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The Action Functional in Non-Commutative Geometry

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Abstract. We establish the equality between the restriction of the Adler-Manin-Wodzicki residue or non-commutative residue to pseudodifferential operators of order $-n$ on an n-dimensional compact manifold M, with the trace which J. Dixmier constructed on the Macaev ideal. We then use the latter trace to recover the Yang Mills interaction in the context of non-commutative differential geometry.

Introduction

The non-commutative residue was discovered in the special case of one dimensional symbols by Adler [1] and Manin [8] in the context of completely integrable systems. In a quite remarkable work [13], Wodzicki proved that it could still be defined in arbitrary dimension and gave the only non-trivial trace, noted Res, for the algebra of pseudodifferential operators of arbitrary order. Given such an operator P on the manifold M, $\operatorname{Res} P$ is the coefficient of $\operatorname{Log} t$ in the asymptotic expansion of $\operatorname{Trace}(P e^{-t\Delta})$, where Δ is a Laplacian. Equivalently it is the residue at $s=0$ of the ζ function $\zeta(s)=\operatorname{Trace}(P\Delta^{-s})$. It is *not* the usual regularisation $\zeta(0)$ of the trace, and it vanishes on any P of order strictly less than $-\dim M$, and on any differential operator. In general this trace: Res, has no positivity property, i.e. one does not have $\operatorname{Res}(P^*P)\geqq 0$. However its restriction to operators of order $-n$, $n=\dim M$ is positive. This restriction of Res to pseudodifferential operators of order $-n$ was discovered and studied by Guillemin [14]. Even though it is easier to handle than the general residue, it will be of great help for our purpose which is to show how conformal geometry fits with [3], the case of Riemannian geometry being treated in [5].

Our first result is the equality between Res and a trace on the dual Macaev ideal, introduced by Dixmier in [6] in order to show that the von Neumann algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators in Hilbert space possessed non-trivial tracial weights. I am grateful to J. Dixmier for explaining his result to me and to D. Voiculescu for helpful conversations on the subject of Macaev ideals. Thus we recall that, given a Hilbert space $\mathscr{H}$, the Macaev ideal $\mathscr{L}^{\omega}(\mathscr{H})$ is the ideal of compact operators T, whose characteristic values satisfy: [7]

$$\sum_1^\infty \frac{1}{n}\mu_n(T) < \infty .$$

It contains all the Schatten classes $\mathcal{L}^p(\mathcal{H})$ for finite p, and the dual ideal, which we denote $\mathcal{L}^{1+}$ consists of all compact operators T, whose characteristic values satisfy:

$$\operatorname*{Sup}_{N>1} \frac{1}{\operatorname{Log} N} \sum_1^N \mu_n(T) < \infty .$$

Gifted with the obvious norm it is a non-separable Banach space containing strictly the ideal $\mathcal{L}^1$ as well as the closure of finite rank operators (thus $\mathcal{L}^1$ is *not* norm dense in $\mathcal{L}^{1+}$ for the natural norm of the latter).

Now in [6], J. Dixmier showed that for any mean ω on the amenable group of upper triangular two by two matrices, one gets a trace on $\mathcal{L}^{1+}$, given by the formula:

$$\operatorname{Tr}_\omega(T) = \lim_\omega \frac{1}{\operatorname{Log} N} \sum_1^N \lambda_n(T)$$

when T is a positive operator, $T \in \mathcal{L}^{1+}$, with eigenvalues $\lambda_n(T)$ in decreasing order, and $\lim_\omega$ is the linear form on bounded sequences defined in [6] using ω.

We shall prove in Sect. 1 that when T is pseudodifferential of order $-\dim(M)$, the value of $\operatorname{Tr}_\omega(T)$ does not depend upon ω and is equal to $\operatorname{Res}(T)$. In Sect. 2 we shall apply the above result to show how one can deduce ordinary differential forms and the natural conformal invariant norm on them from the quantized forms which we introduced in [3]. The key point is that we do not need to take a "classical limit" to achieve this goal but only to use the Dixmier trace appropriately. In particular we obtain a simple formula for the conformal structure in terms of the operator F, $F^2 = 1$, given by the polar decomposition of the Dirac operator.

In Sects. 3 and 4 after discussing the analogue of the Yang Mills action in the context of non-commutative differential geometry and showing, as expected, that 4 is the critical dimension, we exploit the above construction to show that if $d = 4$ the leading divergency of the action is the usual local Yang Mills action. The latter result was announced on several occasions.

1. The Main Equality

Theorem 1. *Let M be a compact n-dimensional manifold, E a complex vector bundle on M, and P a pseudodifferential operator of order $-n$ acting on sections of E. Then the corresponding operator P in $\mathcal{H} = L^2(M, E)$ belongs to the Macaev ideal $\mathcal{L}^{1+}(\mathcal{H})$ and one has:*

$$\operatorname{Trace}_\omega(P) = \frac{1}{n}\operatorname{Res}(P)$$

for any invariant mean ω.

Note first that both $\mathcal{L}^{1+}(\mathcal{H})$ and $\operatorname{Trace}_\omega$ are invariant under similarities $T.\ T^{-1}$ with T and T^{-1} bounded, so that the choice of inner product in the space of L^2 sections of E is irrelevant.

Proof. Since $\mathcal{L}^{1+}(\mathcal{H})$ contains $\mathcal{L}^1(\mathcal{H})$, and any element of the latter is in the kernel of Tr_ω, it follows that we can neglect smoothing operators and we just need to prove the statements locally. Thus to show that $P \in \mathcal{L}^{1+}(\mathcal{H})$ we may assume that M is the standard n torus $\mathbb{T}^n$ and E the trivial line bundle. Then $P = T(1+\Delta)^{-n/2}$, where T is bounded and Δ is the Laplacian of the (flat) torus. Thus as $\mathcal{L}^{1+}$ is an ideal it is enough to check that $(1+\Delta)^{-n/2} \in \mathcal{L}^{1+}$, which is obvious. In fact the characteristic values of $(1+\Delta)^{-n/2}$ are the $(1+l^2)^{-n/2}$, where the l's are the lengths of elements in the lattice $\Gamma = \mathbb{Z}^n$. Thus we see that the limit of $\frac{1}{\mathrm{Log}N} \sum_1^N \lambda_j$ when N goes to ∞, does exist for this operator so that, for any ω:

$$\mathrm{Trace}_\omega((1+\Delta)^{-n/2}) = \frac{1}{n} \int_{S^{n-1}} d\sigma = \frac{1}{n} 2\pi^{\frac{n-1}{2}} \Big/ \Gamma\left(\frac{n-1}{2}\right).$$

Let us now prove the main equality. We may assume that M is the standard n-sphere S^n. Since Trace_ω is positive and vanishes on $\mathcal{L}^1(\mathcal{H})$ it defines a positive linear form on *symbols of order* $-n$, because it only depends upon the principal symbol $\sigma_{-n}(P)$ for P of order $-n$. Since a positive distribution is a measure, we get a measure on the unit sphere cotangent bundle of S^n. But as Tr_ω is a trace, the latter measure is invariant under the action of any isometry of S^n, and hence is proportional to the volume form on $(T^*S^n)_1 = \{(x,\xi) \in T^*S^n;\ \|\xi\| = 1\}$. By the above computation the constant of proportionality is $\frac{1}{n}(2\pi)^{-n}$, thus:

$$\mathrm{Trace}_\omega(P) = \frac{1}{n}(2\pi)^{-n} \int_{(T^*S^n)_1} \sigma_{-n}(P)\, dv$$

for any P of order $-n$ and any ω. As the right-hand side is the formula for $\frac{1}{n}\mathrm{Res}(P)$, we get the conclusion. □

Corollary 2. *All the traces* Tr_ω *agree on pseudodifferential operators of order* $-\dim M$, *on a manifold* M.

One can then conclude that suitable averages of the sequence $\frac{1}{\mathrm{Log}N} \sum_1^N \lambda_j(P)$ do converge, when $N \to \infty$, to this common value.

2. Conformal Geometry

Let M be a compact Riemannian manifold of dimension n, and $A^1 = C^\infty(M, T^*M)$ be the space of smooth 1-forms on M. There is a natural norm on A^1 which depends only upon the *conformal* structure of M. If $\dim M = 2$, it is the ordinary Dirichlet integral: $\int \|\omega\|^2 dv = \int \omega \wedge *\omega$. If $\dim M = n$, it is the L^n norm, given by the (n^{th} root of) following integral:

$$(\|\omega\|)^n = \int \|\omega(x)\|^n d^n x.$$

In [3] we introduced (assuming that M is Spin^c) the quantized differential forms on M, obtained as operators of the form $\sum a\,db$; $a, b \in C^\infty(M)$, in the Hilbert space $\mathcal{H}$ of L^2 spinors on M. Here db is given by the commutator $i[F, b]$, where the operator F, $F^2 = 1$, is the sign $D|D|^{-1}$ of the Dirac operator. (We can ignore the non-invertibility of D, since it only modifies F by a finite rank operator.)

The next result shows how to pass from quantized 1-forms to ordinary forms, not by a classical limit, but by a direct application of the Dixmier trace.

Theorem 3. *Let M be a* Spin^c *Riemannian manifold of dimension $n>1$, $\mathscr{H}=L^2(M,S)$ the Hilbert space of L^2 spinors, $F=D|D|^{-1}$ the sign of the Dirac operator. Let $\mathscr{A}=C^\infty(M)$ be the algebra of smooth functions on M and $\Omega^1=\{\Sigma a[F,b];\, a,b\in\mathscr{A}\}$ be the $\mathscr{A}$-bimodule of quantized forms of degree 1.*

1) *For any $\alpha\in\Omega^1$ one has $|\alpha|^n\in\mathscr{L}^{1+}(\mathscr{H})$.*

2) *There exists a unique bimodule linear map $\Omega^1\xrightarrow{c}A^1$ such that $c(i[F,a])=da\ \forall a\in C^\infty(M)$. This map is surjective and the image of the self adjoint elements of Ω^1 are the real forms.*

3) *For any $\alpha=\alpha^*\in\Omega^1$ one has* $\mathrm{Trace}_\omega(|\alpha|^n)=\lambda_n\int\|c(\alpha)\|^n$ *with* $\lambda_n=2(2\pi)^{-n/2}\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)^{-1}\Gamma(n+1)^{-1}$.

Proof. 1. By construction α is a pseudodifferential operator of order -1, so that $|\alpha|^n$ is also a pseudodifferential operator and is of order $-n$. The conclusion follows from Theorem 1.

2. For $x\in M$ let $C_x=\mathrm{Cliff}_{\mathbb{C}}(T_x^*M)$ be the complexified Clifford algebra of the cotangent space T_x^*M of M at x. One has $C_x=\mathrm{End}(S_x)$, where S is the Spinor bundle. For each $\xi\in T_x^*M$ we let $\gamma(\xi)\in C_x$ be the corresponding γ matrix, $\gamma(\xi)=\gamma(\xi)^*$, $\gamma(\xi)^2=\|\xi\|^2$, and we extend γ to a linear map of $T_{x,\mathbb{C}}^*(M)$ to C_x. Given $a\in\mathscr{A}=C^\infty(M)$, the symbol of order -1 of $[F,a]$ is the Poisson bracket $\{\sigma,a\}$, where $\sigma(x,\xi)=\gamma(\xi)/\|\xi\|$, and thus its restriction to the unit sphere is the transverse part $\varrho(x,\xi)=\gamma(da-\langle da,\xi\rangle\xi)$ of $\gamma(da)$. It is a homogeneous function of degree -1 on T_x^*M with values in C_x. Now provided $n>1$, a vector $\eta\in T_x^*$ is uniquely determined by the transverse part $\xi\to\eta-\langle\eta,\xi\rangle\xi$, as a function of $\xi\in S_x^*$, and this still holds for $\eta\in T_{x,\mathbb{C}}^*$. Thus the map c exists and is characterized by the equality:

$$\sigma_{-1}(\alpha)(x,\xi)=\gamma(c(\alpha)(x)-\langle c(\alpha)(x),\xi\rangle\xi)\ \forall(x,\xi)\in S^*M\,.$$

The image of $\sum ai[F,b]\in\Omega^1$ is $\sum adb\in A^1$ so the surjectivity of c is clear. The image of $ai[F,b]+(ai[F,b])^*$ is $adb+(db^*)a^*$ which is a real form, so 2. follows.

3. The absolute value of $\gamma(\eta)$ for $\eta\in T_x^*(M)$ (but not its complexification) is $\|\eta\|\,1$, where 1 is the unit of C_x. Thus by Theorem 1 we have:

$$\mathrm{Trace}_\omega(|\alpha|)^n)=\frac{(2\pi)^{-n}}{n}\int_{S^*M}\|\alpha_x-\langle\alpha_x,\xi\rangle\xi\|^n\,\mathrm{trace}(1)d^nxd^{n-1}\xi\,.$$

Here $\mathrm{trace}(1)=\dim(S_x)=2^{n/2}$. Thus we just need to show that for any $\eta\in\mathbb{R}^n$ one has $\int_{S^{n-1}}\|\eta-\langle\eta,\xi\rangle\xi\|^n(d^{n-1}\xi)=2^{-n/2}\lambda_n\|\eta\|^n$. By homogeneity and invariance under rotations we are reduced to the computation of an integral, which is obviously >0 for $n>1$. □

As an immediate corollary of the theorem we see that the Fredholm module $(\mathscr{H},F)$ allows us to recover both the bimodule of 1-forms A^1 with the ordinary differentiation: $\mathscr{A}\xrightarrow{d}A^1$ (given by $a\to$Class of $i[F,a]$), and also the conformal structure of M since the L^n norm on A^1 uniquely determines it.

Another equivalent way to formulate the result is to consider for each n the ideal $\mathscr{L}^{n+}$, n^{th} root of $\mathscr{L}^{1+}$, in $\mathscr{L}(\mathscr{H})$,

$$\mathscr{L}^{n+}=\left\{T\in\mathscr{L}(\mathscr{H}),\ T\text{ compact, }\operatorname{Sup}_N\left(\frac{1}{\operatorname{Log}N}\sum_1^N\mu_j(T)^n\right)<\infty\right\},$$

and the ideal $\mathscr{L}_0^{n+}$ which is the norm closure, for the norm of $\mathscr{L}^{n+}$, of operators of finite rank (cf. [7]). Then on an n-dimensional manifold M as above the quantized 1-forms are all in $\mathscr{L}^{n+}$, and the ordinary forms are obtained by moding out $\mathscr{L}_0^{n+}\subset\mathscr{L}^{n+}$. The ordinary differential is obtained in the same way from the quantized differential $a\to i[F,a]\in\Omega^1$.

For forms of arbitrary degree there are two more points which we have to clarify before we can handle the Yang Mills action. Given an n-dimensional Euclidean space E, we let Π_E be the homomorphism of the tensor algebra $T(E)$ in $C^\infty(S_E,\operatorname{Cliff}(E))$, (the algebra of smooth maps from the unit sphere $S_E=\{\xi\in E,\ \|\xi\|=1\}$ to the Clifford algebra of E) obtained from the linear map $\eta\to\varrho(\eta)$, $\varrho(\eta)(\xi)=\gamma(\eta-<\eta,\xi>\xi)\ \forall\xi\in S_E$.

We let $J(E)$ be the kernel of Π_E.

Lemma 4. *With the notations of Theorem 3, let Ω^k be the the $\mathscr{A}$-bimodule of quantized forms of degree k.*

1. *For $1\leq k\leq n$ one has $\Omega^k\subset\mathscr{L}^{n/k+}(\mathscr{H})$ and the direct sum $\bigoplus_0^n\Omega_0^k$, with $\Omega_0^k=\mathscr{L}^{n/k+}\cap\Omega^k$ is a two sided ideal in the algebra $\bigoplus_0^n\Omega^k=\Omega^*$.*

2. *The principal symbol map gives a canonical isomorphism c of graded algebras, from Ω^*/Ω_0^* to the graded algebra of smooth sections of the vector bundle $\bigoplus_0^n E_k$, where E_k is obtained from the cotangent bundle by applying the functor:*

$$E\to T^k(E)/J(E)\cap T^k(E)=f_k(E).$$

Proof. 1. Any element of Ω^k is a pseudodifferential operator P of order $-k$; thus $|P|^{n/k}$ is of order $-n$ and Theorem 1 applies. The Holder inequality also holds for the ideals $\mathscr{L}^{p+}$ and shows that $\mathscr{L}^{p_1+}\times\mathscr{L}^{p_2+}\subset\mathscr{L}^{p_3+}$, $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$ and also that $\mathscr{L}_0^{p_1+}\times\mathscr{L}^{p_2+}\subset\mathscr{L}_0^{p_3+}$, $\mathscr{L}^{p_1+}\times\mathscr{L}_0^{p_2+}\subset\mathscr{L}_0^{p_3+}$ (cf. [7]).

2. First, by Theorem 1, an element P of Ω^k belongs to $\mathscr{L}_0^{n/k+}$ if and only if its principal symbol vanishes. (If it does then the operator is of order $<-k$ and hence even belongs to $\mathscr{L}^{n/k}$; if it does not then the Dixmier trace of $|P|^{n/k}$ does not vanish.) The quotient Ω^k/Ω_0^k is a commutative bimodule over $\mathscr{A}=C^\infty(M)$, and since any element of Ω^k is a finite sum of products of k elements of Ω^1, the symbols $\sigma_{-k}(P)$, $P\in\Omega^k$ are exactly the smooth sections of $f_k(T^*M)$. □

For our purpose we only need to determine f_1 and f_2. For $n>1$ we have seen that $f_1(E)=E$. For $n>2$ let us show that $J(E)\cap T^2(E)=\{0\}$, i.e. that the map Π_E is injective on tensors of rank 2. Since $J(E)$ is invariant under the action of the orthogonal group $O(E)$, it is enough to check that Π_E is non-zero on the three irreducible subspaces of $T^2(E)$, namely a) antisymmetric tensors b) symmetric traceless tensors c) the inner product (viewed as a symmetric tensor). Since $n>2$ we can take $\eta_1, \eta_2 \in E$ linearly independent, and ξ, $\|\xi\|=1$, orthogonal to both, to get that $\Pi_E(\eta_1 \otimes \eta_2 - \eta_2 \otimes \eta_1) \neq 0$. The image by Π_E of the symmetric tensor $\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1 (\eta_i \in E)$ is the scalar valued function on $S_E : \Pi_E(\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1)(\xi) = \langle \eta_1, \eta_2 \rangle - \langle \eta_1, \xi \rangle \langle \eta_2, \xi \rangle$. This is enough to show that Π_E is non-zero and hence injective on tensors of type a) b) or c). Thus we get:

Lemma 5. *If* $\dim E > 2$, $f_2(E) = T^2(E)$.

The next point that we need to clarify is that even though $f = \Omega^*/\Omega_0^*$ is a graded algebra of tensors on the manifold M, and c is a homomorphism from the graded algebra Ω^* to Ω^*/Ω_0^*, we do not have a natural differential in f. The point is that the ideal Ω_0^* is not in general stable under the map:

$$\alpha \in \Omega^k \to d\alpha = i(F\alpha - (-1)^k \alpha F) \in \Omega^{k+1}.$$

However since $d^2 = 0$, this is easily cured:

Lemma 6. 1. *The direct sum* $\Omega_{00}^* = \bigoplus_0^n \Omega_{00}^k$ *with* $\Omega_{00}^k = \{\alpha \in \Omega_0^k, d\alpha \in \Omega_0^{k+1}\}$ *is a graded differential two sided ideal in the graded differential algebra* Ω^*.

2. *The map* $\tilde{c}$, $\tilde{c}(\alpha) = (c(\alpha), c(d\alpha))$ *is a linear injection of the quotient* Ω^k/Ω_{00}^k *in the space of sections of the bundle* $f_k(T^*) \oplus f_{k+1}(T^*)$.

Proof. 1. We just have to check that it is a two sided ideal, which follows from Lemma 4 1) and the equality $d(\alpha_1 \alpha_2) = (d\alpha_1)\alpha_2 + (-1)^{\partial_1} \alpha_1 d\alpha_2$.

2. Apply Lemma 4 2). □

Assuming $n > 2$ let us determine the image $\tilde{c}(\Omega^1)$, i.e. the pairs $(c(\alpha), c(d\alpha))$ when α varies in Ω^1.

Lemma 7. *For* $n > 2$, $\tilde{c}(\Omega^1)$ *consists of all smooth tensors* (ω, β), *where* ω *is of rank* 1, β *of rank* 2 *and one has:*

$$A\beta = d\omega,$$

where A *is the projection on antisymmetric tensors of rank* 2.

Proof. It is enough to check the equation for the pair $\omega = c(\alpha)$, $\beta = c(d\alpha)$ with $\alpha = a\,db$; $a, b \in C^\infty(M)$. Then by Theorem 3 2), $c(\alpha)$ is the 1-form $a\,db$ and since $d\alpha = da\,db$, we see that $A\beta$ is the antisymmetric tensor $\frac{1}{2}(da \otimes db - db \otimes da)$, thus the equality $A\beta = d\omega$. It remains to show that $\tilde{c}(\Omega^1)$ contains all the smooth symmetric tensors of rank 2. Now with $\alpha = a\,db$ as above and $x \in C^\infty(M)$ we have $c(x\alpha - \alpha x) = 0$ and $c(d(x\alpha - \alpha x)) = c((dx)\alpha + \alpha(dx))$. Thus $\tilde{c}(x\alpha - \alpha x)$ is the smooth symmetric two tensor $(dx)\alpha + \alpha(dx)$. As every smooth symmetric two tensor is a finite sum of such terms we get the conclusion. □

3. The Action Functional in Non-Commutative Differential Geometry

We begin this section by a very simple example, the case of the circle S^1, where we show that using our quantized differential forms, the quantized flat connections correspond exactly to the Grassmannian which plays a fundamental role in the theory of totally integrable systems [9].

Thus we let $\mathscr{A} = C^\infty(S^1)$ be the algebra of smooth functions on S^1 and let $(\mathscr{H}, F)$ be the Fredholm module over $\mathscr{A}$ given by $\mathscr{H} = L^2(S^1)$ and $F = 2P - 1$, where P is the Toeplitz projection. In other words the operator F multiplies the n^{th} Fourier component of $\xi \in L^2(S^1)$ by 1 if $n \geqq 0$ and -1 otherwise.

Lemma 8. *The space $\Omega^1 = \{\sum a[F, b];\ a, b \in \mathscr{A}\}$ of 1-forms is dense in the space $\mathscr{L}^2(\mathscr{H})$ of Hilbert Schmidt operators.*

Proof. Let $u \in \mathscr{A}$ be the function $u(\theta) = \exp i\theta$, $\theta \in S^1$. The operator $\frac{1}{2}u^{-1}[F, u]$ is the rank one projection on the subspace $\mathbb{C}e_0$, where $(e_n)_{n\in\mathbb{Z}}$ is the canonical basis of $\mathscr{H} = L^2(S^1)$, $e_n(\theta) = \exp(in\theta)$, $\forall \theta \in S^1$. Thus the quantized forms $\omega_{n,m} = u^n(\frac{1}{2}u^{-1}[F, u])\, u^m$ form the natural orthonormal basis of $\mathscr{L}^2(\mathscr{H})$. □

We cannot entirely justify the choice of the Hilbert Schmidt norm in the above lemma, since it happens in dimension 1, that 1-forms are traceable. (As we saw above, by Theorem 1, it is not true that 1-forms belong to $\mathscr{L}^n$ for an n-dimensional manifold, $n > 1$.) The only sensible justification is that the definition of the character of the Fredholm module only requires that 1-forms be of Hilbert Schmidt class, and is continuous in this norm (cf. [3]). Next consider the trivial line bundle, with fiber $\mathbb{C}$, on S^1, or equivalently the finite projective module $\mathscr{E} = C^\infty(S^1)$ over $\mathscr{A}$. Then as in [3, Definition 18, p. 110] a connection ∇ on $\mathscr{E}$ is given by a linear map $\nabla: \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{A}} \Omega^1$ such that

$$\nabla(\xi \cdot x) = (\nabla\xi)x + \xi \otimes dx,$$

where here $dx = i[F, x]$, according to our definition of the quantized differential. We endow the above line bundle with its obvious metric, i.e. we view $\mathscr{E}$ as a C^* module over $\mathscr{A}$, with $\langle \xi, \eta \rangle(\theta) = \bar{\xi}(\theta)\eta(\theta)$, $\forall \theta \in S^1$, $\forall \xi, \eta \in \mathscr{E}$. Obviously a connection on $\mathscr{E}$ is specified by the 1-form $\alpha = \nabla 1$, and the latter is an arbitrary element of Ω^1. Moreover the connection associated to α is compatible with the metric (cf. [4]), (i.e. such that $\langle \nabla\xi, \eta\rangle + \langle \xi, \nabla\eta\rangle = d\langle \xi, \eta\rangle$ $\forall \xi, \eta \in \mathscr{E}$) iff $\alpha + \alpha^* = 0$.

We thus get the elementary but significant result:

Theorem 9. *The map $\nabla \to \frac{1}{2}(1 + F) - \frac{1}{2}i\nabla(1)$ is a one-to-one bijection from flat compatible and square integrable connections on $\mathscr{E}$ with the restricted Grassmannian. It is equivariant with respect to the natural action of $C^\infty(S^1, U(1))$.*

Proof. First ∇ is characterized by $\alpha = \nabla(1)$ and is compatible iff $\alpha^* = -\alpha$, and square integrable iff $\alpha \in \mathscr{L}^2$; thus by Lemma 8, without the flatness condition the allowed α's are the skew adjoint elements of $\mathscr{L}^2(\mathscr{H})$. Now (cf. [9]) the restricted Grassmannian consists exactly of the idempotents Q, $Q = Q^*$ such that $Q - P \in \mathscr{L}^2$. Thus if we set $Q = \frac{1}{2}(1 + F) - \frac{1}{2}i\alpha$, we just need to check that $Q^2 = Q$ iff ∇_α is flat, i.e. iff one has $i(F\alpha + \alpha F) + \alpha^2 = 0$, which is obvious. The unitary group $\mathscr{U} = C^\infty(S^1, U(1))$ of $\mathrm{End}_{\mathscr{A}}(\mathscr{E})$ acts by gauge transformations on compatible connections (cf. [4]) with $\gamma_u(\nabla) = u\nabla u^{-1}$ for $u \in \mathscr{U}$, or equivalently $\gamma_u(\alpha) = ui[F, u^{-1}] + u\alpha u^{-1}$. Thus the corresponding Q_α is replaced by $uQ_\alpha u^{-1}$. □

A similar statement holds for the bundle with fiber $\mathbb{C}^n$, with $\mathscr{U}$ replaced by $C^\infty(S^1, U(n))$.

In relation with [2] and [12] we also want to point out that on the space of all compatible connections (i.e. all $\alpha = -\alpha^*$ in $\mathscr{L}^2(\mathscr{H})$) one has a natural Chern-

Simons action given by

$$I(\alpha)=\int(\alpha d\alpha+\tfrac{2}{3}\alpha^3),$$

where the integral is the *trace* and as usual $d\alpha$ is the graded commutator $d\alpha=i(F\alpha+\alpha F)$.

But let us now pass to the analogue of the Yang Mills action. The set up is, as in [3] and as above, fixed by a_* algebra $\mathcal{A}$ and a Fredholm module $(\mathcal{H},F)$ over $\mathcal{A}$ which is p-summable, i.e. $[F,x]\in\mathcal{L}^p(\mathcal{H})$ for some finite p, which as explained in [3] has to do with dimension. We are also given the analogue of a Hermitian bundle, i.e. a finite projective module $\mathcal{E}$ over $\mathcal{A}$, with an $\mathcal{A}$ valued inner product (cf. [4]). This latter data can be ignored for a first reading and specialized to $\mathcal{E}=\mathcal{A}$ with $\langle a,b\rangle=a^*b\in\mathcal{A}$.

Then using the differential algebra of quantized differential forms, $\Omega^k=\{\sum a^0da^1\dots da^k; a^j\in\mathcal{A}, da=i[F,a]\}$ (cf. [3]) we get the notions of connection, compatible connection, curvature relative to $\mathcal{E}$. For $\mathcal{E}=\mathcal{A}$ a connection is just an element α of Ω^1, it is compatible iff $\alpha^*=-\alpha$ and its curvature is $\theta=d\alpha+\alpha^2=i(F\alpha+\alpha F)+\alpha^2$. (cf. [3, p. 110] and [4]). Using [3, Lemma 1, p. 56], we get:

Theorem 10. 1. *The action* $I_+(\alpha)=\|\theta\|^2_{HS}$ *is finite if* $p\leqq 4$.

2. *When* $p\leqq 4$, *the action* I_+ *is a quartic positive function of* α *invariant under the action of the gauge group of second kind*

$$\mathcal{U}=\{u\in\mathrm{End}(\mathcal{E}); uu^*=u^*u=1\}.$$

Proof. For the sake of clarity we take $\mathcal{E}=\mathcal{A}$. By construction $\theta=d\alpha+\alpha^2\in\Omega^2$, and by [3, Lemma 1, p. 56] one has $\Omega^k\subset\mathcal{L}^{p/k}$, so that $\Omega^2\subset\mathcal{L}^{p/2}$. Thus θ is Hilbert Schmidt when $p/2\leqq 2$, i.e. when $p\leqq 4$. If we replace α by $\gamma_u(\alpha)=udu^{-1}+u\alpha u^{-1}$, the curvature θ is replaced by $u\theta u^{-1}$ so that the statement 2. is obvious. □

It is well known that the dimension $n=4$ is the relevant dimension for the classical Yang Mills action since it is only for $n=4$ that it is conformally invariant, but for the action I_+ the situation is slightly different: 1. The action I_+ is *finite* only if the degree of summability p is $\leqq 4$, 2. For a 4-dimensional manifold M, the Fredholm module $(\mathcal{H},F)$ on $C^\infty(M)$ given by Theorem 3 is p summable for any $p=4+\varepsilon$, $\varepsilon>0$ but not for $p=4$. Thus in this case the action I_+ is divergent. However by Lemma 4 one has $\Omega^2\subset\mathcal{L}^{2+}$ so that the divergence of $\|\theta\|^2_{HS}=\mathrm{Trace}(\theta^*\theta)$ is only logarithmic ($\theta^*\theta\in\mathcal{L}^{1+}$) and the principal term (i.e. the coefficient of $\mathrm{Log}\,K$ in terms of a cut off K) is given by the Dixmier trace $\mathrm{Trace}_\omega(\theta^*\theta)$. In the next section we shall fully identify this leading term in I_+ with the classical Yang Mills action.

4. The Leading Term of the Action in 4 Dimensions

Let M be a 4 dimensional compact smooth Riemannian manifold. We assume that M is Spin^c and let $(\mathcal{H},F)$ be the Fredholm module over $\mathcal{A}=C^\infty(M)$, with $\mathcal{H}$ the Hilbert space of L^2 spinors and $F=D|D|^{-1}$, where D is the Dirac operator. We let (Ω^*,d) be the graded differential algebra of quantized forms, and define as in Sect. 3 the notion of compatible connection for a Hermitian vector bundle E over M. This involves the module $\mathscr{E}=C^\infty(M,E)$ (of sections of E) over $\mathscr{A}$ and the $\mathscr{A}$-valued inner product given by the metric of E. By construction (cf. [3]) the curvature θ is an element of $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}\otimes_{\mathscr{A}}\Omega^2)$, but since here Ω^2 acts in the Hilbert space $\mathscr{H}$, we can view θ as an operator in the Hilbert space $\mathscr{E}\otimes_{\mathscr{A}}\mathscr{H}$. The inner product of the latter space is given by (cf. [4]) $\langle\xi\otimes\eta,\ \xi'\otimes\eta'\rangle=\langle\langle\xi,\xi'\rangle\eta,\eta'\rangle$ for $\xi,\xi'\in\mathscr{E}$ and $\eta,\ \eta'\in\mathscr{H}$. In the simple case where $\mathscr{E}$ is the free module $\mathscr{A}^q$ (i.e. E is the trivial bundle with fiber $\mathbb{C}^q$), the connection is given by a matrix $\omega=\omega_{ij}$ of elements of Ω^1, with $i,j\in\{1,\dots,q\}$ and the curvature is the operator in $\mathscr{H}^q$ given by the matrix $d\omega+\omega^2$, with $(d\omega+\omega^2)_{ik}=d(\omega_{ik})+\sum\omega_{ij}\omega_{jk}$. In general if θ is the curvature, $\theta=\nabla^2\in\mathrm{Hom}(\mathscr{E},\mathscr{E}\otimes_{\mathscr{A}}\Omega^2)$, of the connection ∇, there exists elements ξ^i of $\mathscr{E}$, $i\in\{1,\dots,q\}$ and $\theta_{ij}\in\Omega^2$; $i,\ j\in\{1,\dots,q\}$ such that $\theta(\xi)=\Sigma(\xi^i\otimes\theta_{ij})\langle\xi^j,\xi\rangle$. The corresponding operator in $\mathscr{E}\otimes_{\mathscr{A}}\mathscr{H}$ is then such that:

$$\theta(\xi\otimes\eta)=\sum\xi^i\otimes\theta_{ij}(\langle\xi^j,\xi>\eta)\ \forall\xi\in\mathscr{E},\eta\in\mathscr{H}.$$

The compatibility of the connection ∇ with the metric implies that θ is a selfadjoint operator in $\mathscr{E}\otimes_{\mathscr{A}}\mathscr{H}$: If $\mathscr{E}=\mathscr{A}^q$, then the connection given by $\omega=(\omega_{ij})\in M_q(\Omega^1)$ is compatible iff $\omega^*=-\omega$ and the curvature $\theta=d\omega+\omega^2$ is then selfadjoint since for $\alpha\in\Omega^1$ one has $d\alpha^*=-(d\alpha)^*\in\Omega^2$. For the sake of clarity, since we are going to relate our notion of connection with the usual notion we shall use the term q-connection for the former and c-connection for the latter.

Lemma 11. a) *Every q-connection $\nabla:\mathscr{E}\to\mathscr{E}\otimes\Omega^1$ determines uniquely a classical connection ∇_c by composition with the bimodule map $c:\Omega^1\to A^1$ of Theorem 3: $\nabla_c=(1\otimes c)\circ\nabla$.*

b) *Let θ be the curvature of the q-connection ∇, then the curvature θ_c of ∇_c is the antisymmetric part $A\ c(\theta)$ of $c(\theta)$.*

Proof. a) One has $c(a\alpha b)=ac(\alpha)b$ for $a,b\in\mathscr{A},\alpha\in\Omega^1$, so $(1\otimes c)\circ\nabla$ is a linear map of $\mathscr{E}=C^\infty(M,E)$ to $\mathscr{E}\otimes_{\mathscr{A}}A^1=C^\infty(M,E\otimes T^*)$ such that $\nabla_c(\xi a)=(\nabla_c\xi)a+\xi\otimes da$ for any $\xi\in\mathscr{E},a\in\mathscr{A}$.

b) Since the ordinary exterior product of two 1-forms is the antisymmetric part of their tensor product, the answer follows from Lemma 7. □

Corollary 12. *The map $\nabla\to\nabla_c$ maps flat q-connections to ordinary flat connections on $\mathscr{E}$.*

Note that the flatness of the q-connection ∇ means as in Theorem 9 that the operator $F_\nabla=1\otimes F-i\nabla$ in the Hilbert space $\mathscr{E}\otimes_{\mathscr{A}}\mathscr{H}$ satisfies $F_\nabla^2=1$, and hence, in the compatible case, yields an element of a suitable Grassmanian. Here F_∇ is defined by: $F_\nabla(\xi\otimes\eta)=\xi\otimes F\eta-i\sum\xi^j\otimes\omega_j\eta$, with $\nabla\xi=\sum\xi^j\otimes\omega_j\in\mathscr{E}\otimes_{\mathscr{A}}\Omega^1$. One checks that the right-hand side is independent of any choice. Now by Lemma 7 we can associate to every q-connexion a classical tensorial data which is a bit more refined than a classical connexion. Indeed the bimodule $\Omega^1/\Omega^1_{00}=\sum$ is by Lemma 7 isomorphic to the space of smooth tensors $C^\infty(M,T^1\oplus T^2)$ which satisfy the equation $d\omega=A\beta$, and the bimodule structure of $\sum$ is given by: $a(\omega,\beta)=(a\omega,da\otimes\omega+a\beta)$; $(\omega,\beta)a=(\omega a,\beta a-\omega\otimes da)$. By the map $(\omega,\beta)\to(\omega,\beta-d\omega)$, we can identify $\sum$ with the space of all smooth tensors $C^\infty(M,T^1\oplus S^2T^1)$ with the bimodule structure given by:

$$a(\omega,\sigma)=(\omega,\sigma)a=(a\omega,a\sigma+\tfrac{1}{2}(da\otimes\omega+\omega\otimes da))$$

$=(a\omega, a\sigma+da\cdot\omega)$, where $da\cdot\omega$ is the product in the symmetric algebra. Note in particular that the map $(\omega,\sigma)\to\omega$ is an $\mathscr{A}$-bimodule map of $\sum$ to A^1, but that the subspace $\{(\omega,\sigma)\in\sum;\sigma=0\}$ is *not* a submodule of $\sum$.

Lemma 13. 1. *The map $\nabla\to(1\otimes\tilde{c})\circ\nabla$ is a surjection of the space of q-connections on $\mathscr{E}$ to the space $\Gamma_{\mathscr{E}}$ of maps $\chi:\mathscr{E}\to\mathscr{E}\otimes_{\mathscr{A}}\sum$ such that $\chi(\xi a)=\chi(\xi)a+\xi\otimes da\ \forall\xi\in\mathscr{E}$, $a\in\mathscr{A}$.*

2. *The map $(\omega,\sigma)\to\omega$ gives a surjection ϱ of $\Gamma_{\mathscr{E}}$ on the space of classical connections on E, and the fibers of ϱ are affine spaces over the vector space $C^\infty(M,\operatorname{End}E\otimes S^2T^*)$ of smooth 2-tensors.*

Proof. 1. To prove 1. one can assume, as in [3, Proposition 19], that $\mathscr{E}=\mathscr{A}^n$, so that a q-connection is an element of $M_n(\Omega^1)$ and $\Gamma_{\mathscr{E}}=M_n(\sum)$, thus 1. follows from Lemma 7.

2. We view $C^\infty(M,S^2T^*)$ as a submodule $\sum_0$ of $\sum$ by the map $\sigma\to(0,\sigma)$. One has $C^\infty(M,\operatorname{End}E\otimes S^2T^*)=\operatorname{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}\otimes_{\mathscr{A}}\sum_0)$. Thus the exact sequence of bimodules:

$$0\to\textstyle\sum_0\to\sum\to A^1\to 0$$

gives the desired answer. □

Theorem 14. *Let M be a 4-dimensional* Spin^c *Riemannian compact manifold, $\mathscr{H}=L^2(M,S)$ and $F=D|D|^{-1}$ as above, and E a hermitian vector bundle over M, $\mathscr{E}=C^\infty(M,E)$.*

1. *For every compatible q-connection ∇ on $\mathscr{E}$, the curvature $\theta\in\mathscr{L}(\mathscr{E}\otimes_{\mathscr{A}}\mathscr{H})$ belongs to $\mathscr{L}^{2+}$ and the value of the Dixmier trace* $\operatorname{Trace}_\omega(\theta^2)=I(\theta)$, *is independent of ω and defines a gauge invariant positive functional I.*

2. *The restriction of I to each (affine space) fiber of the map $\nabla\to\nabla_c$ is Gaussian (i.e. a quadratic form) and one has:*

$$\operatorname*{Inf}_{\nabla_c=A} I(\nabla)=(16\pi^2)^{-1}\mathrm{YM}(A),$$

where A is a classical connection and YM *the classical Yang Mills action.*

In fact we shall prove more since we shall identify the Hilbert space of the Gaussian as $L^2(M,\operatorname{End}E\otimes S^2T^*)$.

Proof. 1. Follows from the inclusion $\Omega^2\subset\mathscr{L}^{2+}$, i.e. Lemma 4, 1) and Theorem 1. The gauge invariance (under the unitary group of $\operatorname{End}_{\mathscr{A}}(\mathscr{E})$) follows from the trace property of $\operatorname{Trace}_\omega$.

2. The value of $I(\theta)$ depends only upon the element χ of Γ associated to the q-connection ∇. In order to see that and to compute $I(\theta)$ we shall for simplicity assume that $\mathscr{E}=\mathscr{A}^n$. Then ∇ is given by a matrix (α_{ij}), $\alpha_{ij}\in\Omega^1$, with $\alpha_{ji}=-\alpha_{ij}^*\ \forall i$, $j\in\{1,\dots,n\}$. The curvature θ is given by the matrix (θ_{ij}), $\theta=d\alpha+\alpha^2$, i.e. $\theta_{ij}=d\alpha_{ij}+\sum_k\alpha_{ik}\alpha_{kj}$. Since $\alpha_{ij}\in\Omega^1$, one has $(d\alpha_{ij})^*=d\alpha_{ji}$ and $\theta_{ij}^*=\theta_{ji}$. Now the value of $\operatorname{Tr}_\omega(\theta^2)$ only depends upon the image of θ in Ω^2/Ω_0^2, and the latter only depends upon the image $\tilde{c}(\alpha_{ij})$ of α_{ij} in Ω^1/Ω_{00}^1, thus our assertion. Now let us write $\tilde{c}(\alpha_{ij})=(\omega_{ij},\beta_{ij})$ with $A\beta_{ij}=d\omega_{ij}$ as in Lemma 7. Then the image $c(\theta_j)$ of θ_{ij} in Ω^2/Ω_0^2, considered as a tensor of rank 2, is given by the following formula:

$$c(\theta_{ij})=\beta_{ij}+\sum_k\omega_{ik}\omega_{kj}.$$

For each ij the antisymmetric part $Ac(\theta_{ij})$ is the i, j component of the curvature of the associated classical connection (cf. 11b)). By 13 2., the symmetric part of the tensors β_{ij} is any smooth symmetric tensor t_{ij} with $t_{ji}=t_{ij}^* \ \forall i, j$, [where $(\xi \otimes \eta)^* = \eta^* \otimes \xi^*$ for any tensors of rank 1, ξ and η]. By Theorem 1, there exists an $O(4)$ invariant inner product on $T^2\mathbb{R}^4 = \Lambda^2\mathbb{R}^4 \oplus S^2\mathbb{R}^4$ such that, with the above notations:

$$I(\nabla) = \mathrm{Trace}_\omega(\theta^2) = \int_M \|c(\theta_{ij})\|^2 .$$

Since in this inner product $\Lambda^2\mathbb{R}^4$ is necessarily orthogonal to $S^2\mathbb{R}^4$, it follows that, while $I(\nabla)$ obviously depends quadratically on the symmetric part of β_{ij}, its minimum over each fiber of $\nabla \to \nabla_c$ is reached when the symmetric part of each tensor $c(\theta_{ij})$ is set equal to 0. But then the value of $I(\nabla)$ is, up to a numerical factor, the standard Yang-Mills action. □

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