^{-t|n|}=-\sum_1^\infty e^{-tk}+1+\sum_1^\infty e^{-tk}=1\,. \tag{105}$$

Thus in that case $a=0$, $b=1$ and,

$$⨍\,\beta^*[D',\beta]|D'|^{-1}=1\,. \tag{106}$$

The second term in (100) is $⨍\,\beta^*\nabla([D',\beta])\,|D'|^{-3}$ where ∇ is the commutator with D'^2. If we let as above δ be the commutator with $|D'|$, one has

$$\nabla T=\delta(T)\,|D'|+|D'|\,\delta(T)\,.$$

Thus, permuting $|D'|$ modulo operators of lower order, we get,

$$\int\!\!\!\!\!- \beta^* \nabla([D',\beta])\,|D'|^{-3} = 2 \int\!\!\!\!\!- \beta^* \delta([D',\beta])\,|D'|^{-2} \tag{107}$$

To compute the r. h. s. we take $T = U^* \delta^2(U)$ and look at the residue at $s = 2$. One has $\delta(U) = UH$, $\delta^2(U) = (UH)H = U$ and $T = 1$. Thus

$$\begin{aligned}\mathrm{Trace}\,(Te^{-t|D_0|}) &= \sum_{\mathbb{Z}} e^{-t|n|} = 1 + 2\sum_1^\infty e^{-tn} \\ &= 1 + \frac{2}{e^t - 1} \sim \frac{2}{t} + 0(t)\end{aligned}$$

so that $a = 2$, $b = 0$. Thus we get,

$$\int\!\!\!\!\!- \beta^* \delta([D',\beta])\,|D'|^{-2} = \mathrm{Res}_{s=2}(\mathrm{Trace}\,((U^*\delta^2(U)\otimes 1)\,|D'|^{-s}) = 2 \tag{108}$$

Thus we get,

$$\int\!\!\!\!\!-_{\mathcal{H}'} \beta^*\,[D,\beta]\,|D|^{-1} = 1\,, \int\!\!\!\!\!-_{\mathcal{H}'} \beta^*\,\nabla([D,\beta])\,|D|^{-3} = 4 \tag{109}$$

and $\varphi_1'(\beta^*,\beta) = 0$ precisely because of the coefficient $\frac{-1}{4}$ in (100).

One proceeds similarly to compute $\varphi_1'(\beta,\beta^*)$. The first term in (100) comes from $T = U\delta(U^*)$. In the canonical basis ε_n of $\ell^2(\mathbb{Z})$ one has $U\delta(U^*)\,\varepsilon_n = (|n-1| - |n|)\,\varepsilon_n = -\mathrm{sign}\,(n-1)\,\varepsilon_n$ and $\mathrm{Trace}\,(Te^{-t|D_0|}) = \sum(-\mathrm{sign}\,(n-1))\,e^{-t|n|} = 1$. Thus,

$$\int\!\!\!\!\!- \beta\;[D',\beta^*]|D'|^{-1} = 1\,. \tag{110}$$

Since $U\delta^2(U^*) = 1$ the computation of the second term of (100) is the same as above and we get,

$$\int\!\!\!\!\!-_{\mathcal{H}'} \beta\,[D,\beta^*]\,|D|^{-1} = 1\,, \int\!\!\!\!\!-_{\mathcal{H}'} \beta\,\nabla([D,\beta^*])\,|D|^{-3} = 4 \tag{111}$$

so that $\varphi_1'(\beta,\beta^*) = 0$.

Let us now take $n > 0$ and compute $\varphi_1'(\beta^{*n},\beta^n)$. The first term of (100) involves $T = U^{*n}\delta(U^n)$. One has,

$$\delta(U^n) = UHU^{n-1} + U^2HU^{n-2} + \cdots + U^jHU^{n-j} + \cdots + U^nH = \sum_{j=1}^n U^jHU^{n-j}\,,$$

$$U^{*n}\delta(U^n) = (U^*)^{n-1}HU^{n-1} + \cdots + H\,.$$

One has $U^{*k}HU^k\varepsilon_\ell = \mathrm{sign}(k+\ell)\,\varepsilon_\ell$ and,

$$\mathrm{Trace}\,(U^{*k}HU^k e^{-t|D_0|}) = \mathrm{Trace}\,(He^{-t|D_0|}) + 2\sum_1^k e^{-t|j|} \sim 1+2k+0(t)\,.$$

Thus,

$$\mathrm{Trace}\,(U^{*n}\delta(U^n)\,e^{-t|D_0|}) \sim \sum_{j=1}^n (2(n-j)+1)+0(t) = \sum_{k=0}^{n-1}(2k+1)+0(t) = n^2+0(t)\,,$$

and $(n>0)$

$$⨍\ \beta^{*n}\,[D',\beta^n]\,|D'|^{-1} = n^2\,, \tag{112}$$

Now modulo finite rank operators one has $\delta(U^n) = nU^nH$ and $\delta^2(U^n) = n^2U^n$, thus as above,

$$⨍\ \beta^{*n}\,\nabla([D',\beta^n])\,|D'|^{-3} = 4n^2, \tag{113}$$

so that $\varphi_1'(\beta^{*n},\beta^n) = 0$.

Finally the computation of $\varphi_1'(\beta^n,\beta^{*n})$ involves $T = U^n\delta(U^{*n})$. One has

$$U^n\delta(U^{*n}) = -\sum_{k=1}^n U^kHU^{*k}\,,$$

and $U^kHU^{*k}\varepsilon_\ell = \mathrm{sign}\,(\ell-k)\,\varepsilon_\ell$ so that,

$$\mathrm{Trace}\,(U^kHU^{*k}e^{-t|D_0|}) - \mathrm{Trace}\,(He^{-t|D_0|}) = -2\sum_0^{k-1} e^{-tj}$$

and,

$$\mathrm{Trace}\,(U^kHU^{*k}e^{-t|D_0|}) \sim 1-2k+0(t)\,.$$

Thus,

$$\begin{aligned}\mathrm{Trace}\,(U^n\delta(U^{*n})\,e^{-t|D_0|}) &= -\sum_{k=1}^n \mathrm{Trace}\,(U^kHU^{*k}e^{-t|D_0|})\\ &\sim -\sum_{k=1}^n (1-2k)+0(t) \sim n^2+0(t)\,,\end{aligned}$$

and,

$$⨍\ \beta^n\,[D',\beta^{*n}]\,|D'|^{-1} = n^2\,. \tag{114}$$

Also as above,

$$\int\!\!\!\!\!\!- \; \beta^n \, \nabla([D', \beta^{*n}]) \, |D'|^{-3} = 4n^2. \tag{115}$$

so that we get the required vanishing,

$$\varphi_1' = 0$$

What is instructive in the above computation is that this vanishing which is required by theorem 1, involves because of the factorisation (99) terms such as "Trace(H)" which appear in equation (105) and are similar to eta-invariants. We have thus shown that $\varphi_1 = \tau_1$ on $C^\infty(\beta)$, or equivalently that,

$$\int\!\!\!\!\!\!- \; \beta^{-n} \, [D, \beta^n] \, |D|^{-1} - \frac{1}{4} \int\!\!\!\!\!\!- \; \beta^{-n} \, \nabla([D, \beta^n]) \, |D|^{-3} = 2n. \tag{116}$$

4.2 Restriction to the ideal $\mathcal{J}$

Let us extend this computation to $\mathcal{J} = \operatorname{Ker} \sigma$. The component φ_3 vanishes on $\mathcal{J}$ and we just need to compute $\varphi_1 = \varphi_1^{(0)} - \frac{1}{4} \varphi_1^{(1)} + \frac{1}{8} \varphi_1^{(2)}$. We begin by $\varphi_1^{(0)}(\mu', \mu) = \int\!\!\!\!- \; \mu'[D, \mu]|D|^{-1}$, and need only consider the case where $\mu = \alpha^{*k} \beta^n \alpha^\ell$ and $\mu' = \alpha^{*k'} \beta^{n'} \alpha^{\ell'}$ are monomials. (As above $\beta^{-n} = \beta^{*n}$ and $\beta^0 = e$). With $F = 2P - 1$ one has $[F, \alpha] = 0$ and

$$[D, \alpha^\ell] = -\ell F \alpha^\ell = -\ell \alpha^\ell F \tag{117}$$

and

$$[D, \alpha^{*k}] = k F \alpha^{*k} = k \alpha^{*k} F \, . \tag{118}$$

Thus,

$$\begin{aligned} [D, \mu] &= [D, \alpha^{*k}] \, \beta^n \alpha^\ell + \alpha^{*k} [D, \beta^n] \, \alpha^\ell + \alpha^{*k} \beta^n [D, \alpha^\ell] \\ &= k \alpha^{*k} F \beta^n \alpha^\ell + \alpha^{*k} [D, \beta^n] \, \alpha^\ell - \ell \alpha^{*k} \beta^n F \alpha^\ell \, , \end{aligned}$$

so that

$$[D, \mu] = \alpha^{*k} (k F \beta^n + [D, \beta^n] - \ell \beta^n F) \, \alpha^\ell \tag{119}$$

The bigrading (66) shows that $\int\!\!\!\!- \; \mu'[D, \mu] \, |D|^{-1}$ vanishes unless both total degrees are 0, i.e.

$$\ell + \ell' - k - k' = 0 \, , \; n' = -n \, . \tag{120}$$

The element $X = (k F \beta^n + [D, \beta^n] - \ell \beta^n F)$ satisfies $e X e = X$ so that the product

$$\mu'[D, \mu] = \alpha^{*k'} \beta^{n'} \alpha^{\ell'} \alpha^{*k} \, X \, \alpha^\ell,$$

vanishes unless $\ell' = k$. Combining with (120) we get $\ell = k'$, and can assume that $\mu' = \alpha^{*\ell}\beta^{*n}\alpha^k$. Then $\mu'[D,\mu] = \alpha^{*\ell}\beta^{*n}\,X\,\alpha^\ell$ so that we just need to compute,

$$\int\!\!\!\!\!- \alpha^{*\ell}\beta^{*n}\,X\,\alpha^\ell\,|D|^{-1}\,.$$

Now by (72) one has,

$$|D|^{-1}\alpha^{*\ell} = \alpha^{*\ell}|D|^{-1} - \ell\alpha^{*\ell}|D|^{-2} + 0(|D|^{-3}) \tag{121}$$

Thus,

$$\int\!\!\!\!\!- \mu'[D,\mu]\,|D|^{-1} = \int\!\!\!\!\!- \beta^{*n}X\,|D|^{-1} - \ell\int\!\!\!\!\!- \beta^{*n}X\,|D|^{-2} \tag{122}$$

where

$$X = (kF\beta^n + [D,\beta^n] - \ell\beta^n F)\,.$$

Note that α, α^* have now disappeared so that we can compute using the subspaces $\mathcal{H}_0$ and $\mathcal{H}'$ of $e\mathcal{H}$. Note also that on $\mathcal{H}'$ one has $F = -1$ since $P = 0$. Only $\mathcal{H}'$ matters for $\int\!\!\!\!- \beta^{*n}X\,|D|^{-2}$. One has $\int\!\!\!\!- \beta^{*n}kF\beta^n|D|^{-2} = -k\int\!\!\!\!-_{\mathcal{H}'}|D|^{-2}$ since $F \sim -1$, and $\int\!\!\!\!- \beta^{*n}(-\ell\beta^n F)\,|D|^{-2} = \ell\int\!\!\!\!-_{\mathcal{H}'}|D|^{-2}$. Also $\int\!\!\!\!- \beta^{*n}[D,\beta^n]\,|D|^{-2} = 0$ since $\int\!\!\!\!-_{L^2(S^1)} U^{-n}[|D_0|, U^n]\,|D_0|^{-1} = 0$. Thus,

$$\int\!\!\!\!\!- \beta^{*n}X\,|D|^{-2} = (\ell - k)\int\!\!\!\!\!-_{\mathcal{H}'}|D|^{-2}\,. \tag{123}$$

One has

$$\int\!\!\!\!\!- \beta^{*n}X\,|D|^{-1} = \int\!\!\!\!\!- \beta^{*n}kF\beta^n|D|^{-1} + \int\!\!\!\!\!- \beta^{*n}[D,\beta^n]\,|D|^{-1} - \ell\int\!\!\!\!\!- eF\,|D|^{-1}$$

where the $\int\!\!\!\!-$ are on $\mathcal{H}'\oplus\mathcal{H}_0$. One has $\int\!\!\!\!-_{\mathcal{H}_0}\beta^{*n}F\beta^n|D|^{-1} = 0$ and $\int\!\!\!\!-_{\mathcal{H}_0}F\,|D|^{-1} = 0$. On $\mathcal{H}'$ one has $F = -1$, thus,

$$\int\!\!\!\!\!- k\beta^{*n}F\beta^n|D|^{-1} = -k\int\!\!\!\!\!-_{\mathcal{H}'}|D|^{-1} \tag{124}$$

and

$$-\,\ell\int\!\!\!\!\!- eF\,|D|^{-1} = \ell\int\!\!\!\!\!-_{\mathcal{H}'}|D|^{-1} \tag{125}$$

so that,

$$\int\!\!\!\!\!- \beta^{*n}X\,|D|^{-1} = \int\!\!\!\!\!- \beta^{*n}[D,\beta^n]\,|D|^{-1} + (\ell - k)\int\!\!\!\!\!-_{\mathcal{H}'}|D|^{-1}\,. \tag{126}$$

We now need to compute $-\frac{1}{4}\int\!\!\!\!- \mu'\nabla([D,\mu])\,|D|^{-3}$ with μ, μ' monomials in $\mathcal{J}$ as above.

Since $|D|^{-2}$ is order 1 on $\mathcal{J}$ we can replace the above by $-\frac{1}{2} \fint \mu' \delta([D,\mu])\,|D|^{-2}$ where $\delta(x) = [|D|, x]$. Moreover, using (119),

$$\delta([D,\mu]) = \delta(\alpha^{*k} X \alpha^\ell) = k\alpha^{*k} X \alpha^\ell + \alpha^{*k}\delta(X)\,\alpha^\ell - \ell\alpha^{*k} X\alpha^\ell\,,$$

$$\delta([D,\mu]) = (k-\ell)\,\alpha^{*k} X\alpha^\ell + \alpha^{*k}\delta(X)\,\alpha^\ell \tag{127}$$

with $X = kF\beta^n + [D,\beta^n] - \ell\beta^n F$.
As above for $\varphi_1^{(0)}$ we get that $\varphi_1^{(1)}$ vanishes unless $\ell' = k$, $k' = \ell$, $n' = -n$ so that $\mu' = \alpha^{*\ell}\beta^{*n}\alpha^k$ and we can replace $\mu'\delta([D,\mu])\,|D|^{-2}$ by

$$(k-\ell)\,\beta^{*n} X\,|D|^{-2} + \beta^{*n}\delta(X)\,|D|^{-2}\,. \tag{128}$$

Now by (123),

$$\fint \beta^{*n} X\,|D|^{-2} = (\ell - k) \fint_{\mathcal{H}'} |D|^{-2}\,.$$

Moreover one has $\fint \beta^{*n}\delta(F\beta^n)\,|D|^{-2} = \fint \beta^{*n}\delta(\beta^n F)\,|D|^{-2} = 0$ since only the $\fint$ on $\mathcal{H}'$ matters and $\fint_{L^2(S^1)} U^{-n}[|D_0|, U^n]\,|D_0|^{-1} = 0$. Thus,

$$\begin{aligned} -\frac{1}{4} \fint \mu'\nabla([D,\mu])\,|D|^{-3} \;&=\; -\frac{1}{2}\,(k-\ell)(\ell-k)\fint_{\mathcal{H}'}|D|^{-2} \\ &-\; \frac{1}{2} \fint \beta^{*n}\delta([D,\beta^n])\,|D|^{-2}\,. \end{aligned} \tag{129}$$

Since $\varphi_1^{(2)} = 0$ on $\mathcal{J}$ we get,

$$\begin{aligned} \varphi_1(\mu',\mu) \;&=\; \fint \beta^{*n}[D,\beta^n]\,|D|^{-1} + (\ell-k)\fint_{\mathcal{H}'}|D|^{-1} - \ell\,(\ell-k)\fint_{\mathcal{H}'}|D|^{-2} \\ &-\; \frac{1}{2}\,(k-\ell)(\ell-k)\fint_{\mathcal{H}'}|D|^{-2} - \frac{1}{2}\fint \beta^{*n}\delta([D,\beta^n])\,|D|^{-2}\,. \end{aligned}$$

Now by (116),

$$\fint \beta^{*n}[D,\beta^n]\,|D|^{-1} - \frac{1}{2}\fint \beta^{*n}\delta([D,\beta^n])\,|D|^{-2} = 2n \tag{130}$$

Thus we get,

$$\varphi_1(\mu',\mu) = 2n + (\ell-k)\fint_{\mathcal{H}'}|D|^{-1} - \frac{1}{2}\,(\ell^2-k^2)\fint_{\mathcal{H}'}|D|^{-2}\,. \tag{131}$$

Let us show that φ_1 is cohomologous to τ_1 on $\mathcal{J}$. Indeed, let $\rho(k)$ be an arbitrary sequence of polynomial growth and φ_0 be the 0-cochain given by,

$$\varphi_0(\alpha^{*k}\beta^n\alpha^\ell) = 0 \text{ unless } k=\ell, \qquad \varphi_0(\alpha^{*k}\beta^0\alpha^k) = \rho(k)\,. \tag{132}$$

Then

$$(b\varphi_0)(\mu',\mu) = \varphi_0(\mu'\mu) - \varphi_0(\mu\mu')$$

and both terms vanish unless $k=\ell'$, $k'=\ell$, $n'=-n$. Moreover in that case

$$\mu\mu' = \alpha^{*k}\beta^n\alpha^\ell\alpha^{*k'}\beta^{n'}\alpha^{\ell'} = \alpha^{*k}\beta^0\alpha^k$$

while

$$\mu'\mu = \alpha^{*\ell}\beta^{-n}\alpha^k\alpha^{*k}\beta^n\alpha^\ell = \alpha^{*\ell}\beta^0\alpha^\ell$$

so that

$$(b\varphi_0)(\mu',\mu) = \rho(\ell) - \rho(k)\,.$$

Thus, with,

$$\rho(k) = k \,⨍_{\mathcal{H}'} |D|^{-1} - \frac{1}{2}\, k^2 \,⨍_{\mathcal{H}'} |D|^{-2} \tag{133}$$

we have, on the ideal $\mathcal{J}$,

$$\varphi_1 = \tau_1 + b\varphi_0\,. \tag{134}$$

Let us now extend this equality to the case when only one of the variables μ,μ' belongs to the ideal $\mathcal{J}$.

Assuming first that μ belongs to the ideal $\mathcal{J}$, we just need to compute $\varphi_1(\mu',\mu)$ for $\mu=\alpha^{*k}\beta^0\alpha^\ell$ and $\mu'=\alpha^{*k'}$ if $k'=\ell-k\geq 0$ or $\mu'=\alpha^{\ell'}$ if $\ell'=k-\ell>0$. One has by (119), $[D,\mu]=\alpha^{*k}X\alpha^\ell$, $X=(k-\ell)\,\beta^0F$ since $[D,\beta^0]=0$ and $[F,\beta^0]=0$. Thus, for $k'\geq 0$,

$$⨍ \mu'[D,\mu]\,|D|^{-1} = ⨍ \alpha^{*k'}\alpha^{*k}(k-\ell)\,\beta^0F\alpha^\ell\,|D|^{-1} = (k-\ell)⨍ \alpha^{*\ell}\beta^0F\alpha^\ell\,|D|^{-1}$$

since $k+k'=\ell$.

For $k'<0$, $\ell'=k-\ell>0$ one gets

$$⨍ \alpha^{\ell'}\alpha^{*k}(k-\ell)\beta^0F\alpha^\ell\,|D|^{-1} = (k-\ell)⨍ \alpha^{*\ell}\beta^0F\alpha^\ell\,|D|^{-1}\,.$$

Thus in both cases we get, using,

$$\begin{aligned} ⨍ \alpha^{*\ell}\beta^0F\alpha^\ell\,|D|^{-1} &= ⨍ \beta^0F\,|D|^{-1} - \ell ⨍ \beta_0F\,|D|^{-2} \\ &= -⨍_{\mathcal{H}'}|D|^{-1} + \ell ⨍_{\mathcal{H}'}|D|^{-2} \end{aligned} \tag{135}$$

the formula,

$$⨍ \mu'[D,\mu]\,|D|^{-1} = (\ell-k)⨍_{\mathcal{H}'}|D|^{-1} + \ell(k-\ell)⨍_{\mathcal{H}'}|D|^{-2}\,. \tag{136}$$

One has

$$\delta([D,\mu]) = \delta((k-\ell)\,\alpha^{*k}\beta^0F\alpha^\ell) = (k-\ell)^2\alpha^{*k}\beta^0F\alpha^\ell$$

so that

$$\fint \mu' \delta([D,\mu])\,|D|^{-2} = -(k-\ell)^2 \fint_{\mathcal{H}'} |D|^{-2}\,. \tag{137}$$

Thus

$$\begin{aligned}\varphi_1(\mu',\mu) &= (\ell-k)\fint_{\mathcal{H}'}|D|^{-1} + \left(\ell(k-\ell)+\frac{1}{2}\,(k-\ell)^2\right)\fint_{\mathcal{H}'}|D|^{-2} \\ &= (\ell-k)\fint_{\mathcal{H}'}|D|^{-1} + \frac{1}{2}\,(k^2-\ell^2)\fint_{\mathcal{H}'}|D|^{-2}\,.\end{aligned}$$

Now one has $\tau_1(\mu',\mu)=0$ and,

$$b\varphi_0(\mu',\mu) = \varphi_0(\mu'\mu) - \varphi_0(\mu\mu') = \varphi_0(\alpha^{*\ell}\beta^0\alpha^\ell) - \varphi_0(\alpha^{*k}\beta^0\alpha^k) = \rho(\ell)-\rho(k)\,.$$

Thus we check that,

$$\varphi_1(\mu',\mu) = \tau_1(\mu',\mu) + b\varphi_0(\mu',\mu)\,. \tag{138}$$

Let us now assume that μ' belongs to the ideal $\mathcal{J}$. We take $\mu' = \alpha^{*k'}\beta^0\alpha^{\ell'}$ and μ to be α^{*k} if $k=\ell'-k'\geq 0$ and α^ℓ if $\ell = k'-\ell' > 0$. Assume first $k\geq 0$. One has $[D,\mu] = k\alpha^{*k}F$ and $\fint \mu'[D,\mu]\,|D|^{-1} = k\fint \alpha^{*k'}\beta^0\alpha^{k'}F\,|D|^{-1}$. Thus using (135) we get,

$$\fint \mu'[D,\mu]\,|D|^{-1} = k\left(-\fint_{\mathcal{H}'}|D|^{-1} + k'\fint_{\mathcal{H}'}|D|^{-2}\right)\,.$$

But $k=\ell'-k'$ so that,

$$\fint \mu'[D,\mu]\,|D|^{-1} = (k'-\ell')\fint_{\mathcal{H}'}|D|^{-1} + k'(\ell'-k')\fint_{\mathcal{H}'}|D|^{-2}\,. \tag{139}$$

Also $\delta([D,\mu]) = k^2\alpha^{*k}F$ and

$$\fint \mu'\delta([D,\mu])\,|D|^{-2} = k^2\fint \alpha^{*k'}\beta^0\alpha^{k'}F\,|D|^{-2} = -k^2\fint_{\mathcal{H}'}|D|^{-2}\,.$$

Thus

$$\varphi_1(\mu',\mu) = (k'-\ell')\fint_{\mathcal{H}'}|D|^{-1} + \left(k'(\ell'-k')+\frac{1}{2}\,k^2\right)\fint_{\mathcal{H}'}|D|^{-2}\,.$$

One has

$$k'(\ell'-k')+\frac{1}{2}\,k^2 = k'(\ell'-k') + \frac{1}{2}\,(\ell'-k')^2 = \frac{1}{2}\,\ell'^2 - \frac{1}{2}\,k'^2\,,$$

so that,

$$\varphi_1(\mu',\mu) = (k'-\ell')\fint_{\mathcal{H}'}|D|^{-1} + \frac{1}{2}\,(\ell'^2-k'^2)\fint_{\mathcal{H}'}|D|^{-2}\,. \tag{140}$$

Now $\mu'\mu = \alpha^{*k'}\beta^0\alpha^{k'}$, $\mu\mu' = \alpha^{*\ell'}\beta^0\alpha^{\ell'}$ so that,

$b\varphi_0(\mu',\mu) = \varphi_0(\mu'\mu) - \varphi_0(\mu\mu') = \varphi_0(\alpha^{*k'}\beta^0\alpha^{k'}) - \varphi_0(\alpha^{*\ell'}\beta^0\alpha^{\ell'}) = \rho(k') - \rho(\ell')$.

Thus, since $\tau_1(\mu',\mu) = 0$, we get,

$$\varphi_1(\mu',\mu) = \tau_1(\mu',\mu) + (b\varphi_0)(\mu',\mu)\,. \tag{141}$$

Next, let us assume that $\ell = k' - \ell' > 0$. Then $\mu = \alpha^\ell$, $[D,\mu] = -\ell\alpha^\ell F$, and $\mu'[D,\mu] = -\ell\alpha^{*k'}\beta^0\alpha^{k'}F$ so that by (135),

$$⨍ \mu'[D,\mu]\,|D|^{-1} = -\ell\left(-⨍_{\mathcal{H}'}|D|^{-1} + k'⨍_{\mathcal{H}'}|D|^{-2}\right),$$

$$⨍ \mu'[D,\mu]\,|D|^{-1} = (k'-\ell')⨍_{\mathcal{H}'}|D|^{-1} + k'(\ell'-k')⨍_{\mathcal{H}'}|D|^{-2}\,. \tag{142}$$

One has $\delta([D,\mu]) = \ell^2\alpha^\ell F$ and,

$$\begin{aligned} ⨍ \mu'\delta([D,\mu])\,|D|^{-2} &= \ell^2 ⨍ \alpha^{*k'}\beta^0\alpha^{k'}F\,|D|^{-1} \\ &= -\ell^2⨍_{\mathcal{H}'}|D|^{-2} = -(k'-\ell')^2⨍_{\mathcal{H}'}|D|^{-2}\,. \end{aligned}$$

Thus as above the coefficient of $⨍_{\mathcal{H}'}|D|^{-2}$ in $\varphi_1(\mu',\mu)$ is $k'(\ell'-k') + \frac{1}{2}(k'-\ell')^2 = \frac{1}{2}\ell'^2 - \frac{1}{2}k'^2$ and,

$$\varphi_1(\mu',\mu) = (k'-\ell')⨍_{\mathcal{H}'}|D|^{-1} + \frac{1}{2}(\ell'^2 - k'^2)⨍_{\mathcal{H}'}|D|^{-2}\,. \tag{143}$$

Thus, as above we get,

$$\varphi_1(\mu',\mu) = \tau_1(\mu',\mu) + b\varphi_0(\mu',\mu)\,. \tag{144}$$

Before we proceed, note that (132) which defines φ_0 is only determined up to the addition of an arbitrary constant to ρ. As it turns out this constant will play a role and will be uniquely specified by equation (85) with the value $\frac{2}{3}$. Also in order to show that the above computation was largely independent of the specific numerical values of $⨍_{\mathcal{H}'}|D|^{-1}$ and $⨍_{\mathcal{H}'}|D|^{-2}$ we did not replace these expressions by their values which are,

$$⨍_{\mathcal{H}'}|D|^{-1} = -1 \qquad ⨍_{\mathcal{H}'}|D|^{-2} = 2\,. \tag{145}$$

(To get (145) we use (97) and (98) and compute

$$\begin{aligned} \mathrm{Trace}_{\mathcal{H}'}(e^{-t|D|}) &= \left(\sum_{k\in\mathbb{Z}} e^{-t|k|}\right)\left(\sum_1^\infty e^{-t\ell}\right) \\ &= \left(1 + \frac{2}{e^t-1}\right)\left(\frac{1}{e^t-1}\right) \sim \frac{2}{t^2} - \frac{1}{t} + \frac{1}{3} - \frac{t}{12} + \cdots) \end{aligned}$$

Thus (up to an additive constant), (133) gives,

$$\rho(j) = -(j + j^2) \tag{146}$$

We extend the definition of φ_0 to $\mathcal{A}$ by $\varphi_0(1) = 0$ while, as above, $\varphi_0(a)$ vanishes if the bidegree of a is $\neq (0,0)$.

4.3 Three dimensional components

It follows from (144) that $\psi = \varphi_1 - \tau_1 - b\varphi_0$ vanishes if one of the arguments is in $\mathcal{J}$ and thus $\psi(a_0, a_1)$ only depends on the symbols $\sigma(a_i) \in C^\infty(S^1)$ where

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} C^\infty(S^1) \longrightarrow 0 \tag{147}$$

is the natural exact sequence, with $\sigma(\alpha) = u$ and $\sigma(\beta) = 0$.
But the same holds for the component φ_3,

$$\varphi_3(a_0, a_1, a_2, a_3) = ⨍ a_0\,[D, a_1]\,[D, a_2]\,[D, a_3]\,|D|^{-3}\,. \tag{148}$$

Indeed if one of the a_j belongs to the two sided ideal $\mathcal{J}$ one is dealing with a trace class operator since $|D|^{-3}$ is trace class on the support of β. Thus $\varphi_3(a_0, a_1, a_2, a_3)$ only depends on the symbols $f_j = \sigma(a_j)$, and is given by,

$$\varphi_3(a_0, a_1, a_2, a_3) = \frac{-1}{2\pi i}\int_{S^1} f_0 f_1' f_2' f_3' \mathrm{d}\theta\,, \tag{149}$$

where $f' = \frac{\partial}{\partial\theta}$. Since $F = 2P - 1$ introduces a minus sign, we use (118) and can replace $[D, a_j]$ by $-if_j'$, so that (149) follows from,

$$⨍ |D|^{-3} = 1\,. \tag{150}$$

Thus to get the complete control of the cocycle φ it remains only to compute $\psi(\alpha^k, \alpha^{*k})$ and $\psi(\alpha^{*\ell}, \alpha^\ell)$.
Let us compute $\psi(\alpha^k, \alpha^{*k})$. One has

$$b\varphi_0(\alpha^k, \alpha^{*k}) = \varphi_0(\alpha^k\alpha^{*k}) - \varphi_0(\alpha^{*k}\alpha^k) = -\,\varphi_0(\alpha^{*k}\alpha^k)$$

Let $\lambda_k = \varphi_0(\alpha^{*k}\alpha^k)$, then

$$\lambda_k - \lambda_{k-1} = \varphi_0(\alpha^{*k-1}(\alpha^*\alpha - 1)\,\alpha^{k-1}) = -\varphi_0(\alpha^{*k-1}\beta^0\alpha^{k-1}) = -\rho(k-1)$$

since $\alpha^*\alpha - 1 = -\beta^0$. We get,

$$\lambda_k = \sum_0^{k-1}(j + j^2) = \frac{k^3}{3} - \frac{k}{3}\,, \tag{151}$$

and,

$$b\varphi_0(\alpha^k, \alpha^{*k}) = -\frac{k^3}{3} + \frac{k}{3} \tag{152}$$

Let us compute $\varphi_1(\alpha^k, \alpha^{*k})$. With $\varphi_1 = \varphi_1^{(0)} - \frac{1}{4}\varphi_1^{(1)} + \frac{1}{8}\varphi_1^{(2)}$, one has,

$$\varphi_1^{(0)}(\alpha^k, \alpha^{*k}) = ⨍ \alpha^k[D, \alpha^{*k}]\,|D|^{-1} = ⨍ k\alpha^k\alpha^{*k}F\,|D|^{-1} = k ⨍ F\,|D|^{-1}$$

using (118). Next,

$$\varphi_1^{(1)}(\alpha^k, \alpha^{*k}) = ⨍ \alpha^k\nabla([D, \alpha^{*k}])\,|D|^{-3} = k ⨍ \alpha^k\nabla(\alpha^{*k}F)\,|D|^{-3}.$$

But $\nabla(x) = |D|\,\delta(x) + \delta(x)\,|D| = \delta^2(x) + 2\delta(x)\,|D|$ and $\delta^2(\alpha^{*k}F) = k^2\alpha^{*k}F$, thus,

$$\varphi_1^{(1)}(\alpha^k, \alpha^{*k}) = k^3 ⨍ F\,|D|^{-3} + 2k^2 ⨍ F\,|D|^{-2}$$

Finally,

$$\begin{aligned}\varphi_1^{(2)}(\alpha^k, \alpha^{*k}) &= ⨍ \alpha^k\nabla^2([D, \alpha^{*k}])\,|D|^{-5} \\ &= 4 ⨍ \alpha^k\delta^2([D, \alpha^{*k}])\,|D|^{-3} = 4k^3 ⨍ F\,|D|^{-3}.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi_1(\alpha^k, \alpha^{*k}) &= k ⨍ F\,|D|^{-1} - \frac{1}{2}k^2 ⨍ F\,|D|^{-2} \\ &+ \left(-\frac{1}{4}k^3 + \frac{1}{2}k^3\right) ⨍ F\,|D|^{-3}.\end{aligned} \tag{153}$$

One has $⨍ F\,|D|^{-3} = -1$ and thus the term in k^3 is $-\frac{k^3}{4}$. Thus the term in k^3 in $\psi = \varphi_1 - \tau_1 - b\varphi_0$ is, using (152),

$$\left(\frac{1}{3} - \frac{1}{4}\right)k^3 = \frac{k^3}{12}. \tag{154}$$

As we shall see now, this $\frac{1}{12}$ corresponds exactly to the coefficient $\frac{1}{12}$ in the universal index formula (theorem 1).

Indeed the term in k^3 corresponds to the cochain ψ_3 given in terms of the symbols f_0, f_1 by,

$$\psi_3(a_0, a_1) = \frac{1}{2\pi i}\int f_0 f_1'''\mathrm{d}\theta. \tag{155}$$

Let us compute $b\psi_3$ where we only involve the symbols. One has

$$f_0 f_1 f_2''' - f_0(f_1 f_2)''' + f_2 f_0 f_1''' = f_0(-3f_1''f_2' - 3f_1'f_2'')$$

thus we get,

$$b\psi_3(a_0, a_1, a_2) = \frac{1}{2\pi i} \int f_0(-3f_1'' f_2' - 3f_1' f_2'')\, \mathrm{d}\theta\,. \tag{156}$$

We have,

$$B_0\varphi_3(a_0, a_1, a_2) = -\frac{1}{2\pi i} \int f_0' f_1' f_2' \mathrm{d}\theta = \frac{1}{2\pi i} \int f_0(f_1'' f_2' + f_1' f_2'')\, \mathrm{d}\theta\,.$$

This is already cyclic so that,

$$B\varphi_3(a_0, a_1, a_2) = \frac{3}{2\pi i} \int f_0(f_1'' f_2' + f_1' f_2'')\, \mathrm{d}\theta\,. \tag{157}$$

and we get,

$$b\psi_3 \,+\, B\varphi_3 \,=\, 0. \tag{158}$$

In fact,

$$\psi_3 \,=\, B\varphi_2\ ,\ \varphi_3 \,=\, b\varphi_2\,, \tag{159}$$

where φ_2 is given by,

$$\varphi_2(a_0, a_1, a_2) = \frac{-1}{2}\frac{1}{2\pi i} \int f_0 f_1' f_2''\, \mathrm{d}\theta. \tag{160}$$

Let us now compute $⨍\, F\,|D|^{-2}$. One has,

$$⨍\, |D|^{-2} = 2\,. \tag{161}$$

and,

$$⨍\, P\,|D|^{-2} = 1\,. \tag{162}$$

Thus we get, since $F = 2P - 1$,

$$⨍\, F\,|D|^{-2} = 0\,. \tag{163}$$

and by a similar computation,

$$⨍\, F\,|D|^{-1} = 1\,. \tag{164}$$

This gives,

$$\psi(\alpha^k, \alpha^{*k}) = \frac{2k}{3} + \frac{k^3}{12}\,. \tag{165}$$

The computation of $\psi(\alpha^{\ell *}, \alpha^{\ell})$ is entirely similar and gives $\psi(\alpha^{\ell *}, \alpha^{\ell}) = -\frac{2\ell}{3} - \frac{\ell^3}{12}$. We thus have $\psi = -\frac{2}{3}\psi_1 + \frac{1}{12}\psi_3$ where,

$$\psi_1(a_0, a_1) = \frac{1}{2\pi i} \int f_0 f_1' \mathrm{d}\theta \,. \tag{166}$$

It just remains to see why adding a constant to ρ allows to eliminate $-\frac{2}{3}\psi_1$ from ψ. This follows from (151) and (152) i. e.

$$b\varphi_0(\alpha^k, \alpha^{*k}) = \sum_0^{k-1} \rho(j) \,, \tag{167}$$

Thus adding $\frac{2}{3}$ to ρ gives $\psi = \frac{1}{12}\psi_3$ and ends the proof of theorem 3. □
We shall now understand the conceptual meaning of the above concrete computation.

5 The η-Cochain.

In this section we shall give two general formulas. The first will provide the conceptual explanation of theorem 3, and of the cochain (φ_0, φ_2) which appears there. The second will prepare for the computation of the local index formula in the general case $q \in]0, 1[$.
The explanation of theorem 3 and of the cochains,

$$\varphi_0(\alpha^{*j} f(\beta)\, \alpha^j) = (\frac{2}{3} - j - j^2)\; \frac{1}{2\pi} \int_{S^1} f \; \mathrm{d}\theta, \tag{168}$$

$$\varphi_2(a_0, a_1, a_2) = \frac{-1}{24} \frac{1}{2\pi i} \int f_0 f_1' f_2'' \, \mathrm{d}\theta, \;\; f_j = \sigma(a_j), \tag{169}$$

is given by the following,

Proposition 2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete simple dimension spectrum not containing 0 and upper bounded by 3. Assume that $[F, a]$ is trace class for all $a \in \mathcal{A}$. Let $\tau_1(a_0, a_1) = \mathrm{Trace}(a_0[F, a_1])$.*
Then the local Chern Character (φ_1, φ_3) of $(\mathcal{A}, \mathcal{H}, D)$ is equal to $\tau_1 + (b + B)\varphi$ where (φ_0, φ_2) is the cochain given by,

$$\varphi_0(a) = \mathrm{Trace}(F\, a\, |D|^{-s})_{s=0},$$
$$\varphi_2(a_0, a_1, a_2) = \frac{1}{24} ⨍ a_0\, \delta(a_1)\, \delta^2(a_2)\, F\, |D|^{-3} \,.$$

Note that φ_0 makes sense by the absence of pole at $s = 0$, i.e. the hypothesis $0 \notin$ Dimension Spectrum. Its value for $a = 1$ coincides with the classical η-invariant ([1],[2]) and justifies the terminology of η-cochain to qualify the cochain (φ_0, φ_2).

The proof of the proposition is a simple calculation based on the expansion ([9])

$$|D|^{-s}a \sim a\ |D|^{-s} - s\ \delta(a)|D|^{-s-1} + \frac{(-s)(-s-1)}{2!}\ \delta^2(a)|D|^{-s-2}$$
$$+\ \frac{(-s)(-s-1)(-s-3)}{3!}\ \delta^3(a)|D|^{-s-3} + s\ \ o(|D|^{-s-3}), \tag{170}$$

which allows to express $b\varphi_0$ in terms of residues. More specifically one gets,

$$b\varphi_0(a_0,a_1) = -\tau_1(a_0,a_1) + {\int\!\!\!\!\!\!-} a_0\,\delta(a_1)\,F\,|D|^{-1}$$
$$-\frac{1}{2}{\int\!\!\!\!\!\!-} a_0\,\delta^2(a_1)\,F\,|D|^{-2} + \frac{1}{3}{\int\!\!\!\!\!\!-} a_0\,\delta^3(a_1)\,F\,|D|^{-3} \tag{171}$$

using the hypothesis $[F,a]$ trace class for all $a \in \mathcal{A}$. This hypothesis also shows that,

$$\varphi_1(a_0,a_1) = {\int\!\!\!\!\!\!-} a_0\,\delta(a_1)\,F\,|D|^{-1}$$
$$-\frac{1}{2}{\int\!\!\!\!\!\!-} a_0\,\delta^2(a_1)\,F\,|D|^{-2} + \frac{1}{4}{\int\!\!\!\!\!\!-} a_0\,\delta^3(a_1)\,F\,|D|^{-3} \tag{172}$$

Comparing (171) with (172) gives the required $\frac{1}{12}$ and allows to check that $\varphi_{odd} = \tau_1 + (b+B)\varphi_{ev}$.
Let us compute φ_{ev} in the above example. One has, as in (73),

$$\varphi_0(\alpha^{*k} e\,\alpha^k) = (\mathrm{Trace}\,(F\,e\,(|D|+k)^{-s}))_{s=0} \tag{173}$$

Using $F = 2P - 1$ this gives,

$$\varphi_0(\alpha^{*k} e\,\alpha^k) = 2(\sum_0^\infty \frac{1}{(n+k)^s})_{s=0} - (\sum_0^\infty \frac{2n+1}{(n+k)^s})_{s=0} \tag{174}$$

One has,

$$(\sum_0^\infty \frac{1}{(n+k)^s})_{s=0} = \zeta(0) - (k-1).$$

Also,

$$\sum_0^\infty \frac{2n+1}{(n+k)^s})_{s=0} =\ 2\zeta(-1) + (1-2k)\,\zeta(0) - \sum_1^{k-1}(2\ell - 2k + 1)$$
$$=\ 2\,\zeta(-1) + (1-2k)\,\zeta(0) + (k-1)^2. \tag{175}$$

Thus we get,

$$\varphi_0(\alpha^{*k} e\,\alpha^k) = 2(\zeta(0) - (k-1)) - (2\,\zeta(-1) + (1-2k)\,\zeta(0) + (k-1)^2) \tag{176}$$

which using the values,

$$\zeta(0) = -\frac{1}{2}\,, \quad \zeta(-1) = -\frac{1}{12} \tag{177}$$

gives the desired result,

$$\varphi_0(\alpha^{*k} e\, \alpha^k) = \frac{2}{3} - k - k^2. \tag{178}$$

The only other non-trivial value of φ_0 is $\eta = \varphi_0(1)$, and the computation gives $\eta = \frac{1}{2}$. Finally the equality,

$$⨍ a_0\, \delta(a_1)\, \delta^2(a_2)\, F\, |D|^{-3} = \frac{1}{2\pi i} \int f_0 f_1' f_2''\, \mathrm{d}\theta, \;\; f_j = \sigma(a_j), \tag{179}$$

and the coincidence of the functional τ_1 of theorem 3 with $\mathrm{Trace}(a_0[F, a_1])$ give a perfect account of theorem 3.
In order to lighten the general computation, for $q \in]0,1[$, we shall state a small variant of proposition 2, proved in a similar way. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ let us define the metric dimension $\mathrm{Dm}(P)$ of a projection P commuting with D as the lower bound of all $d \in \mathbb{R}$ such that $P(D+i)^{-1}$ is in the Schatten class L^d. We then have as above,

Proposition 3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete dimension spectrum not containing* 0*. Assume that* $\mathrm{Dm}(\mathcal{H}) \le 3$*, and* $\mathrm{Dm}(P) \le 2$, $P = (1+F)/2$*, and that $[F,a]$ is trace class for all $a \in \mathcal{A}$. Then the local Chern Character (φ_1, φ_3) of $(\mathcal{A}, \mathcal{H}, D)$ is equal to $\psi_1 - (b+B)\varphi$ where ψ_1 is the cyclic cocycle,*

$$\psi_1(a_0, a_1) = 2 ⨍ a_0\, \delta(a_1)\, P\, |D|^{-1} - ⨍ a_0\, \delta^2(a_1)\, P\, |D|^{-1}$$

and (φ_0, φ_2) is the cochain given by,

$$\varphi_0(a) = \mathrm{Trace}(\, a\, |D|^{-s})_{s=0},$$
$$\varphi_2(a_0, a_1, a_2) = \frac{1}{24} ⨍ a_0\, \delta(a_1)\, \delta^2(a_2)\; |D|^{-3}\,.$$

Combining Propositions 2 and 3 one obtains under the hypothesis of Proposition 3 the equality,

$$\psi_1 - \tau_1 = b(\psi_0) \tag{180}$$

where the cochain ψ_0 is given by,

$$\psi_0(a) = 2\ \mathrm{Trace}(\, a\, P\, |D|^{-s})_{s=0} \tag{181}$$

6 Pseudo-differential calculus and the cosphere bundle on $\mathrm{SU}_q(2)$, $q \in]0,1[$

In this section we shall construct the pseudo-differential calculus on $\mathrm{SU}_q(2)$ following the general theory of [9]. We shall determine the algebra of complete symbols by computing the quotient by smoothing operators. This will give the cosphere bundle S_q^* of $\mathrm{SU}_q(2)$ and the analogue of the geodesic flow will yield a one-parameter group of automorphisms γ_t of $C^\infty(S_q^*)$. We shall also construct the restriction morphism r to the product of two 2-disks,

$$r : C^\infty(S_q^*) \to C^\infty(D^2_{q_+} \times D^2_{q_-}) \tag{182}$$

Our goal is to prepare for the computation in the next section of the dimension spectrum and of residues. Let us recall from [9] that given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we say that an operator P in $\mathcal{H}$ is of order α when,

$$|D|^{-\alpha}\, P \in \bigcap_{n=1}^{\infty} \mathrm{Dom}\, \delta^n \tag{183}$$

where δ is the unbounded derivation given by,

$$\delta(T) = |D|\, T - T\, |D|\,. \tag{184}$$

Thus $OP^0 = \bigcap\limits_{n=1}^{\infty} \mathrm{Dom}\, \delta^n$ is the algebra of operators of order 0 and $OP^{-\infty} = \bigcap\limits_{k>0} OP^{-k}$ is a two sided ideal in OP^0.

We let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple of [3] and we first determine the algebra $\mathcal{B}$ generated by the $\delta^k(a)$, $a \in \mathcal{A}$.

Recall that D is the diagonal operator in $\mathcal{H}$ given by,

$$D(e_{ij}^{(n)}) = (2\, \delta_0(n-i) - 1)\, 2n\; e_{ij}^{(n)} \tag{185}$$

where $\delta_0(k) = 0$ if $k \neq 0$ and $\delta_0(0) = 1$.

By construction, the generators α, β of $\mathcal{A}$ are of the form,

$$\alpha = \alpha_+ + \alpha_-\,, \qquad \beta = \beta_+ + \beta_- \tag{186}$$

where,

$$\delta(\alpha_\pm) = \pm\alpha_\pm\,, \qquad \delta(\beta_\pm) = \pm\beta_\pm\,. \tag{187}$$

The explicit form of $\alpha_\pm$, $\beta_\pm$ is, using $\frac{n}{2}$ instead of n for the notation of the $\frac{1}{2}$ integer,

$$\alpha_\pm(e_{(i,j)}^{(n/2)}) = a_\pm\,(n/2, i, j)\; e^{\left(\frac{n\pm 1}{2}\right)}_{\left(i-\frac{1}{2}, j-\frac{1}{2}\right)} \tag{188}$$

$$\beta_\pm(e^{(n/2)}_{(i,j)}) = b_\pm\,(n/2,i,j)\; e^{\left(\frac{n\pm 1}{2}\right)}_{\left(i+\frac{1}{2},j-\frac{1}{2}\right)} \tag{189}$$

where $a_\pm$, $b_\pm$ are as in (20) and (21) above.

Thus the algebra $\mathcal{B}$ is generated by the operators $\alpha_\pm$, $\beta_\pm$ and their adjoints.

We shall now see that, modulo the smoothing operators, we can strip the complicated formulas for the coefficients $a_\pm$, $b_\pm$ and replace them by extremely simple ones. Since we are computing *local formulas* we are indeed entitled to mod out by smoothing operators and this is exactly where great simplifications do occur.

Let us first relabel the indices i, j using,

$$x = \frac{n}{2} + i\,, \qquad y = \frac{n}{2} + j\,. \tag{190}$$

By construction x and y are *integers* which vary exactly in $\{0, 1, \ldots, n\}$.

Working modulo $OP^{-\infty}$ means that we can neglect in the formulas for $a_\pm$, $b_\pm$ any modification by a sequence of rapid decay in the set:

$$\Lambda = \{(n,x,y)\,;\; n \in \mathbb{N}\,,\; x,y \in \{0,\ldots,n\}\}\,. \tag{191}$$

Thus first, we can get rid of the denominators, since both $(1-q^{2n})^{-1/2}$ or $(1-q^{(2n+2)})^{-1/2}$ are equivalent to 1 and the numerators are bounded.

Next, when we rewrite the numerators in terms of the variables n, x, y we get, say for a_+, the simplified form,

$$a'_+(n,x,y) = q^{1+x+y}(1-q^{2+2(n-x)})^{1/2}(1-q^{2+2(n-y)})^{1/2}\,. \tag{192}$$

Modulo sequences of rapid decay one has,

$$q^x(1-q^{2+2(n-x)})^{1/2} \sim q^x\,,$$

as one sees from the inequality $(1-(1-u)^{1/2}) \leq u$ valid for $u \in [0,1]$, and the fact that

$$q^x\, q^{2(n-x)} \leq q^x\, q^{(n-x)} = q^n\,.$$

Thus we see that modulo sequences of rapid decay we can replace a'_+ by,

$$a''_+(n,x,y) = q^{1+x+y}\,. \tag{193}$$

To simplify formulas let us relabel the basis as,

$$f^{(n)}_{x,y} = e^{(n/2)}_{(x-n/2,y-n/2)}\,, \tag{194}$$

then the following operator agrees with α_+ modulo $OP^{-\infty}$,

$$\alpha'_+(f^{(n)}_{x,y}) = q^{1+x+y}\, f^{(n+1)}_{x,y}\,. \tag{195}$$

For α_- one has, as above,

$$a'_-(n,x,y) = (1-q^{2x})^{1/2}\,(1-q^{2y})^{1/2} \tag{196}$$

and the corresponding operator α'_- is,

$$\alpha'_-(f^{(n)}_{x,y}) = (1-q^{2x})^{1/2}(1-q^{2y})^{1/2}\,f^{(n-1)}_{x-1,y-1}\,. \tag{197}$$

Note that α'_- makes sense for $x=0$, $y=0$. For β_+ one gets,

$$b'_+(n,x,y) = -q^y(1-q^{2+2(n-y)})^{1/2}(1-q^{2+2x})^{1/2} \tag{198}$$

and as above we can replace it by,

$$b''_+(n,x,y) = -q^y(1-q^{2+2x})^{1/2} \tag{199}$$

which gives,

$$\beta'_+(f^{(n)}_{x,y}) = -q^y(1-q^{2+2x})^{1/2}\,f^{(n+1)}_{x+1,y}\,. \tag{200}$$

In a similar way one gets,

$$\beta'_-(f^{(n)}_{x,y}) = q^x(1-q^{2y})^{1/2}\,f^{(n-1)}_{x,y-1} \tag{201}$$

which makes sense even for $y=0$.

It is conspicuous in the above formulas that the new and much simpler coefficients *no longer depend upon the variable* n.

To understand these formulas we introduce the following representations $\pi_\pm$ of $\mathcal{A} = C^\infty(\mathrm{SU}_q(2))$ [1]. In both cases the Hilbert spaces are $\mathcal{H}_\pm = \ell^2(\mathbb{N})$ with basis $(\varepsilon_x)_{x\in\mathbb{N}}$ and the representations are given by,

$$\pi_\pm(\alpha)\,\varepsilon_x = (1-q^{2x})^{1/2}\,\varepsilon_{x-1} \qquad \forall x\in\mathbb{N} \tag{202}$$

$$\pi_\pm(\beta)\,\varepsilon_x = \pm\,q^x\,\varepsilon_x \qquad \forall x\in\mathbb{N}\,. \tag{203}$$

With these notations, and if we ignore the n-dependence in the above formulas we have the correspondence,

$$\begin{aligned} \alpha'_+ &\cong -q\,\beta^*\otimes\beta \\ \alpha'_- &\cong \alpha\otimes\alpha \\ \beta'_+ &\cong \alpha^*\otimes\beta \\ \beta'_- &\cong \beta\otimes\alpha\,, \end{aligned} \tag{204}$$

through the representation $\pi = \pi_+\otimes\pi_-$. Now recall that $\mathcal{A}$ is a Hopf algebra, with coproduct corresponding to matrix tensor multiplication for the following 2×2 matrix,

$$U = \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \tag{205}$$

[1] see the appendix for the notation

which gives,

$$\Delta\alpha = \alpha \otimes \alpha - q\beta^* \otimes \beta \tag{206}$$
$$\Delta\beta = \beta \otimes \alpha + \alpha^* \otimes \beta\,.$$

This shows of course that $\alpha' = \alpha'_+ + \alpha'_-$ and $\beta' = \beta'_+ + \beta'_-$ provide a representation of $\mathcal{A}$ which is the tensor product in the sense of Hopf algebras of the representations π_+ and π_- of $\mathcal{A}$. However to really understand the algebra $\mathcal{B}$ modulo $OP^{-\infty}$ an its action in $\mathcal{H}$ we need to keep track of the shift of n in the formulas for $\alpha'_\pm$ and $\beta'_\pm$.

One can encode these shifts using the $\mathbb{Z}$-grading of $\mathcal{B}$ coming from the one parameter group of automorphisms $\gamma(t)$ which plays the role of the geodesic flow,

$$\gamma(t)(P) = e^{it|D|}\, P\, e^{-it|D|}\,. \tag{207}$$

For the corresponding $\mathbb{Z}$-grading one has,

$$\deg(\alpha_\pm) = \pm 1\,, \qquad \deg(\beta_\pm) = \pm 1\,, \tag{208}$$

which are the correct powers of the shifts of n in the above formulas for $\alpha'_\pm, \beta'_\pm$. To γ we associate the algebra morphism,

$$\gamma : \mathcal{B} \to \mathcal{B} \otimes C^\infty(S^1) = C^\infty(S^1, \mathcal{B}) \tag{209}$$

given by $\gamma(b)(t) = \gamma_t(b)$, $\forall t \in S^1$.

Finally, note that the representations $\pi_\pm$ are not faithful on $C^\infty(SU_q(2))$ since the spectrum of β is real and positive in π_+ and real negative for π_-. We let $C^\infty(D^2_{q\pm})$ be the corresponding quotient algebras and $r_\pm$ the restriction morphisms.

Proposition 4. *The following equalities define an algebra homomorphism* ρ *from* $\mathcal{B}$ *to*

$$C^\infty(D^2_{q+}) \otimes C^\infty(D^2_{q-}) \otimes C^\infty(S^1)\,,$$
$$\rho(\alpha_+) = -q\beta^* \otimes \beta \otimes u\,, \quad \rho(\alpha_-) = \alpha \otimes \alpha \otimes u^*\,,$$
$$\rho(\beta_+) = \alpha^* \otimes \beta \otimes u\,, \quad \rho(\beta_-) = \beta \otimes \alpha \otimes u^*\,,$$

where we omitted $r_+ \otimes r_-$.

Proof. Using (209) it is enough to show that the formulas,

$$\rho_1(\alpha_+) = -q\, \pi_+(\beta^*) \otimes \pi_-(\beta)\,, \quad \rho_1(\alpha_-) = \pi_+(\alpha) \otimes \pi_-(\alpha)$$
$$\rho_1(\beta_+) = \pi_+(\alpha^*) \otimes \pi_-(\beta)\,, \quad \rho_1(\beta_-) = \pi_+(\beta) \otimes \pi_-(\alpha)$$

define a representation of $\mathcal{B}$.

But this representation is weakly contained in the natural representation of $\mathcal{B}$ in $\mathcal{H}$. To obtain ρ_1 from the latter, one just considers vectors $\varepsilon^N_{x,y}$ in $\mathcal{H}$, of the form,

$$\varepsilon^N_{x,y} = \sum h^N_{(n)}\, f^{(n)}_{x,y} \tag{210}$$

where $h^N \in \ell^2(\mathbb{N})$ corresponds to the amenability of the group $\mathbb{Z}$, i.e. to the weak containement of the trivial representation of $\mathbb{Z}$ by the regular one. Thus h^N depends on a large integer N and is $1/\sqrt{N}$ for $0 \leq n < N$ and 0 for $n \geq N$. The almost invariance of h^N under translation of n shows that the n-dependence of the formulas (198)–(201) disappears when $N \to \infty$ and that ρ_1 is a representation of $\mathcal{B}$. Finally ρ is its amplification using (209) $\Box$

Definition 1. *Let $C^\infty(S^*_q)$ be the range of ρ in $C^\infty(D^2_{q+} \times D^2_{q-} \times S^1)$.*

By construction $C^\infty(S^*_q)$ is topologically generated by $\rho(\alpha_\pm)$, $\rho(\beta_\pm)$. The NC-space S^*_q plays the role of the *cosphere bundle*. The algebra $C^\infty(S^*_q)$ is strictly contained in $C^\infty(D^2_{q+} \times D^2_{q-} \times S^1)$ since its image under $\sigma \otimes \sigma \otimes \mathrm{Id}$ is the subalgebra of $C^\infty(S^1 \times S^1 \times S^1)$ generated by $u \otimes u \otimes u^*$. Let ν_t be the S^1-action on S^*_q given by the restriction of the derivation $1 \otimes 1 \otimes \partial_u$ where $\partial_u(u) = u$. By construction,

$$\rho(\gamma(t)(P)) = \nu_t(\rho(P) \tag{211}$$

so that ν_t is the analogue of the action of the geodesic flow on the cosphere bundle. We let,

$$r : C^\infty(S^*_q) \to C^\infty(D^2_{q_+} \times D^2_{q_-}) \tag{212}$$

be the natural restriction morphism.
Viewing ρ as the total symbol map we shall now define a natural lifting from symbols to operators. This will only be relevant on the range of ρ but to define it we start from the representation $\pi = \pi_+ \otimes \pi_- \otimes s$ of $C^\infty(D^2_{q_+}) \otimes C^\infty(D^2_{q_-}) \otimes C^\infty(S^1)$ in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ where $s(u)$ is the shift S in $\ell^2(\mathbb{Z})$. We let Q be the orthogonal projection on the subset Λ of the basis $f^{(n)}_{x,y}$ determined by $n \geq \sup(x, y)$ and identify the range of Q with the Hilbert space $\mathcal{H}$. By definition the lifting λ is the compression,

$$\lambda(g) = Q\, \pi(g)\, Q \tag{213}$$

For g of the form $\mu \otimes u^n$, one has,

$$\lambda(g)\, f^{(\ell)}_{x,y} = \sum \mu^{(x',y')}_{(x,y)}\, f^{(\ell+n)}_{x',y'} \tag{214}$$

where $\mu^{(x',y')}_{(x,y)}$ are the matrix elements for the action of μ in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$,

$$\mu\, \varepsilon_{x,y} = \sum \mu^{(x',y')}_{(x,y)}\, \varepsilon_{x',y'}\,. \tag{215}$$

It may happen in formula (214) that the indices in $f^{(\ell+n)}_{x',y'}$ do not make sense, i.e. that $f^{(\ell+n)}_{x',y'}$ does not belong to Λ. In that case the corresponding term is 0. We have now restaured the shift of n in the formulas for $\alpha'_\pm$ and $\beta'_\pm$ and get,

Lemma 1. *For any $b \in \mathcal{B}$ one has,*

$$b - \lambda(\rho(b)) \in OP^{-\infty}\,.$$

We refer to the appendix for the implications of this lemma. We give there another general lemma proving the stability under holomorphic functional calculus for the natural smooth algebras involved in our discussion.

7 Dimension Spectrum and Residues for $\mathrm{SU}_q(2)$, $q \in]0,1[$

Let as above S^*_q be the cosphere bundle of $\mathrm{SU}(2)_q$, γ_t its geodesic flow and,

$$r : C^\infty(S^*_q) \to C^\infty(D^2_{q_+} \times D^2_{q_-}) \tag{216}$$

be the natural restriction morphism.
For $C^\infty(D^2_{q_\pm})$ we have an exact sequence of the form,

$$0 \longrightarrow \mathcal{S} \longrightarrow C^\infty(D^2_q) \stackrel{\sigma}{\longrightarrow} C^\infty(S^1) \longrightarrow 0 \tag{217}$$

where the ideal $\mathcal{S}$ is isomorphic to the algebra of matrices of rapid decay. Using the representations $\pi_\pm$ of $C^\infty(D^2_{q_\pm})$ in $\ell^2(\mathbb{N})$ with basis (ε_x), $x \in \mathbb{N}$, we define two linear functionals τ_0 and τ_1 by,

$$\tau_1(a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(a)\,\mathrm{d}\theta \qquad \forall a \in C^\infty(D^2_q) \tag{218}$$

and

$$\tau_0(a) = \lim_{N\to\infty} \mathrm{Trace}_N(\pi(a)) - \tau_1(a)\,N \tag{219}$$

where,

$$\mathrm{Trace}_N(a) = \sum_0^N \langle a\,\varepsilon_x, \varepsilon_x\rangle\,. \tag{220}$$

(where we omitted $\pm$ in the above formulas). For $a \in \mathcal{S}$ one has $\sigma(a) = 0$ and $\tau_1(a) = 0$, $\tau_0(a) = \mathrm{Trace}\,(a)$. In general both τ_0 and τ_1 are invariant under the one parameter group generated by ∂_α and on the fixed points of this group, one has,

$$\tau_0(a) = \mathrm{Trace}\,(\pi(a) - \tau_1(a)\,1) + \tau_1(a)\,. \tag{221}$$

For all $a \in \mathcal{A}$ one has, (for all $k > 0$),

$$\mathrm{Trace}_N(\pi(a)) = \tau_1(a)\, N + \tau_0(a) + 0(N^{-k})\,. \tag{222}$$

We shall now prove a general formula computing residues of pseudo-differential operators in terms of their symbols,

Theorem 4.

1. *The dimension spectrum of* $\mathrm{SU}_q(2)$ *is* $\{1,2,3\}$.

2. *Let* $b \in \mathcal{B}$, $\rho(b) \in C^\infty(S_q^*)$ *its symbol. Then let* $\rho(b)^0$ *be the component of degree* 0 *for the geodesic flow* γ_t. *One has,*

$$⨍ b\,|D|^{-3} = (\tau_1 \otimes \tau_1)(r\rho(b)^0)$$

$$⨍ b\,|D|^{-2} = (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r\rho(b)^0)$$

$$⨍ b\,|D|^{-1} = (\tau_0 \otimes \tau_0)(r\rho(b)^0)\,.$$

Proof. By lemma the operator $b - \lambda\rho(b)$ belongs to $OP^{-\infty}$, thus $\zeta_b(s) - \mathrm{Trace}\,(\lambda\rho(b)\ |D|^{-s})$ is a holomorphic function of $s \in \mathbb{C}$.
One has $\mathrm{Trace}\,(\lambda\rho(b)\,|D|^{-s}) = \mathrm{Trace}\,(\lambda\rho(b)^0|D|^{-s})$ and with $\rho(b)^0 = T$,

$$\mathrm{Trace}\,(\lambda(T)\,|D|^{-s}) = \sum_{n=0}^{\infty}\left(\sum_{x=0}^{n}\ \sum_{y=0}^{n}\langle\pi(T)\,\varepsilon_{x,y},\varepsilon_{x,y}\rangle\right) n^{-s}\,. \tag{223}$$

Thus by (222) we get, modulo holomorphic functions of $s \in \mathbb{C}$,

$$\begin{aligned} \mathrm{Trace}\,(\lambda(T)\,|D|^{-s}) \ &\cong\ (\tau_1 \otimes \tau_1)(T)\,\zeta(s-2) \qquad (224)\\ &+\ (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(T)\,\zeta(s-1) + (\tau_0 \otimes \tau_0)(T)\,\zeta(s)\,. \end{aligned}$$

This shows that $\zeta_b(s)$ extends to a meromorphic function of s with simple poles at $s \in \{1,2,3\}$ and gives the above values for the residues.
To show 1) we still need to adjoin $F = \mathrm{Sign}\, D$ to the algebra $\mathcal{B}$, but by [3] one has,

$$[F,a] \in OP^{-\infty} \qquad \forall a \in \mathcal{B} \tag{225}$$

so that the only elements which were not handled above are those of the form,

$$bP\,, \quad b \in \mathcal{B}\,, \quad P = \frac{1+F}{2}\,. \tag{226}$$

Thus with the above notation we still need to analyse,

$$\text{Trace}\,(\lambda(T)\,P\,|D|^{-s})\,. \tag{227}$$

Since P corresponds to the subset of the basis $f_{x,y}^{(n)}$ given by $\{x = n\}$ in the above notations, the trace (227) can be expressed as,

$$\text{Trace}\,(\lambda(T)\,P\,|D|^{-s}) = \sum_{n=0}^{\infty}\sum_{y=0}^{n}\langle \pi(T)\,\varepsilon_{n,y}, \varepsilon_{n,y}\rangle\, n^{-s} \tag{228}$$

and the structure of the representation π_+ shows that the r.h.s. gives, modulo holomorphic function of $s \in \mathbb{C}$,

$$\text{Trace}\,(\lambda(T)\,P\,|D|^{-s}) \cong (\tau_1 \otimes \tau_1)(T)\,\zeta(s-1) + (\tau_1 \otimes \tau_0)(T)\,\zeta(s)\,. \tag{229}$$

This shows (216) and also gives the two formulas,

$$⨍\, b\,P\,|D|^{-2} = (\tau_1 \otimes \tau_0)(\rho(b)^0) \tag{230}$$

and,

$$⨍\, b\,P\,|D|^{-1} = (\tau_0 \otimes \tau_0)\,\rho(b)^0 \tag{231}$$

which we shall now exploit to do the computation of the local index formula for $\text{SU}_q(2)$.

8 The local index formula for $\text{SU}_q(2)$, $q \in]0,1[$

The local index formula for the spectral triple of $\text{SU}_q(2)$ uniquely determines a cyclic 1-cocycle and hence by ([5]) a corresponding one dimensional cycle. We shall first describe independently the obtained cycle since the NC-differential calculus it exhibits is of independent interest.
Let $\mathcal{A} = C^\infty(\text{SU}_q(2))$ and ∂ the derivation,

$$\partial = \partial_\beta - \partial_\alpha\,. \tag{232}$$

We extend the functional τ_0 of (219) to $\mathcal{A}$ by,

$$\tau(a) = \tau_0\,(r_-(a^{(0)})) \qquad \forall a \in \mathcal{A} \tag{233}$$

where $a^{(0)}$ is the component of degree 0 for ∂. By construction τ is ∂-invariant but fails to be a trace. It is the average of the transformed of $\tau_0 \circ r_-$ by the automorphism $\nu_t \in \text{Aut}(\mathcal{A})$,

$$\nu_t = \exp(it\partial)\,. \tag{234}$$

Thus τ fails to be a trace because τ_0 does. However we can compute the Hochschild coboundary $b\,\tau_0$, $b\,\tau_0(a_0,a_1)=\tau_0(a_0\,a_1)-\tau_0(a_1\,a_0)$. It only depends upon the symbols $\sigma(a_j)\in C^\infty(S^1)$ and is given by,

$$b\,\tau_0(a_0,a_1)=\frac{-1}{2\pi i}\int\sigma(a_0)\,\mathrm{d}\sigma(a_1)\,. \tag{235}$$

One has

$$b\,\tau(a_0,a_1)=\frac{1}{2\pi}\int b\,\tau_0(a_0(t),a_1(t))\,\mathrm{d}t$$

where $a(t)=\nu(t)(a)$, and for homogeneous elements of $\mathcal{A}$, $b\,\tau(a_0,a_1)=0$ unless the total degree is $(0,0)$. For such elements we thus get,

$$b\,\tau(a_0,a_1)=\frac{1}{2\pi}\int b\,\tau_0(a_0(t),a_1(t))\,\mathrm{d}t=b\,\tau_0(a_0,a_1)=-\frac{1}{2\pi i}\int\sigma(a_0)\,\mathrm{d}\sigma(a_1)$$

so that,

$$b\,\tau(a_0,a_1)=\frac{-1}{2\pi i}\int\sigma(a_0)\,\mathrm{d}\sigma(a_1)\qquad\forall a_j\in\mathcal{A}\,. \tag{236}$$

Thus, even though τ is not a trace we do control by how much it fails to be a trace and this allows us to define a *cycle* in the sense of [5] using both first and second derivatives to define the differential,

$$\mathcal{A}\xrightarrow{d}\Omega^1\,. \tag{237}$$

More precisely let us define the $\mathcal{A}$-bimodule Ω^1 with underlying linear space the direct sum, $\Omega^1=\mathcal{A}\oplus\Omega^{(2)}(S^1)$ where $\Omega^{(2)}(S^1)$ is the space of differential forms $f(\theta)\,\mathrm{d}\theta^2$ of weight 2 on S^1. The bimodule structure is defined by,

$$\begin{aligned} a(\xi,f)&=(a\xi,\sigma(a)f) \\ (\xi,f)b&=(\xi b,-i\,\sigma(\xi)\,\sigma(b)'+f\sigma(b)) \end{aligned} \tag{238}$$

for $a,b\in\mathcal{A}$, $\xi\in\mathcal{A}$ and $f\in\Omega^{(2)}(S^1)$.
The differential d of (237) is then given by,

$$da=\partial a+\frac{1}{2}\,\sigma(a)''\,\mathrm{d}\theta^2 \tag{239}$$

as in a Taylor expansion.
The functional $\int$ is defined by,

$$\int(\xi,f)=\tau(\xi)+\frac{1}{2\pi i}\int f\,\mathrm{d}\theta\,. \tag{240}$$

We then have,

Proposition 5. *The triple* $(\Omega, d, \int)$ *is a* cycle, *i.e.* $\Omega = \mathcal{A} \oplus \Omega^1$ *equipped with* d *is a graded differential algebra (with* $\Omega^0 = \mathcal{A}$*) and the functional* $\int$ *is a closed graded trace on* Ω*.*

Proof. One checks directly that Ω^1 is an $\mathcal{A}$-bimodule so that $\Omega = \mathcal{A} \oplus \Omega^1$ is a graded algebra. The equality $\sigma(\partial a) = i\,\sigma(a)'$ together with (238) show that $d(ab) = (da)\,b + a\,db \quad \forall a, b \in \mathcal{A}$. It is clear also that $\int \mathrm{d}a = 0 \quad \forall a \in \mathcal{A}$. It remains to show that $\int$ is a (graded) trace, i.e. that $\int a\omega = \int \omega a \quad \forall \omega \in \Omega^1$, $a \in \mathcal{A}$.
With $\omega = (a_1, f)$ one has

$$a\omega - \omega a = (aa_1 - a_1 a, \sigma(a) f + i\sigma(a_1)\,\sigma(a)' - f\sigma(a)) = (aa_1 - a_1 a, i\sigma(a_1)\,\sigma(a)')\,.$$

Thus (236) shows that $\tau(aa_1 - a_1 a) + \frac{1}{2\pi}\int i\sigma(a_1)\,\sigma(a)'\,\mathrm{d}\theta = 0$. □

We let χ be the cyclic 1-cocycle which is the character of the above cycle, explicitly,

$$\chi(a_0, a_1) = \int a_0\,\mathrm{d}a_1 \qquad \forall a_0, a_1 \in \mathcal{A}\,. \tag{241}$$

As above in proposition 3, we let φ be the cochain,

$$\varphi_0(a) = \mathrm{Trace}(\,a\,|D|^{-s})_{s=0},$$
$$\varphi_2(a_0, a_1, a_2) = \frac{1}{24} ⨍ a_0\,\delta(a_1)\,\delta^2(a_2)\ |D|^{-3}\,.$$

Theorem 5. *The local index formula of the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is given by the cyclic cocycle* χ *up to the coboundary of the cochain* (φ_0,φ_2)*.*

In other words the cocycle ψ_1 of proposition 3 is equal to χ. This follows from (230), (231).
We leave it as an exercice for the reader to compute the (non-zero) pairing between the above cyclic cocycle χ and the K-theory class of the basic unitary,

$$U = \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \tag{242}$$

Applying (181) we obtain the following corollary,

Corollary 1. *The character* $\mathrm{Trace}(a_0[F, a_1])$ *of the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is given by the cyclic cocycle* χ *up to the coboundary of the cochain* ψ_0 *given by* $\psi_0(a) = 2\ \mathrm{Trace}(\,a\,P\,|D|^{-s})_{s=0}$.

The cochain ψ_0 is only non-zero on elements which are functions of $\beta^*\beta$ as one sees for homogeneity reasons using the bigrading and the natural basis of $\mathcal{A}$ given by the $\alpha^k(\beta^*)^n\beta^m$. It is thus entirely determined by the values $\psi_0((\beta^*\beta)^n)$. It is an interesting problem to compute these functions of q. In order to state the result we recall that the Dedekind eta-function for the modulus q^2 is given by,

$$\eta(q^2) = q^{\frac{1}{12}} \prod_1^\infty (1 - q^{2n}). \tag{243}$$

We let G be its logarithmic derivative $q^2 \partial_{q^2} \eta(q^2)$, (up to sign and after substraction of the constant term),

$$G(q^2) = \sum_1^\infty n\, q^{2n} (1 - q^{2n})^{-1}. \tag{244}$$

Theorem 6. *The functions $\frac{1}{2}\psi_0((\beta^*\beta)^r)$ of the variable q are of the form $q^{-2r}(q^2 R_r(q^2) - G(q^2))$ where $R_r(q^2)$ are rational fractions of q^2 with poles only at roots of unity.*

Proof. The first step is to prove that the diagonal terms $d(n,i,j)$ of the matrix $(\beta^*\beta)^r$ fulfill the equality,

$$d(n,n,j) = \prod_{l=0}^{r-1} \frac{q^{2n+2j} - q^{4n+2+2l}}{1 - q^{4n+4+2l}}. \tag{245}$$

This follows by writing $(\beta^*\beta)^r = \beta^{*r}\beta^r = (\beta_+^* + \beta_-^*)^r(\beta_+ + \beta_-)^r$ and observing that, since $i = n$, the only term which contributes is $(\beta_+^*)^r(\beta_+)^r$.
We then change variables as above replacing n by $n/2$ and j by $-n/2 + y$, which gives for the value of $\frac{1}{2}\psi_0((\beta^*\beta)^r)$ the value at $s = 0$ of the sum,

$$Z(s) = \sum_{n=0}^\infty n^{-s} \sum_{y=0}^n \prod_{l=0}^{r-1} \frac{q^{2y} - q^{2n+2+2l}}{1 - q^{2n+4+2l}}. \tag{246}$$

(with the usual convention for $n = 0$).
To understand the appearance of $G(q^2)$ and the corresponding coefficient, let us take the constant term in

$$P(q^{2y}) = \prod_{l=0}^{r-1} \frac{q^{2y} - q^{2n+2+2l}}{1 - q^{2n+4+2l}} \tag{247}$$

viewed as a polynomial in q^{2y}, which gives, with $x = q^{2n}$,

$$R(x) = \prod_{l=0}^{r-1} \frac{(-q^{2+2l}x)}{1 - q^{4+2l}x} \tag{248}$$

The fraction $R(1/z)$ has simple distinct poles at $z = q^{4+2l}$ and vanishes at ∞, thus we can express it in the form,

$$R(1/z) = \sum_{l=0}^{r-1} \frac{\lambda_l}{z - q^{4+2l}} \tag{249}$$

Each term contributes to (246) by the value at $s = 0$ of,

$$\lambda_l \sum_{n=0}^{\infty} n^{-s}(n+1)\, \frac{q^{2n}}{1 - q^{2n+4+2l}}. \tag{250}$$

The value at $s = 0$ makes sense as a convergent series, and the coefficient of $G(q^2)$ is obtained by setting $n' = n+l+2$ which gives $\lambda_l\, q^{-4-2l}\, G(q^2)$. Thus the overall coefficient for $G(q^2)$ is, using (248), (249) and the behaviour for $z = 0$,

$$\sum_{l=0}^{r-1} \frac{\lambda_l}{q^{4+2l}} = -q^{-2r}. \tag{251}$$

One has $n+1 = n' - l - 1$ and the above terms also generate a non-zero multiple of the function,

$$G_0(q^2) = \frac{q^{2n}}{1 - q^{2n}}. \tag{252}$$

The coefficient is given by,

$$c_0 = -\sum_{l=0}^{r-1} \lambda_l\, (l+1)\, q^{-2l-4}. \tag{253}$$

We need to show that the other terms coming from the non-constant terms in $P(q^{2y})$ exactly cancell the above multiple of $G_0(q^2)$, modulo rational functions of q^2.

Using the q^2-binomial coefficients $\binom{r}{k}_{q^2}$, one obtains, with P as in (247), that,

$$P(z) = N(z) \prod_{l=0}^{r-1} (1 - q^{2n+4+2l})^{-1} \tag{254}$$

where,

$$N(z) = \sum_{k=0}^{r} (-1)^k q^{k(k+1)} q^{2kn} \binom{r}{k}_{q^2} z^{r-k} \tag{255}$$

The constant term (in z^0) has already been taken care of, and for the others the effect of the summation $\sum_{y=0}^{n}$ in (246) is to replace z^{r-k} in the above sum by,

$$\sum_{y=0}^{n} q^{2y(r-k)} = \frac{1 - q^{2(r-k)(n+1)}}{1 - q^{2(r-k)}} \tag{256}$$

Thus the contribution of the other terms is governed by the rational fraction of $x = q^{2n}$

$$Q(x) = \left(\sum_{k=0}^{r-1} (-1)^k q^{k(k+1)} \left(\begin{array}{c} r \\ k \end{array}\right)_{q^2} \frac{x^k - x^r q^{2(r-k)}}{1 - q^{2(r-k)}}\right) \prod_{l=0}^{r-1} (1 - x\, q^{4+2l})^{-1} \tag{257}$$

The degree of the numerator is the same as the degree of the denominator, all poles are simple, and we can thus expand $Q(x)$ as,

$$Q(x) = \mu + \sum_{l=0}^{r-1} \frac{\mu_l}{1 - x\, q^{4+2l}} \tag{258}$$

The same reasoning as above shows that modulo rational functions of q^2, each term contributes to (246) by a multiple of $G_0(q^2)$, while the overall coefficient is the sum of the μ_l,

$$c_1 = Q(0) - Q(\infty) \tag{259}$$

Using (257) one obtains,

$$c_1 = (1 - q^{2r})^{-1} + (-1)^r q^{-r(r+1)} \sum_{k=0}^{r-1} (-1)^k \left(\begin{array}{c} r \\ k \end{array}\right)_{q^2} \frac{q^{k(k-1)}}{1 - q^{2(r-k)}} \tag{260}$$

To compute the λ_l one takes the residues of (249) which gives the formula,

$$\lambda_j = -(-1)^j\, q^{(4+j^2+j(3-2r)-3r+r^2)} \left(\begin{array}{c} r-1 \\ j \end{array}\right)_{q^2} \rho(r-1)^{-1} \tag{261}$$

where,

$$\rho(r-1) = \prod_{a=1}^{r-1} (q^{2a} - 1) \tag{262}$$

This gives the following formula for the coefficient c_0,

$$c_0 = \rho(r-1)^{-1} \sum_{j=0}^{r-1} (-1)^j\, (j+1)\, q^{(j^2+j-2rj-3r+r^2)} \left(\begin{array}{c} r-1 \\ j \end{array}\right)_{q^2} \tag{263}$$

The fundamental cancellation now is the identity

$$c_0 + c_1 = 0 \tag{264}$$

which is proved by differentiation of the q^2-binomial formula.
The above discussion provides an explicit formula for the rational fractions R_r which allows to check that their only poles are roots of unity. $\square$

The simple expression $q^{-2r}G(q^2)$ blows up exponentially for $r \to \infty$ and if it were alone it would be impossible to extend the cochain ψ_0 from the purely algebraic to the smooth framework. However,

$$\psi_0((\beta^*\beta)^r) \to 1 + 2q^2/(q^2-1) \text{ when } r \to \infty.$$

Thus it is only by the virtue of the rational approximations $q^2\, R_r(q^2)$ of $G(q^2)$ that the tempered behaviour of $\psi_0((\beta^*\beta)^r)$ is insured.
The list of the first $R_r(q)$ is as follows,

$$\begin{aligned} R_1[q] &= \frac{3}{2(1-q)}, \qquad R_2[q] = \frac{2+5q-3q^2}{2(-1+q)^2(1+q)}, \\ R_3[q] &= \frac{2+8q+13q^2+11q^3-q^4-3q^5}{2(-1+q^2)^2(1+q+q^2)}, \\ R_4[q] &= \frac{2+10q+24q^2+43q^3+50q^4+46q^5+24q^6+4q^7-4q^8-3q^9}{2(1+q^2)(-1-q+q^3+q^4)^2}. \end{aligned} \tag{265}$$

Finally, note that the appearance of the function $G(q^2)$ in ψ_0 is not an artefact which could be eliminated by a better choice of cochain with the same coboundary. Indeed since

$$\alpha\alpha^* - \alpha^*\alpha = (1-q^2)\,\beta^*\beta$$

the coboundary $b\psi_0(\alpha, \alpha^*)$ still involves $G(q^2)$.

9 Quantum groups and invariant cyclic cohomology

The main virtue of the above spectral triple for $\mathrm{SU}_q(2)$ is its invariance under left translations (cf. [3]). More precisely the following equalities define an action of the envelopping algebra $\mathcal{U} = U_q(\mathrm{SL}(2))$ on $\mathcal{H}$ which commutes with D and implements the translations on $C^\infty(\mathrm{SU}_q(2))$,

$$k\, e_{ij}^{(n)} = q^j\, e_{ij}^{(n)} \tag{266}$$

$$e\, e_{ij}^{(n)} = q^{-n+\frac{1}{2}}(1-q^{2(n+j+1)})^{1/2}(1-q^{2(n-j)})^{1/2}(1-q^2)^{-1}\, e_{ij+1}^{(n)} \tag{267}$$

while $f = e^*$.
With these notations one has,

$$ke = q\, e\, k\,, \;\; kf = q^{-1} f k\,, \;\; [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}}\,. \tag{268}$$

The vector $\Omega = e^{(0)}_{(0,0)}$ is preserved by the action and one has a natural densely defined action of $\mathcal{U}$ on $\mathcal{A} = C^\infty(\mathrm{SU}_q(2))$ such that,

$$h(x)\Omega = h(x\Omega) \qquad \forall x \in \mathcal{A}\,,\ h \in \mathcal{U}\,. \tag{269}$$

The coproduct is given by,

$$\Delta k = k \otimes k\,,\ \Delta e = k^{-1} \otimes e + e \otimes k\,,\ \Delta f = k^{-1} \otimes f + f \otimes k \tag{270}$$

and the action of $\mathcal{U}$ on $\mathcal{A}$ fulfills,

$$h(xy) = \sum h_{(1)}(x)\, h_{(2)}(y) \qquad \forall x, y \in \mathcal{A}\,, \tag{271}$$

$\forall h \in \mathcal{U}$ with $\Delta h = \sum h_{(1)} \otimes h_{(2)}$.
On the generators $\alpha, \alpha^*, \beta, \beta^*$ of $\mathcal{A}$ one has,

$$k(\alpha) = q^{-1/2}\alpha\,,\ k(\beta) = q^{-1/2}\beta\,,\ e(\alpha) = q\beta^*\,, \tag{272}$$

$$e(\beta) = -\alpha\,,\ e(\alpha^*) = 0\,,\ e(\beta^*) = 0\,.$$

This representation of $\mathcal{U}$ in $\mathcal{H}$ generates the regular representation of the compact quantum group $\mathrm{SU}_q(2)$ and we let $M = \mathcal{U}''$ be the von Neumann algebra it generates in $\mathcal{H}$. It is a product $M = \prod_{n \in \frac{1}{2}\mathbb{N}} M_{2n+1}(\mathbb{C})$ of matrix algebras, where $M_{2n+1}(\mathbb{C})$ acts with multiplicity $2n+1$ in the space $\mathcal{H}_{(n)} = \{\text{span of } e^{(n)}_{i,j}\}$.
The elements of $\mathcal{U}$ are unbounded operators affiliated to M and at the qualitative level we shall leave the freedom to choose a weakly dense subalgebra $\mathcal{C}$ of M. Since all the constructions performed so far in this paper were canonically dependent on the spectral triple, the $\mathrm{SU}_q(2)$ equivariance of $(\mathcal{A}, \mathcal{H}, D)$ should entail a corresponding *invariance* of all the objects we delt with. We shall concentrate on the cyclic cohomology aspect and show that indeed there is a fairly natural and simple notion of *invariance* fulfilled by all cochains involved in the above computation.

The main point is that we can enlarge the algebra $\mathcal{A}$ to the algebra $\mathcal{D} = \mathcal{A} \bowtie \mathcal{C}$ generated by $\mathcal{A}$ and $\mathcal{C}$, extend the cochains on $\mathcal{D}$ by similar formulas and use the commutation,

$$[D, c] = 0 \qquad \forall c \in \mathcal{C} \tag{273}$$

to conclude that the extended cochains fulfill the following key property,

Definition 2. *Let $\mathcal{D}$ be a unital algebra, $\mathcal{C} \subset \mathcal{D}$ a (unital) subalgebra and $\varphi \in C^n(\mathcal{D})$ an* n-*cochain. We shall say that φ is $\mathcal{C}$-*constant *iff both $\varphi(a^0, \ldots, a^n)$ and $(b\varphi)(a^0, \ldots, a^{n+1})$ vanish if one of the a^j, $j \geq 1$ is in $\mathcal{C}$.*

When $\mathcal{C} = \mathbb{C}$ this is a normalization condition.

When φ is $\mathcal{C}$-constant then $B_0\varphi(a^0,\dots,a^{n-1}) = \varphi(1,a^0,\dots,a^{n-1})$ so that $B\varphi = AB_0\varphi$ is also $\mathcal{C}$-constant. It follows that $\mathcal{C}$-constant cochains form a subcomplex of the (b,B) bicomplex of $\mathcal{D}$ and we can develop cyclic cohomology in that context, parallel to ([5],[6]). We shall denote by $HC^*_{\mathcal{C}}(\mathcal{D})$ the corresponding theory.
In the above context we take for $\mathcal{D}$ the algebra $\mathcal{A} > \triangleleft \mathcal{C}$ and use the lighter notation $HC^*_{\mathcal{C}}(\mathcal{A})$ for the corresponding theory.
Let us now give examples of specific cochains on $\mathcal{A} = C^\infty(\mathrm{SU}_q(2))$ which extend to $\mathcal{C}$-constant cochains on $\mathcal{A} \rtimes \mathcal{C} = \mathcal{D}$. We start with the non local form of the Chern character of the spectral triple,

$$\psi_1(a^0,a^1) = \mathrm{Trace}\,(a^0\,[F,a^1]) \qquad \forall a^0,a^1 \in \mathcal{A}\,. \tag{274}$$

Let us show how to extend ψ_1 to an M-constant cochain on $\mathcal{A} \rtimes M = \mathcal{D}$. An element of $\mathcal{D}$ is a finite linear combination of monomials $a_1\,m_1\,a_2\,m_2\dots a_\ell\,m_\ell$, where $a_j \in \mathcal{A}$, $m_\ell \in M$. But $[F,a]$ is a trace class operator for any $a \in \mathcal{A}$, while $[F,m] = 0 \quad \forall m \in M$, thus we get,

$$[F,x] \in \mathcal{L}^1 \qquad \forall x \in \mathcal{D} = \mathcal{A} \rtimes M\,. \tag{275}$$

We can thus define $\widetilde{\psi}_1$ as the character of the module $(\mathcal{H},F)$ on $\mathcal{D}$, namely,

$$\widetilde{\psi}_1(x_0,x_1) = \mathrm{Trace}\,(x_0\,[F,x_1])\,. \tag{276}$$

It is clear that $\widetilde{\psi}_1$ is M-constant and that $b\widetilde{\psi}_1 = 0$ so that $b\widetilde{\psi}_1$ is also M-constant, $\widetilde{\psi}_1 \in HC^1_M(\mathcal{A})$. This example is quite striking in that we could extend ψ_1 to a very large algebra. Indeed if we stick to bounded operators M is the largest possible choice for $\mathcal{C}$. A similar surprising extension of a cyclic 1-cocycle in a von-Neumann algebra context already occured in the anabelian 1-traces of ([14]). As a next example let us take the functional on $\mathcal{A}$ which is the natural trace,

$$\psi_0(x) = \frac{1}{2\pi}\int \sigma(x)\,\mathrm{d}\theta \qquad \forall x \in C^\infty(\mathrm{SU}_q(2))\,. \tag{277}$$

When written like this, its $\mathrm{SU}_q(2)$-invariance is not clear and in fact cannot hold in the simplest sense since this would contradict the uniqueness of the Haar state on $C^\infty(\mathrm{SU}_q(2))$. Let us however show that ψ_0 extends to an M-constant cochain (in fact an M-constant trace) on $\mathcal{D} = \mathcal{A} \rtimes M$ as above. To do this we rewrite (277) as,

$$\psi_0(x) = \mathrm{Tr}_\omega(x\,|D|^{-3}) \qquad \forall x \in C^\infty(\mathrm{SU}_q(2)) \tag{278}$$

where Tr_ω is the Dixmier trace ([16])([13]) and simply write the extension as,

$$\widetilde{\psi}_0(x) = \mathrm{Tr}_\omega(x\,|D|^{-3})\,. \tag{279}$$

For any monomial $\mu = a_1\, m_1 \dots a_m\, m_m$ as above one has $[D, \mu]$ bounded and $[|D|, \mu]$ bounded. Thus it follows from the general properties of Tr_ω that,

$$\widetilde{\psi}_0(xy) = \widetilde{\psi}_0(yx) \qquad \forall x, y \in \mathcal{D} = \mathcal{A} \rtimes M\,. \tag{280}$$

This shows of course that $\widetilde{\psi}_0$ is a 0-cycle in the invariant cyclic cohomology $HC^0_M(\mathcal{A})$. After giving these simple examples it is natural to wonder wether the above notion of $\mathcal{C}$-constant cochain is restrictive enough. Here is a simple consequence of this hypothesis:

Proposition 6. *Let $\mathcal{C} \subset \mathcal{D}$ be unital algebra and $\varphi \in C^n_{\mathcal{C}}$ be a $\mathcal{C}$-constant cochain on $\mathcal{D}$. Then for any invertible element $u \in \mathcal{C}$ one has,*

$$\varphi(u\, a^0 u^{-1}, u\, a^1 u^{-1}, \dots, u\, a^n u^{-1}) = \varphi(a^0, \dots, a^n)\,.$$

Proof. One has $b\varphi(a^0, u, a^1, \dots, a^n) = 0$ so that $\varphi(a^0 u, a^1, \dots, a^n) - \varphi(a^0, ua^1, \dots, a^n) = 0$ since all other terms have u as an argument and hence vanish. Similarly $\varphi(a^0, \dots, a^{j-1}u, a^j, \dots, a^n) = \varphi(a^0, \dots, a^{j-1}, ua^j, \dots, a^n)$ for all $j \in \{1, \dots, n\}$ and $\varphi(ua^0, \dots, a^n) = \varphi(a^0, \dots, a^n u)$. Applying these equalities yields the statement. □

Let us now consider the more sophisticated cochains which appeared throughout and show how to extend them to $\mathcal{C}$-constant cochains on $\mathcal{D} = \mathcal{A} \rtimes \mathcal{C}$ for suitable algebra $\mathcal{C}$ describing the quantum group $\mathrm{SU}_q(2)$.

We first note that the action of the envelopping algebra $\mathcal{U} = U_q(\mathrm{SL}(2))$ on $\mathcal{A}$ extends to an action on the algebra of pseudo-differential operators. First it extends to $\mathcal{B}$ with the following action on the generators $\alpha_\pm$, $\alpha^*_\pm$, $\beta_\pm$, $\beta^*_\pm$,

$$k(\alpha_\pm) = q^{-1/2}\alpha_\pm\,,\ k(\beta_\pm) = q^{-1/2}\beta_\pm\,, \tag{281}$$

and

$$e(\alpha_\pm) = q\beta^*_\mp\,,\ e(\beta_\pm) = -\alpha_\mp\,,\ e(\alpha^*_\pm) = 0\,,\ e(\beta^*_\pm) = 0\,. \tag{282}$$

Moreover $\mathcal{U}$ acts through the trivial representation on D, $|D|$ and F.

In fact it is important to describe the action of $\mathcal{U}$ on arbitrary pseudo-differential operators by a closed formula and this is achieved by,

Proposition 7. *The action of the generators k, e, f of $\mathcal{U}$ on pseudo-differential operators P is given by,* a) $k(P) = kPk^{-1}$, b) $e(P) = ePk^{-1} - qk^{-1}Pe$, c) $f(P) = fPk^{-1} - q^{-1}k^{-1}Pf$.

Proof. These formulas just describe the tensor product of the action of $\mathcal{U}$ in $\mathcal{H}$ by the contragredient representation, since the antipode S in $\mathcal{U}$ fulfills

$$S(k) = k^{-1}\,,\ S(e) = -qe\,,\ S(f) = -q^{-1}f\,. \tag{283}$$

One checks directly that they agree with (281) and (282) on the generators $\alpha_\pm, \ldots, \beta^*_\pm$ as well as on D, $|D|$ and F. Thus we are just using the natural implementation of the action of $\mathcal{U}$ which extends this action to operators. $\Box$

The only technical difficulty is that the generators of $\mathcal{U}$ are unbounded operators in $\mathcal{H}$ so that to extend cochains to $\mathcal{A} \bowtie \mathcal{U}$ requires a little more work. In fact the only needed extension is for the residue,

$$⨍ P = \mathrm{Res}_{s=0} \mathrm{Trace}\,(P\,|D|^{-s})\,. \tag{284}$$

Using formula (θ) we can reexpress (284) as follows,

$$⨍ P = \frac{1}{2} \text{ coefficient of } \log t^{-1} \text{ in } \mathrm{Trace}\,(Pe^{-tD^2})\,. \tag{285}$$

Thus more precisely we let $\theta_P(t) = \mathrm{Trace}\,(Pe^{-tD^2})$ and assume that it has an asymptotic expansion for $t \to 0$ of the form

$$\theta_P(t) \sim \sum a_\alpha\, t^{-\alpha} + \lambda \log t^{-1} + a_0 + \cdots\,, \tag{286}$$

then the equality between (284) and (285) holds, both formulas giving $\lambda/2$. In our context we could use (285) above instead of (284) since we always controlled the size of $\zeta_b(s)$ on vertical strips to perform the inverse Mellin transform.

Let now L be an arbitrary extension of the linear form on function $f \in C^\infty$ $(]0,\infty[)$ which satisfies,

$$L(f) = \frac{1}{2} \text{ coefficient of } \log t^{-1} \text{ if } f \text{ admits} \tag{287}$$

an asymptotic expansion (286).

We then extend the definition (284) by,

$$⨍_L P = L(\theta_P(t))\,. \tag{288}$$

With these notations we then have,

Proposition 8. *Let $(k_1, \ldots, k_n)$ be a multi-index, then the formula*

$$\widetilde{\psi}(a^0, \ldots, a^n) = ⨍_L a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)} |D|^{-n-|k|}$$

where $T^{(k)} = \delta^k(T)$, defines a $\mathcal{U}$-constant extension of the restriction ψ of $\widetilde{\psi}$ to $\mathcal{A}$ to the algebra $\mathcal{D} = \mathcal{A} \bowtie \mathcal{U}$.

Proof. In computing $b\widetilde{\psi}$ one uses the equality

$$\begin{aligned}\delta^k([D,a\,b]) &= \delta^k([D,a])\,b + a\,\delta^k([D,b]) + \sum_{j=0}^{k-1} C_k^j\,\delta^j([D,a])\,\delta^{k-j}(b) \\ &+ \sum_{\ell=1}^{k} \delta^\ell(a)\,\delta^{k-\ell}([D,b])\,. \end{aligned} \tag{289}$$

Thus in $b\,\widetilde{\psi}(a_0,\ldots,a_{n+1})$ the only term which does not involve a derivative of a is of the form,

$$⨍ a_{n+1}\,T|D|^{-n-|k|} - ⨍ T|D|^{-n-|k|}\,a_{n+1}\,. \tag{290}$$

This shows that $b\,\widetilde{\psi}$ vanishes if any of the $a_j \in \mathcal{U}$ for $j=1,\ldots,n$. For $j=n+1$, i.e. for $a_{n+1}=v\in\mathcal{U}$ one has the term (290) but since v commutes with D one has,

$$\theta_{vT} = \theta_{Tv}\,, \tag{291}$$

and one gets the desired result. □

This proposition shows the richness of the space of $\mathcal{U}$-constant cochains, but it does not address the more delicate issue of computing the cyclic cohomology $HC^*_{\mathcal{U}}\,(\mathcal{A})$. A much more careful choice of L would be necessary if one wanted to lift cocycles to cocycles.

We shall now show that $HC^*_{\mathcal{U}}(\mathcal{A})$ which obviously maps to the ordinary cyclic theory $HC^*(\mathcal{A})$,

$$HC^*_{\mathcal{U}}(\mathcal{A}) \xrightarrow{\rho} HC^*(\mathcal{A})\,, \tag{292}$$

also maps in fact to the "twisted" cyclic cohomology $HC^*_{\mathrm{inv}}(\mathcal{A},\,\theta)$ proposed in [21], where θ is the inner automorphism implemented by k^2. This will allow to put the latter proposal in the correct perspective. Indeed the drawback of this simple variation on ([6]) is that it lacks the relation to K-theory which is the back-bone of cyclic cohomology. This was a good reason to refrain from developping such a "twisted" form of the general theory in spite of its previous appearance in ([11] cf. equation 2.28 p.14) and of its merit which is to connect with the various "differential calculi" on quantum groups ([23],[24]). However the next proposition shows that it would be very interesting to use it as a "detector" of classes in $HC^*_{\mathcal{U}}(\mathcal{A})$.

To see what happens, let us start with a $\mathcal{U}$-constant 0-dimensional cochain ψ on $\mathcal{A}\rtimes\mathcal{U}$ and get an analogue of the group invariance provided by proposition 6. One has of course $\psi(kak^{-1})=\psi(a)$ but this is not much. We would like a similar statement for the other generator e of $U_q(SL(2)$. Now by proposition 7 one has $e(a) = eak^{-1} - qk^{-1}ae$ so that $e(a)k^2 = eak - qk^{-1}aek^2$. But $\mathcal{U}$ is in the centraliser of ψ by proposition 6 and thus,

$$\psi(eak) = \psi(kea)\,,\ \psi(k^{-1}aek^2) = \psi(eka)\,,$$

hence $\psi(e(a)\,k^2) = 0$. One gets in general,

$$\psi(h(a)k^2) = \varepsilon(h)\,\psi(a\,k^2) \tag{293}$$

which is the usual invariance of a linear form. More generally one has,

Proposition 9. *The equality* $\rho_\theta(\psi)(a_0, \ldots, a_n) = \psi(a_0, \ldots, a_n k^2)$ *defines a morphism,*

$$HC^*_{\mathcal{U}}(\mathcal{A}) \xrightarrow{\rho_\theta} HC^*_{\rm inv}(\mathcal{A},\, \theta)\,.$$

where θ *is the inner automorphism implemented by* k^2.

We have seen above (276),(279) that the basic cohomology classes in the ordinary cyclic theory $HC^*(\mathcal{A})$ of $\mathcal{A}$ lift to actual cocycles in $HC^*_M(\mathcal{A})$ where M is the von-Neumann algebra bicommutant of $\mathcal{U}$. It is however not clear that they lift to $HC^*_{\mathcal{U}}(\mathcal{A})$ since the generators of $\mathcal{U}$ are unbounded operators. We can however insure that such liftings exist in the entire cyclic cohomology ([12])([19]) since the θ-summability of the spectral triple continues to hold for the algebra $\mathcal{A} \rtimes \mathcal{U}$. This point is not unrelated to the attempt by Goswami in ([18]).
What we have shown here is that the local formulas work perfectly well in the context of quantum groups, and that the framework of NCG needs no change whatsoever, at least as far as $SU_q(2)$ is concerned. The only notion that requires more work is that of invariance in the q-group context.
Finally the above notion of invariant cyclic cohomology is complementary to the theory developped in ([10],[11]). In the latter the Hopf action is used to construct ordinary cyclic cocycles from twisted-traces. In the q-group situation, cocycles thus constructed from the right translations should be left-invariant in the above sense.

10 Appendix

We have not defined carefully the smooth algebras C^∞ involved in section 6. A careful definition can however be deduced from their structure and the exact sequence involving $OP^{-\infty}$ and the symbol map provided by lemma 1. What really matters is that the obtained algebras are stable under holomorphic functional calculus (h.f.c.) and we shall now provide the technical lemma which allows to check this point.
Let $(B, \mathcal{H}, D)$ be a spectral triple. As above we say that an operator P in $\mathcal{H}$ is of order α when,

$$|D|^{-\alpha}\, P \in \bigcap_{n=1}^{\infty} \mathrm{Dom}\,\delta^n \tag{294}$$

where δ is the unbounded derivation given by,

$$\delta(T) = |D| \, T - T \, |D| \,. \tag{295}$$

Thus $OP^0 = \bigcap_{n=1}^{\infty} \operatorname{Dom} \delta^n$ is the algebra of operators of order 0 and $OP^{-\infty}$ is a two sided ideal in OP^0.
Let $\rho : B \to C$ a morphism of C^*-algebras, $\mathcal{C} \subset C$ be a subalgebra stable under h. f. c. and $\lambda : \mathcal{C} \to \mathcal{L}(\mathcal{H})$ be a linear map such that $\lambda(1) = 1$ and,

$$\begin{aligned} &\lambda(c) \in OP^0, \ \forall c \in \mathcal{C} \\ &\lambda(a\, b) - \lambda(a)\lambda(b) \in OP^{-\infty}, \quad \forall a, b \in \mathcal{C} \,. \end{aligned} \tag{296}$$

We then have the following,

Lemma 2. *Let $\mathcal{B} = \{x \in B \, ; \, x \in OP^0, \, \rho(x) \in \mathcal{C}, \, x - \lambda(\rho(x)) \in OP^{-\infty}\}$. Then $\mathcal{B} \subset B$ is a subalgebra stable under holomorphic functional calculus.*

Proof. Let $x \in \mathcal{B}$ be invertible in B, let us show that $x^{-1} \in \mathcal{B}$. Let $a = \rho(x)$, then since $\mathcal{C}$ is stable under h.f.c. the inverse $b = \rho(x^{-1})$ of a belongs to $\mathcal{C}$. Also since $x \in OP^0$ we have $x^{-1} \in OP^0$. Let us show that $x^{-1} - \lambda(b) \in OP^{-\infty}$. Since $a\, b = 1$ one has by (296), $\lambda(a)\lambda(b) - 1 \in OP^{-\infty}$. But $x - \lambda(a) \in OP^{-\infty}$ and $OP^{-\infty}$ is a two-sided ideal in OP^0, thus multiplying $x - \lambda(a)$ by $\lambda(b)$ on the right, we get $x\, \lambda(b) - 1 \in OP^{-\infty}$. Finally since $x^{-1} \in OP^0$ we get, multiplying $x\, \lambda(b) - 1$ on the left by x^{-1} that $\lambda(b) - x^{-1} \in OP^{-\infty}$.

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