

# ON THE NOTION OF GEOMETRY OVER $\mathbb{F}_1$

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ABSTRACT. We refine the notion of variety over the “field with one element” developed by C. Soulé by introducing a grading in the associated functor to the category of sets, and show that this notion becomes compatible with the geometric viewpoint developed by J. Tits. We then solve an open question of C. Soulé by proving, using results of J. Tits and C. Chevalley, that Chevalley group schemes are examples of varieties over a quadratic extension of the above “field”.

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## 1. INTRODUCTION

In his theory of buildings J. Tits obtained a broad generalization of the celebrated von Staudt reconstruction theorem in projective geometry, involving as groups of symmetries not only  $GL_n$  but the full collection of Chevalley algebraic groups. Among the axioms ([28]) which characterize these constructions, a relevant one is played by the condition of “thickness” which states, in its simplest form, that a projective line contains at least three points. By replacing this requirement with its strong negation, *i.e.* by imposing that a line contains exactly two points, one still obtains a coherent “geometry” which is a degenerate form of classical projective geometry. In the case of buildings, this degenerate case is described by the theory of “thin” complexes and in particular by the structure of the apartments, which are the basic constituents of the theory of buildings. The degeneracy of the von Staudt field inspired to Tits the conviction that these degenerate forms of geometries are a manifestation of the existence of a hypothetical algebraic object that he named “the field of characteristic one” ([26]). The richness and beauty of this geometric picture gives convincing evidence for the pertinence of a separate study of the degenerate case.

For completely independent reasons the need for a “field of characteristic one” (also called the field with one element) has also emerged in Arakelov geometry and *e.g.* in [22], in the context of a geometric interpretation of the zeros of zeta and  $L$ -functions. These speculative constructions aim for the description of a geometric framework analogous to the one used by Weil in the proof of the Riemann Hypothesis for function fields. More precisely, one seeks for a replacement of the surface  $C \times_{\mathbb{F}_q} C$ , where  $C$  is a (projective, smooth algebraic) curve over a finite field  $\mathbb{F}_q$  and whose field of functions is the given global field. The main idea is to postulate the existence of the “absolute point”  $\text{Spec } \mathbb{F}_1$  over which any algebraic scheme would sit. In the particular case of  $\text{Spec } \mathbb{Z}$ , one would then be able to use the spectrum of the tensor product  $\mathbb{Z} \times_{\mathbb{F}_1} \mathbb{Z}$  as a substitute for the surface  $C \times_{\mathbb{F}_q} C$ . This viewpoint has given rise, in the recent past, to a number of interesting constructions (*cf.* [18], [22], [24], [25], [15], [16], [29]).

Our interest in the quest for  $\mathbb{F}_1$  arose from the following equation

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[T]/(T^n - 1), \quad n \in \mathbb{N} \tag{1}$$

which was introduced in [25] and supplies a definition of the finite extension  $\mathbb{F}_{1^n}$  of  $\mathbb{F}_1$ , after base change to  $\mathbb{Z}$ . The main point promoted in [9] is that the above equation (1) yields (without knowing the algebraic nature of  $\mathbb{F}_1$ , and after base change to  $\mathbb{Z}$ ), an algebraic object which reflects the structure of the inductive limit  $\mathbb{F}_{1^\infty} = \varinjlim \mathbb{F}_{1^n}$ , by supplying also an analogue of the geometric Frobenius correspondence. This object is the integral model of a rational Hecke algebra which defines the quantum statistical mechanical system of [1]. It is known that our construction determines, after passing to the dual system, a spectral realization of the zeros of the Riemann zeta function, as well as a trace formula interpretation of the Riemann-Weil explicit formulas (see [1], [5], [7], [8], [23]).

In [9], we made use of the general definition of an algebraic variety over  $\mathbb{F}_1$  as introduced by C. Soulé in [25]. Our goal in this paper is to give an answer to the following two natural questions

- Are Chevalley group schemes examples of varieties that can be defined over  $\mathbb{F}_1$ ?

- Is the notion of variety over  $\mathbb{F}_1$  as in *op.cit.* compatible with the geometric viewpoint developed by Tits?

The first question was formulated in [25]. In this article we show that Chevalley group schemes can be defined<sup>1</sup> over the quadratic extension  $\mathbb{F}_{1^2}$  (*cf.* Theorem 4.10). The second question originates naturally by working with the simplest example of a projective variety, namely the projective spaces  $\mathbb{P}^d$ .

At first sight, a very serious problem emerges in [25], since the definition of the set  $\mathbb{P}^d(\mathbb{F}_1)$  does *not* appear to be naturally linked with the notion of a geometry over  $\mathbb{F}_1$  as in [26]. In fact, in § 6 of [25] the cardinality of the set  $\mathbb{P}^d(\mathbb{F}_{1^n})$  is shown to be  $N(2n + 1)$ , where  $N(x)$  is a polynomial function whose values at prime powers  $q = p^r$  are given by the classical formula  $N(q) = q^d + \dots + q + 1$  giving the cardinality of  $\mathbb{P}^d(\mathbb{F}_q)$ . When  $n = 1$ , one obtains the integer  $|\mathbb{P}^d(\mathbb{F}_1)| = N(3) = \frac{3^{d+1}-1}{2}$  which is incompatible with (and much larger than) the number  $d + 1$  of points of the set  $\mathcal{P}_d$  on which Tits defines his notion of projective geometry of dimension  $d$  over  $\mathbb{F}_1$  (*cf.* [26], § 13 p. 285).

After clarifying a few statements taken from [25] on the notion of variety over  $\mathbb{F}_1$  and on the meaning of a natural transformation of functors (*cf.* § 2), we show how to resolve the aforementioned problem by introducing a suitable refinement of the notion of affine algebraic variety over  $\mathbb{F}_1$ . The main idea is to replace the category of sets (in which the covariant functor  $\underline{X}$  of *op.cit.* takes values) by the category of  $\mathbb{Z}_{\geq 0}$ -graded sets. In § 2.2 (*cf.* Definition 2.6), we explain how to refine the covariant functor  $\underline{X}$  into a graded functor  $\underline{X} = \coprod_k \underline{X}^{(k)}$  defined by a disjoint union of homogeneous components which correspond, at the intuitive level, to the terms of the Taylor expansion, at  $q = 1$ , of the counting function  $N(q)$ . The condition that  $N(q)$  is a polynomial is very restrictive (*e.g.* it fails in general for elliptic curves) and was required in [25] (§6, Condition Z) to define the zeta function<sup>2</sup>. In § 3 we check that in the case of a projective space, the set  $\mathbb{P}^d(\mathbb{F}_{1^n})$  coincides *in degree zero* with the  $d + 1$  points of the set  $\mathcal{P}_d$ , and this result shows the sought for agreement with the theory of Tits.

Our new definition of an affine variety over  $\mathbb{F}_1$  is described by the following data

- A covariant functor from the category of finite abelian groups to the category of graded sets

$$\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \rightarrow \mathcal{S}ets.$$

- An affine variety  $X_{\mathbb{C}}$  over  $\mathbb{C}$ .
- A natural transformation  $e_X$  connecting  $\underline{X}$  to the functor

$$\mathcal{F}_{ab} \rightarrow \mathcal{S}ets, \quad D \mapsto \text{Hom}(\text{Spec } \mathbb{C}[D], X_{\mathbb{C}}).$$

These data need to fulfill also a strong condition (*cf.* Definition 2.8) which determines uniquely a variety over  $\mathbb{Z}$ . To a point of  $\text{Spec } \mathbb{C}[D]$  is associated a character  $\chi : D \rightarrow \mathbb{C}^*$  which assigns to a group element  $g \in D$  a root of unity  $\chi(g)$  in  $\mathbb{C}$ . For each such character, the map  $e_X$  provides a concrete interpretation of the elements of  $\underline{X}(D)$  as points of  $X_{\mathbb{C}}$ .

<sup>1</sup>as varieties but not as groups

<sup>2</sup>The definition of the zeta function given there is “upside down” with respect to Manin’s, and should be replaced by its inverse to get *e.g.*  $\zeta_{\mathbb{P}^1}(s) = 1/(s(s-1))$  instead of  $s(s-1)$ .

In § 4.5 we test these ideas with Chevalley groups  $G$  and show that they can be defined over  $\mathbb{F}_{12}$ . Let  $\mathfrak{G}$  be the algebraic group scheme over  $\mathbb{Z}$  associated by Chevalley ([3], [14]) to a root system  $\{L, \Phi, n_r\}$  of  $G$  (cf. § 4.1 and [27], § 4.1). In § 4.6 we prove that  $G$  can be defined over  $\mathbb{F}_{12}$  in the above sense. For the proof, one needs to verify that the following conditions are satisfied

- The functor  $\underline{G}$  (to graded sets) contains enough points so that, together with  $G_{\mathbb{C}}$ , it characterizes  $\mathfrak{G}$ .
- The cardinality of  $\underline{G}(\mathbb{F}_{1^n})$  is given by a polynomial  $P(n)$  whose value, for  $q$  a prime power and  $n = q - 1$ , coincides with the cardinality of  $\mathfrak{G}(\mathbb{F}_q)$ .
- The terms of lowest degree in  $\underline{G}$  have degree equal to the rank of  $G$  and determine the group extension of the Weyl group of  $G$  by  $\text{Hom}(L, D)$ , as constructed by Tits in [27].

The first condition ensures the compatibility with Soulé’s original notion of variety over  $\mathbb{F}_1$ . The second statement sets a connection with the theory of zeta functions as in [22]. Finally, the third condition guarantees a link with the constructions of Tits. In fact, in [26] it was originally promoted the idea that the Weyl group of a Chevalley group  $G$  should be interpreted as the points of  $G$  which are rational over  $\mathbb{F}_1$ . For  $G = \text{GL}_d$ , it was then shown in [18] that the points of  $G$  over  $\mathbb{F}_{1^n}$  are described by the wreath product of the group of permutations of  $d$  letters with  $\mu_n^d$ . It is important to notice that in our theory these groups are recast as the terms of lowest degree of  $\underline{G}$ . The terms of higher degree are more subtle to describe; to construct them we make use of the detailed theory of Chevalley as in [2], [3].

If  $G$  is the Chevalley group associated to a root system, the cardinality of the set  $\mathfrak{G}(\mathbb{F}_q)$  (i.e. the number of points of  $\mathfrak{G}$  which are rational<sup>3</sup> over  $\mathbb{F}_q$ ), is given by the formula

$$|\mathfrak{G}(\mathbb{F}_q)| = (q - 1)^\ell q^N \sum_{w \in W} q^{N(w)}, \quad (2)$$

where  $\ell$  denotes the rank of  $G$ ,  $N$  is the number of positive roots,  $W$  is the Weyl group and  $N(w)$  is the number of positive roots  $r$ , such that  $w(r) < 0$ . The above formula (2) corresponds to a decomposition of  $\mathfrak{G}(\mathbb{F}_q)$  as a disjoint union (over the Weyl group  $W$ ) of products of the form

$$\mathfrak{G}(\mathbb{F}_q) = \coprod_{w \in W} \mathbb{G}_m(\mathbb{F}_q)^\ell \mathbb{A}^N(\mathbb{F}_q) \mathbb{A}^{N(w)}(\mathbb{F}_q). \quad (3)$$

This equality suggests the definition of the functor  $\underline{G}$  by means of a sum of products of powers of the graded functors  $\mathbb{G}_m$  and  $\mathbb{A}$  (cf. § 3).

The most technical part of our construction is the definition of the natural transformation  $e_G$  as in (c), which involves the introduction of a lifting procedure from the Weyl group  $W$  to the complex group  $G_{\mathbb{C}} = \mathfrak{G}(\mathbb{C})$ . The solution to this problem is in fact already contained in [27]. Indeed, Tits introduced in that paper a functor  $\mathcal{N}_{D, \epsilon}(L, \Phi)$  from pairs  $(D, \epsilon)$  of an abelian group and an element of square one in  $D$ , to group extensions of the form

$$1 \rightarrow \text{Hom}(L, D) \rightarrow \mathcal{N}_{D, \epsilon}(L, \Phi) \rightarrow W \rightarrow 1.$$

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<sup>3</sup>There are in general more points in  $\mathfrak{G}(\mathbb{F}_q)$  than in the Chevalley group  $G_{\mathbb{F}_q}$  which is the commutator subgroup of  $\mathfrak{G}(\mathbb{F}_q)$ .

The definition of the graded functor  $\underline{G}$  and the natural transformation  $e_G$  then follow by applying the original construction of Tits together with the Bruhat decomposition of  $G$  and working with a specific parametrization of its cells.

Incidentally, we find rather remarkable that the image of this lift of the Weyl group in  $G_{\mathbb{C}}$  consists only of finite products of elements in the maximal torus with elements of the form  $x_r(\mu)$ , where  $\mu$  is a root of unity in  $\mathbb{C}$  and where the  $x_r(t)$  generate (over any field  $k$ ) unipotent one-parameter subgroups associated to the roots  $r$ . The fact that  $\underline{G}$  contains enough points so that, together with  $G_{\mathbb{C}}$ , it characterizes  $\mathfrak{G}$ , follows from an important result of Chevalley [3], by working only with the points in the big cell of  $G$ .

The original definition of varieties over  $\mathbb{F}_1$  in [25], as well as the variant used here, is based on a covariant construction of enough points with cyclotomic coordinates, however a precise control on the size of this set is also needed. In §5, we show that the above examples of varieties  $X$  over  $\mathbb{F}_1$  (more precisely  $\mathbb{F}_{1^2}$ ) fulfill much stronger properties than those originally required. That is:

- The construction of the functor  $\underline{X}$  extends to the category of pairs  $(D, \epsilon)$  of an arbitrary abelian semi-group<sup>4</sup> and an element of square one.
- The definition of the natural transformation  $e_X$  extends to arbitrary commutative rings  $A$  and determines a map  $e_{X,A} : \underline{X}(A_s) \rightarrow X(A)$ , where  $A_s$  is the abelian semi-group given by the multiplication in  $A$ .
- The natural transformation  $e_{X,A}$  is a bijection when  $A = K$  is a field.

In the final part of the paper (*cf.* § 5), we explain how this enriched construction yields a new notion of schemes over  $\mathbb{F}_1$  which reconciles Soulé's original viewpoint with the approach taken up by Deitmar in [12] (following Kurokawa, Ochiai, and Wakayama [20]) and the log-geometry of monoids of K. Kato [19], with also the extra advantage of not being limited to the case of toric varieties. In the Noetherian case, the local representability of the functor  $\underline{X}$  implies that its restriction to finite abelian groups is automatically a functor to graded sets, thus clarifying the role of the grading in our construction. This new notion of schemes over  $\mathbb{F}_1$  supplies also a conceptual reason for the equality of the number of points of  $X(\mathbb{F}_q)$  and the cardinality of the set  $\underline{X}(D)$  ( $D = \mathbb{F}_q^*$ ) and for the polynomial nature of the counting function (*cf.* [6]).

**Acknowledgment.** The authors are partially supported by the NSF grant DMS-FRG-0652164. We are grateful to P. Cartier and C. Soulé for useful comments.

## 2. ON THE NOTION OF VARIETY OVER $\mathbb{F}_1$

In this section we review the notion of variety over  $\mathbb{F}_1$  as in [25] and develop a refinement of this concept that will be applied to the case of Chevalley group schemes in §4.5, to show that these varieties can be defined over (an extension of)  $\mathbb{F}_1$ , thus establishing a link with Tits' geometries.

**2.1. Extension of scalars.** In this paragraph we shall use the same notation as in [25]. Let  $k$  be a field and  $\Omega$  be a commutative  $k$ -algebra. One considers the functor of extension of scalars:

$$\cdot \otimes_k \Omega : \mathcal{A}_k \longrightarrow \mathcal{A}_\Omega, \quad R \mapsto R_\Omega = R \otimes_k \Omega \tag{4}$$

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<sup>4</sup>with a unit and a zero element

from the category  $\mathcal{A}_k$  of unital commutative  $k$ -algebras to the corresponding category  $\mathcal{A}_\Omega$ . This functor extends to the category of schemes over  $k$  and we use the same notation to denote it. If  $X$  is a scheme (of finite type) over  $k$ , one lets  $X_\Omega = X \otimes_k \Omega$  the corresponding scheme over  $\Omega$ . If  $X = \text{Spec}(R)$  is affine and corresponds to the  $k$ -algebra  $R$ , then  $X_\Omega$  is also affine and corresponds to the  $\Omega$ -algebra  $R_\Omega$ . The natural homomorphism of algebras  $R \rightarrow R_\Omega$  corresponds at the level of schemes to a surjective morphism

$$X_\Omega = \text{Spec}(R_\Omega) \rightarrow X = \text{Spec}(R). \quad (5)$$

Let  $\mathcal{S}et$  be the category of sets. Then we view a scheme  $X$  over  $k$  as a covariant functor

$$\underline{X} : \mathcal{A}_k \longrightarrow \mathcal{S}ets, \quad R \mapsto \underline{X}(R). \quad (6)$$

For affine schemes  $X = \text{Spec}(A)$ , one has  $\underline{X}(R) = \text{Hom}(A, R)$ . Note that the functor  $X \rightarrow \underline{X}$  on schemes is covariant.

In [25] (*cf.* Proposition 1), one makes use of the following statement

**Proposition 2.1.** (i) *There exists a natural transformation of functors*

$$i : \underline{X} \rightarrow \underline{X}_\Omega, \quad X(R) \subset X_\Omega(R_\Omega) \quad (7)$$

(ii) *For any scheme  $S$  over  $\Omega$  and any natural transformation*

$$\varphi : \underline{X} \rightarrow \underline{S}, \quad (8)$$

*there exists a unique morphism  $\varphi_\Omega$  (over  $\Omega$ ) from  $X_\Omega$  to  $S$  such that  $\varphi = \varphi_\Omega \circ i$ .*

Notice that (7) seems to imply that by means of the covariance property of the functor  $X \mapsto \underline{X}$ , one should obtain a natural morphism of schemes  $X \rightarrow X_\Omega$ , and this is in evident contradiction with (5). This apparent inconsistency is due to an abuse of notation and it is easily fixed as follows. The functors  $\underline{X}_\Omega$  and  $\underline{S}$  which in *op.cit.* are defined as functors from  $\mathcal{A}_k$  to  $\mathcal{S}ets$  (*cf.* equation above Proposition 1), should instead be properly introduced as functors from the category  $\mathcal{A}_\Omega$  to  $\mathcal{S}ets$ . The “hidden” operation in [25] is the composition with the functor of extension of scalars

$$\beta : \mathcal{A}_k \longrightarrow \mathcal{A}_\Omega, \quad \beta(R) = R_\Omega \quad (9)$$

Up-to replacing in the above proposition  $\underline{X}_\Omega$  by  $\underline{X}_\Omega \circ \beta$  and  $\underline{S}$  by  $\underline{S} \circ \beta$ , one then obtains the following correct statement

**Proposition 2.2.** (i) *There exists a natural transformation of functors*

$$i : \underline{X} \rightarrow \underline{X}_\Omega \circ \beta, \quad X(R) \subset X_\Omega(R_\Omega) \quad (10)$$

(ii) *For any scheme  $S$  over  $\Omega$  and any natural transformation*

$$\varphi : \underline{X} \rightarrow \underline{S} \circ \beta, \quad (11)$$

*there exists a unique morphism  $\varphi_\Omega$  (over  $\Omega$ ) from  $X_\Omega$  to  $S$  such that  $\varphi = \varphi_\Omega \circ i$ .*

The proof is a simple translation of the one given in [25], with only a better use of notation.

*Proof.* We first consider the case of an affine scheme  $X = \text{Spec}(A)$ . The proof in that case applies to any functor  $\underline{S}$  from  $\mathcal{A}_\Omega$  to  $\mathcal{S}ets$ . The functor  $X_\Omega$  is represented by  $A_\Omega$  and the inclusion  $X(R) \subset X_\Omega(R_\Omega)$  is simply described by the inclusion

$$i : \text{Hom}_k(A, R) \subset \text{Hom}_\Omega(A \otimes_k \Omega, R \otimes_k \Omega), \quad f \mapsto i(f) = f \otimes_k id_\Omega = \beta(f).$$

Next, since  $\underline{X}$  is represented by  $A$ , Yoneda's lemma shows that the natural transformation  $\varphi$  of (11) is characterized by

$$h = \varphi(A)(id_A) \in \underline{S}(A_\Omega) \quad (12)$$

(in the displayed formula just after Proposition 1 in [25] there is a typo: the term  $X_\Omega(A_\Omega)$  should be replaced by  $\underline{S}(A_\Omega)$ ). Similarly, the functor  $X_\Omega$  is represented by  $A_\Omega$ , and a morphism of functors  $\psi$  from  $X_\Omega$  to  $\underline{S}$  is uniquely determined by an element of  $\underline{S}(A_\Omega)$ . Thus  $h$  determines a unique morphism  $\psi = \varphi_\Omega$  from  $X_\Omega$  to  $\underline{S}$  such that

$$\psi(A_\Omega)(id_{A_\Omega}) = h.$$

The equality  $\varphi = \varphi_\Omega \circ i$  follows again from Yoneda's lemma since  $\underline{X}$  is represented by  $A$  and one just needs to check the equality on  $id_A$  and it follows from  $i(id_A) = id_{A_\Omega}$ . Similarly, to prove the uniqueness, since the functor  $X_\Omega$  is represented by  $A_\Omega$ , it is enough to show that the equality  $\varphi = \psi \circ i$  uniquely determines  $\psi_{A_\Omega}(id_{A_\Omega})$ .

The extension to schemes which are no longer affine follows as in [25], but the proof given there is unprecise since it is not true in general that the inverse image of an open affine subscheme of a scheme  $X$  by a morphism  $\text{Spec} A \rightarrow X$  is affine. As a functor from  $\mathcal{A}_\Omega$  to *Sets*,  $\underline{X}_\Omega$  is the composition  $\underline{X} \circ \alpha$  of  $\underline{X}$  with the restriction of scalars  $\alpha$  from  $\Omega$  to  $k$  which is the right adjoint of  $\beta$ . The inclusion  $i$  comes from the canonical morphism  $R \rightarrow \alpha(\beta(R))$  for any object  $R$  of  $\mathcal{A}_k$ . We leave it to the reader to clarify the proof and show the result using on the functor  $\underline{S}$  the only assumption that it is *local* in the sense of [13] Definition 3.11.  $\square$

**2.2. Gadgets.** We keep the same notation as in the previous paragraph. Let us first recall the definition of “truc” given in [25]. Let  $\mathcal{R}$  be the full subcategory of the category of commutative rings whose objects are the group rings  $\mathbb{Z}[H]$  of finite abelian groups (*cf.* [25] 3. Définitions, Remarques).

**Definition 2.3.** *A truc over  $\mathbb{F}_1$  consists of the following data*

- a pair  $X = (\underline{X}, \mathcal{A}_X)$  of a covariant functor  $\underline{X} : \mathcal{R} \rightarrow \text{Sets}$  and a  $\mathbb{C}$ -algebra  $\mathcal{A}_X$
- for each object  $R$  of  $\mathcal{R}$  and each homomorphism  $\sigma : R \rightarrow \mathbb{C}$ , an evaluation morphism ( $\mathbb{C}$ -algebra homomorphism)

$$e_{x,\sigma} : \mathcal{A}_X \rightarrow \mathbb{C}, \quad \forall x \in \underline{X}(R),$$

which satisfies the functorial compatibility  $e_{f(y),\sigma} = e_{y,\sigma \circ f}$ ,  $\forall f : R' \rightarrow R$  morphism in  $\mathcal{R}$  and  $\forall y \in \underline{X}(R')$ .

We shall reformulate slightly the above definition with the goal to:

- treat the archimedean place simultaneously with  $\text{Spec } \mathbb{Z}$
- replace the category  $\mathcal{R}$  by the category  $\mathcal{F}_{ab}$  of finite abelian groups<sup>5</sup>
- put in evidence the role of the functor  $\beta$ .

Thus, we replace  $\mathcal{R}$  by the category  $\mathcal{F}_{ab}$  of finite abelian groups. There is a natural functor of extension of scalars from  $\mathbb{F}_1$  to  $\mathbb{Z}$  which is given by

$$\beta : \mathcal{F}_{ab} \rightarrow \mathcal{R}, \quad \beta(D) = D \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[D] \quad (13)$$

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<sup>5</sup>This replacement was suggested by C. Soulé

and associates to an abelian group its convolution algebra over  $\mathbb{Z}$ . Let us understand the evaluation morphism as a natural transformation. We introduce the functor

$$\mathrm{Spec}_\infty(\mathcal{A}_X) : \mathcal{R} \longrightarrow \mathcal{S}ets, \quad R \mapsto \mathrm{Hom}(\mathcal{A}_X, R \otimes_{\mathbb{Z}} \mathbb{C}) \quad (14)$$

and compose it with the functor  $\beta : \mathcal{F}_{ab} \rightarrow \mathcal{R}$ .

**Lemma 2.4.** *The evaluation morphism  $e$  as in Definition 2.3, determines a natural transformation of (covariant) functors*

$$e : \underline{X} \rightarrow \mathrm{Spec}_\infty(\mathcal{A}_X) \circ \beta.$$

*Proof.* For each object  $D$  of  $\mathcal{F}_{ab}$  the evaluation map  $e_{x,\sigma}$  can be viewed as a map of sets  $\underline{X}(D) \rightarrow \mathrm{Hom}(\mathcal{A}_X, R \otimes_{\mathbb{Z}} \mathbb{C})$ , where  $R = \beta(D)$ . Indeed, for  $x \in \underline{X}(D)$ , we get a map  $e_x$  from characters  $\sigma$  of  $R$  to characters of  $\mathcal{A}_X$  which determines a morphism from  $\mathcal{A}_X$  to  $R \otimes_{\mathbb{Z}} \mathbb{C}$ .  $\square$

We now reformulate Definition 2.3 as follows

**Definition 2.5.** *A gadget over  $\mathbb{F}_1$  is a triple  $X = (\underline{X}, X_{\mathbb{C}}, e_X)$  consisting of*

- (a) *a covariant functor  $\underline{X} : \mathcal{F}_{ab} \rightarrow \mathcal{S}ets$  to the category of sets*
- (b) *a variety  $X_{\mathbb{C}}$  over  $\mathbb{C}$*
- (c) *a natural transformation  $e_X : \underline{X} \rightarrow \mathrm{Hom}(\mathrm{Spec} \mathbb{C}[-], X_{\mathbb{C}})$  from the functor  $\underline{X}$  to the functor*

$$\mathrm{Hom}(\mathrm{Spec} \mathbb{C}[-], X_{\mathbb{C}}), \quad D \mapsto \mathrm{Hom}(\mathrm{Spec} \mathbb{C}[D], X_{\mathbb{C}}). \quad (15)$$

2.2.1. *Example.* An affine variety  $V$  over  $\mathbb{Z}$  defines a gadget  $X = \mathcal{G}(V)$  over  $\mathbb{F}_1$  by letting  $X_{\mathbb{C}} = V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C}$ .  $\underline{X}(D) = \mathrm{Hom}(\mathcal{O}, \mathbb{Z}[D])$  is the set of points of  $V$  in the convolution algebra  $\mathbb{Z}[D]$  with the natural transformation to  $\mathrm{Hom}(\mathrm{Spec} \mathbb{C}[D], V_{\mathbb{C}}) = \mathrm{Hom}(\mathcal{O}_{\mathbb{C}}, \mathbb{C}[D])$  obtained by applying the functor  $\otimes_{\mathbb{Z}} \mathbb{C}$ .

**Definition 2.6.** *A gadget  $X$  over  $\mathbb{F}_1$  is said to be graded when*

$$\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \rightarrow \mathcal{S}ets$$

*takes values in the category of  $\mathbb{Z}_{\geq 0}$ -graded sets. It is finite when the set  $\underline{X}(D)$  is finite  $\forall D \in \mathcal{F}_{ab}$ .*

2.3. **Varieties over  $\mathbb{F}_1$ .** The notion of morphism of gadgets  $\phi : X \rightarrow Y$  is essentially that of a natural transformation. More precisely,  $\phi$  is determined by a pair  $\phi = (\underline{\phi}, \phi_{\mathbb{C}})$ , with

$$\underline{\phi} : \underline{X} \rightarrow \underline{Y}, \quad \phi_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}. \quad (16)$$

$\underline{\phi}$  is a natural transformation of functors and  $\phi_{\mathbb{C}}$  a morphism of varieties over  $\mathbb{C}$ . A *required* compatibility with the evaluation maps gives rise to a commutative diagram

$$\begin{array}{ccc} \underline{X}(D) & \xrightarrow{\underline{\phi}^{(D)}} & \underline{Y}(D) \\ e_X(D) \downarrow & & \downarrow e_Y(D) \\ \mathrm{Hom}(\mathrm{Spec} \mathbb{C}[D], X_{\mathbb{C}}) & \xrightarrow{\phi_{\mathbb{C}}} & \mathrm{Hom}(\mathrm{Spec} \mathbb{C}[D], Y_{\mathbb{C}}). \end{array} \quad (17)$$

As in [25], we introduce the following notion of *immersion* of gadgets



**Definition 2.7.** A morphism of gadgets  $\phi : X \rightarrow Y$  is said to be an immersion if  $\phi_{\mathbb{C}}$  is an embedding and for any object  $D$  of  $\mathcal{F}_{ab}$ , the map  $\underline{\phi} : \underline{X}(D) \rightarrow \underline{Y}(D)$  is injective.

We can now re-state the key definition of an *affine* variety  $X$  over  $\mathbb{F}_1$ . In the formulation given in [25], it postulates the existence of a variety (of finite type) over  $\mathbb{Z}$  which plays the role of the scheme  $X \otimes_{\mathbb{F}_1} \mathbb{Z}$  and fulfills the universal property of Proposition 2.2.

**Definition 2.8.** An affine variety  $X$  over  $\mathbb{F}_1$  is a finite, graded gadget  $X$  such that there exists an affine variety  $X_{\mathbb{Z}}$  over  $\mathbb{Z}$  and an immersion  $i : X \rightarrow \mathcal{G}(X_{\mathbb{Z}})$  of gadgets satisfying the following universal property: for any affine variety  $V$  over  $\mathbb{Z}$  and any morphism of gadgets  $\varphi : X \rightarrow \mathcal{G}(V)$ , there exists a unique algebraic morphism

$$\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \rightarrow V$$

of affine varieties such that  $\varphi = \mathcal{G}(\varphi_{\mathbb{Z}}) \circ i$ .

**2.4. Varieties over  $\mathbb{F}_1^n$ .** This small variant is obtained (following [25] §3.8.2) by replacing the category  $\mathcal{F}_{ab}$  of finite abelian groups by the finer one  $\mathcal{F}_{ab}^{(n)}$  whose objects are pairs  $(D, \epsilon)$ , where  $D$  is a finite abelian group and  $\epsilon \in D$  is of order exactly  $n$ . A morphism in  $\mathcal{F}_{ab}^{(n)}$  is a homomorphism of abelian groups which sends  $\epsilon \in D$  to  $\epsilon' \in D'$ . Let  $R_n = \mathbb{Z}[T]/(T^n - 1)$  then the whole discussion takes place over  $\text{Spec}R_n$ . We shall only use in this paper the case  $n = 2$ . In that case, the two homomorphisms  $\rho_{\pm} : R_2 \rightarrow \mathbb{Z}$  given by  $\rho_{\pm}(T) = \pm 1$  show that  $\text{Spec}R_2$  is the union of two copies  $(\text{Spec}\mathbb{Z})_{\pm}$  of  $\text{Spec}\mathbb{Z}$  which cross at the prime 2. We shall concentrate on the non-trivial copy  $(\text{Spec}\mathbb{Z})_- \subset \text{Spec}R_2$ . In Definition 2.5 one replaces  $\mathcal{F}_{ab}$  by  $\mathcal{F}_{ab}^{(2)}$  and one substitutes everywhere the group ring  $\beta[D]$  by the reduced group ring which is the tensor product of rings

$$\beta[D, \epsilon] = \mathbb{Z}[D] \otimes_{\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{Z}, \quad (18)$$

in which  $\epsilon = -1$ . Thus the characters  $\chi$  of the algebra  $\mathbb{C}[D, \epsilon]$  are the characters of  $\mathbb{C}[D]$  such that  $\chi(\epsilon) = -1$ . They still separate the elements of  $D$ :

$$\forall g_1 \neq g_2 \in D, \quad \exists \chi \in \text{Spec}\mathbb{C}[D, \epsilon], \quad \chi(g_1) \neq \chi(g_2) \quad (19)$$

An affine variety  $V$  over  $\mathbb{Z}$  defines a gadget  $X = \mathcal{G}(V)$  over  $\mathbb{F}_1^2$  by letting  $X_{\mathbb{C}} = V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C}$  sit over the non-trivial copy  $(\text{Spec}\mathbb{Z})_- \subset \text{Spec}R_2$ . The functor  $\underline{X}(D) = \text{Hom}(\mathcal{O}, \beta[D, \epsilon])$  is the set of points of  $V$  in the algebra  $\beta[D, \epsilon]$  with the natural transformation to  $\text{Hom}(\text{Spec}\mathbb{C}[D, \epsilon], V_{\mathbb{C}}) = \text{Hom}(\mathcal{O}_{\mathbb{C}}, \mathbb{C}[D, \epsilon])$  obtained by applying the functor  $\otimes_{\mathbb{Z}} \mathbb{C}$ .

### 3. ELEMENTARY EXAMPLES

In this section we apply Definition 2.8 by working out the explicit description of the graded functor  $\underline{X}$  in several elementary examples of algebraic varieties over  $\mathbb{F}_1$ . The main new feature, with respect to [25], is the introduction of a grading. At the intuitive level, the underlying principle in the definition of the graded functor  $\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)}$  is that of considering the Taylor expansion, at  $q = 1$ , of the function  $N(q)$  counting the number of points of the scheme  $X$  over the finite field  $\mathbb{F}_q$ . The

term of degree  $k$  (i.e.  $a_k(q-1)^k$ ) in the expansion should agree with the cardinality of the set  $\underline{X}^{(k)}(D)$ , for  $q-1 = |D|$ ,  $D \in \text{obj}(\mathcal{F}_{ab})$ .

The requirement that the function  $N(q)$  counting the number of points of the scheme  $X$  over the finite field  $\mathbb{F}_q$  is a polynomial in  $q$  is imposed in [25] in order to deal with the zeta function and is very restrictive. It fails for instance in general for elliptic curves but it holds for instance for Chevalley group schemes. We shall first deal with a few concrete examples of simple geometric spaces for which  $N(q)$  is easily computable. These are

- (1)  $\mathbb{G}_m$ ,  $N(q) = q - 1$ .
- (2) The affine line  $\mathbb{A}^1$ ,  $N(q) = q$ .
- (3) The projective space  $\mathbb{P}^d$ ,  $N(q) = 1 + q + \dots + q^d$ .

In the following we shall consider each of these cases in details.

**3.1. Spec $D$ .** Let  $D$  be a finite abelian group. We let  $\text{Spec}D$  be the gadget given by  $\text{Spec}D(D') = \text{Hom}(D, D')$  and  $(\text{Spec}D)_{\mathbb{C}} = \mathbb{C}[D]$  with the obvious natural transformation. One checks that it defines a variety over  $\mathbb{F}_1$ . It is graded by the grading concentrated in degree 0.

**3.2. The multiplicative group  $\mathbb{G}_m$ .** For the multiplicative group  $\mathbb{G}_m$ , the counting function is  $N(q) = q - 1$ : its Taylor expansion at  $q = 1$  has just one term in degree 1. We define the functor  $\underline{\mathbb{G}}_m$  from abelian groups to  $\mathbb{Z}_{\geq 0}$ -graded sets accordingly i.e.

$$\underline{\mathbb{G}}_m : \mathcal{F}_{ab} \longrightarrow \text{Sets}, \quad \underline{\mathbb{G}}_m(D)^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ D & \text{if } k = 1. \end{cases} \quad (20)$$

In particular one sets

$$\underline{\mathbb{G}}_m(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1. \end{cases} \quad (21)$$

Except for the introduction of the grading and for the replacement of the category of (commutative) rings finite and flat over  $\mathbb{Z}$  (as in [25]) by that of finite abelian groups, the definition (20) is the same as the corresponding functor in *op.cit.*

We denote by  $e_m : \underline{\mathbb{G}}_m \rightarrow \text{Hom}(\text{Spec } \mathbb{C}[-], \mathbb{G}_m(\mathbb{C}))$  the natural transformation from the functor  $\underline{\mathbb{G}}_m$  to the functor

$$\text{Hom}(\text{Spec } \mathbb{C}[-], \mathbb{G}_m(\mathbb{C})), \quad D \mapsto \text{Hom}(\text{Spec } \mathbb{C}[D], \mathbb{G}_m(\mathbb{C})),$$

which assigns to a character  $\chi$  associated to a point of  $\text{Spec } \mathbb{C}[D]$  the group homomorphism

$$D \rightarrow \mathbb{C}^*, \quad e_m(D)(g) = \chi(g). \quad (22)$$

It is now possible to adapt the proof of [25] (as in 5.2.2) and show that this gadget defines a variety over  $\mathbb{F}_1$ .

**Proposition 3.1.** *The gadget  $\mathbb{G}_m = (\underline{\mathbb{G}}_m, \mathbb{G}_m(\mathbb{C}), e_m)$  defines a variety over  $\mathbb{F}_1$ .*

*Proof.* By construction  $\mathbb{G}_m$  is a finite and graded gadget. It is easy to guess that  $\mathbb{G}_{m, \mathbb{Z}} = \text{Spec}(\mathbb{Z}[T, T^{-1}])$  while the immersion  $i$  is given by the injection  $D \rightarrow \text{Hom}(\mathbb{Z}[T, T^{-1}], \mathbb{Z}[D])$ . Let us see that the condition of Definition 2.8 is also fulfilled. Let  $V = \text{Spec}(\mathcal{O})$  be an affine variety over  $\mathbb{Z}$ . Let  $\phi : \mathbb{G}_m \rightarrow \mathcal{G}(V)$ , be a morphism of (affine) gadgets. This means that we are given a pair  $(\underline{\phi}, \phi_{\mathbb{C}})$ , where  $\phi_{\mathbb{C}}$  can

be equivalently interpreted by means of the corresponding homomorphism of  $\mathbb{C}$ -algebras ( $\mathcal{O}_{\mathbb{C}} = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}$ )

$$\phi_{\mathbb{C}}^* : \mathcal{O}_{\mathbb{C}} \rightarrow \mathbb{C}[T, T^{-1}].$$

Furthermore,  $\underline{\phi}$  is a morphism of functors (natural transformation)

$$\underline{\phi}(D) : \underline{\mathbb{G}}_m(D) \rightarrow \text{Hom}(\mathcal{O}, \beta(D))$$

which fulfills the following compatibility condition (*cf.* (17)). For any finite abelian group  $D$  the following diagram is commutative

$$\begin{array}{ccc} \underline{\mathbb{G}}_m(D) & \xrightarrow{\underline{\phi}(D)} & \text{Hom}(\mathcal{O}, \beta(D)) \\ e_m(D) \downarrow & & \downarrow e_{\mathcal{G}(V)}(D) \\ \text{Hom}(\mathbb{C}[T, T^{-1}], \beta(D)_{\mathbb{C}}) & \xrightarrow{\phi^*} & \text{Hom}(\mathcal{O}_{\mathbb{C}}, \beta(D)_{\mathbb{C}}). \end{array} \quad (23)$$

To construct  $\psi = \phi_{\mathbb{Z}}$  let us show that  $\phi^*(\mathcal{O}) \subset \mathbb{Z}[T, T^{-1}]$ . Let  $h \in \mathcal{O}$  and  $f = \phi^*(h)$ . Then by construction  $f \in \mathbb{C}[T, T^{-1}]$ . Let  $D = \mathbb{Z}/n\mathbb{Z}$  be the cyclic group of order  $n$  with generator  $\xi \in \mathbb{Z}/n\mathbb{Z}$ ; one has  $\xi \in \underline{\mathbb{G}}_m(D)$  and  $\underline{\phi}(D)(\xi) \in \text{Hom}(\mathcal{O}, \beta(D))$ . By evaluating on  $h \in \mathcal{O}$  one gets

$$\underline{\phi}(D)(\xi)(h) \in \mathbb{Z}[D] \subset \mathbb{C}[D].$$

It follows from the commutativity of the diagram (23) that this is the same as evaluating on  $f \in \mathbb{C}[T, T^{-1}]$  the homomorphism  $e_m(D)(\xi)$  which coincides with the quotient map

$$\theta_n : \mathbb{C}[T, T^{-1}] \rightarrow \mathbb{C}[\mathbb{Z}/n\mathbb{Z}], \quad T \mapsto \xi \in \mathbb{Z}/n\mathbb{Z}.$$

This means that  $\theta_n(f) \in \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ , for all  $n$ . For  $f \in \mathbb{C}[T, T^{-1}]$  one can compute the coefficient of  $T^k$  as the limit

$$b_k = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{a=1}^n f(e^{2\pi i \frac{a}{n}}) e^{-2\pi i k \frac{a}{n}}.$$

When  $f(x) = x^m$ , the sum  $\sum_{a=1}^n f(e^{2\pi i \frac{a}{n}}) e^{-2\pi i k \frac{a}{n}}$  is either zero or  $n$  and the latter case only happens if  $m - k$  is a multiple of  $n$ . Thus,  $\frac{1}{n} \sum_{a=1}^n f(e^{2\pi i \frac{a}{n}}) e^{-2\pi i k \frac{a}{n}}$ , which only depends on  $\theta_n(f)$ , is a relative integer if  $\theta_n(f) \in \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ . It follows that all the  $b_k$  are in  $\mathbb{Z}$  and hence that  $f \in \mathbb{Z}[T, T^{-1}]$ . Thus  $\phi^*(\mathcal{O}) \subset \mathbb{Z}[T, T^{-1}]$  and this uniquely defines  $\psi = \phi_{\mathbb{Z}} \in \text{Hom}(\underline{\mathbb{G}}_{m, \mathbb{Z}}, V)$ . To check that  $\phi = \mathcal{G}(\psi) \circ i$  one uses the injectivity of the map  $\text{Hom}_{\mathbb{Z}}(\mathcal{O}, \beta(D)) \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}}, \beta(D)_{\mathbb{C}})$ .  $\square$

**3.3. The affine space  $\mathbb{A}^F$ .** For the affine line  $\mathbb{A}^1$ , the number of points of  $\mathbb{A}^1(\mathbb{F}_q)$  is  $N(q) = q$ . Thus, the Taylor expansion of  $N(q)$  at  $q = 1$  is  $q = 1 + (q - 1)$  and has two terms in degree 0 and 1. This suggests to refine the definition of the corresponding functor of [25] as follows. We define  $\underline{\mathbb{A}}^1$  as the graded functor

$$\underline{\mathbb{A}}^1 : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets, \quad \underline{\mathbb{A}}^1(D)^{(k)} = \begin{cases} \{0\} & \text{if } k = 0 \\ D & \text{if } k = 1 \\ \emptyset & \text{if } k \geq 2. \end{cases} \quad (24)$$

More generally, one may introduce for any finite set  $F$  the graded functor

$$\underline{\mathbb{A}}^F(D)^{(k)} = \coprod_{\substack{Y \subset F, \\ |Y|=k}} D^Y. \quad (25)$$

which is just the graded product  $(\{0\} \cup D)^F$ .

**Proposition 3.2.** *Let  $e_F : \underline{\mathbb{A}}^F \rightarrow \text{Hom}(\text{Spec } \mathbb{C}[-], \mathbb{C}^F)$  be the natural transformation from the functor  $\underline{\mathbb{A}}^F$  to the functor  $D \mapsto \text{Hom}(\text{Spec } \mathbb{C}[D], \mathbb{C}^F)$  which assigns to a point in  $\text{Spec } \mathbb{C}[D]$ , i.e. to a character  $\chi : \mathbb{C}[D] \rightarrow \mathbb{C}^*$ , the following map*

$$\coprod_{Y \subset F} D^Y \rightarrow \mathbb{C}^F, \quad e_F(D)((g_j)_{j \in Y}) = (\xi_j)_{j \in F}, \quad \xi_j = \begin{cases} \chi(g_j) & \text{if } j \in Y; \\ 0 & \text{if } j \notin Y. \end{cases} \quad (26)$$

Then, the gadget  $\mathbb{A}^F = (\underline{\mathbb{A}}^F, \mathbb{C}^F, e_F)$  defines a variety over  $\mathbb{F}_1$ .

The proof is identical to that of [25].

**3.4. Projective space  $\mathbb{P}^d$ .** In [25], after defining the category  $\mathcal{A}$  of affine varieties over  $\mathbb{F}_1$ , the general case is obtained using contravariant functors from  $\mathcal{A}$  to  $\mathcal{S}ets$ , together with a global  $\mathbb{C}$ -algebra of functions. In the present paper we deal exclusively with the affine case and only briefly mention how the counting of points is affected by the grading in the case of projective space. We adopt the Definition 2.5 in the non-affine case. As for schemes, it is natural to require the existence of an open covering by affine open sets  $U_\alpha$  such that, on each of them, the subfunctor

$$\underline{X}_\alpha(D) = \{x \in \underline{X}(D) \mid e_X(x) \in \text{Hom}(\text{Spec}(\mathbb{C}[D], U_\alpha)\}$$

of the functor  $\underline{X}$  (together with the affine variety  $U_\alpha$  over  $\mathbb{C}$  and the restriction of the natural transformation  $e_X$ ) defines an affine variety over  $\mathbb{F}_1$ . One also requires that the  $\underline{X}_\alpha$  cover  $\underline{X}$  i.e. that for any  $D$  one has  $\underline{X}(D) = \cup \underline{X}_\alpha(D)$ . One can then rely on Proposition 5 of §4.4 of [25] to do the patching. In fact we shall obtain in §5 a general notion of scheme over  $\mathbb{F}_1$ . We shall simply explain here, in the case of projective space  $\mathbb{P}^d$ , how to implement the grading. More precisely, we define  $\underline{\mathbb{P}}^d$  as the following graded functor

$$\underline{\mathbb{P}}^d : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets, \quad \underline{\mathbb{P}}^d(D)^{(k)} = \coprod_{\substack{Y \subset \{1, 2, \dots, d+1\} \\ |Y|=k+1}} D^Y / D, \quad k \geq 0 \quad (27)$$

where the right action of  $D$  on  $D^Y$  is the diagonal action. It follows that the points of lowest degree in  $\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$  are simply labeled by  $\{1, 2, \dots, d+1\}$ . Their number is evidently

$$\# \underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)} = d+1. \quad (28)$$

In particular, this shows that  $\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$  coincides *in degree zero* with the  $d+1$  points of the set  $\mathcal{P}_d$  on which Tits defines his notion of projective geometry of dimension  $d$  over  $\mathbb{F}_1$ . It is striking that the right hand side of the formula (28) is independent of  $n$ . This result is also in agreement with the evaluation at  $q = 1^n$  of the counting function of the set  $\mathbb{P}^d(\mathbb{F}_q)$ , namely (with the evaluation at  $q = 1^n$ ) of the function  $N(q) = \sum_{j=0}^d q^j$ .

#### 4. CHEVALLEY GROUP SCHEMES

The main result of this section (Theorem 4.10) is the proof that Chevalley groups give rise naturally to affine varieties over (an extension of)  $\mathbb{F}_1$ . To achieve this result we shall need to apply the full theory of Chevalley groups both in the classical (*i.e.* Lie-theoretical) and algebraic group theoretical development.

If  $\mathbb{K}$  is an algebraically closed field, a Chevalley group  $G$  over  $\mathbb{K}$  is a connected, semi-simple, linear algebraic group over  $\mathbb{K}$ . By definition of a linear algebraic group over  $\mathbb{K}$ ,  $G$  is isomorphic to a closed subgroup of some  $GL_n(\mathbb{K})$ . The coordinate ring of  $G$ , as affine linear algebraic variety over  $\mathbb{K}$ , is then  $\mathbb{K}[G] = \mathbb{K}[x_{ij}, d^{-1}]/I$  *i.e.* a quotient ring of polynomials in  $n^2$  variables with determinant  $d$  inverted by a prime ideal  $I$ .

As an algebraic group over  $\mathbb{K}$ ,  $G$  is also endowed with a group structure respecting the algebraic structure, *i.e.*  $G$  is endowed with the following two morphisms of varieties over  $\mathbb{K}$

$$\mu : G \times G \rightarrow G, \mu(x, y) = xy; \quad \iota : G \rightarrow G, \iota(x) = x^{-1} \quad (29)$$

Notice that by construction  $\mathbb{K}[G]$  is a Hopf algebra whose coproduct encodes the group structure.

Let  $k$  be the prime field of  $\mathbb{K}$  and  $K$  an intermediate field:  $k \subset K \subset \mathbb{K}$ . Then, the group  $G$  is said to be *defined* over  $K$  if the affine variety  $G$  and the group structure are defined over  $K$  and, by extension of scalars, also over any field above it (*cf.* [21], Chapter 2 (2.1.1), p. 63). In terms of the Hopf algebra structure, one is given a Hopf algebra  $A$  over  $K$  such that, as Hopf algebras

$$\mathbb{K}[G] = A \otimes_K \mathbb{K}.$$

The property for the group  $G$  to be *split* over  $K$  means that some maximal torus  $T \subset G$  is  $K$ -isomorphic to  $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$  ( $d$  copies,  $d = \dim T$ ) (*cf.* [17], Chapter XII, §§ 34.3, 34.4 p. 219-20).

If  $G$  is a linear algebraic group over  $K$ , the set

$$G(K) = \text{Hom}_K(A, K)$$

is a group called the group of  *$K$ -rational points* of  $G$ . One has an identification  $G(K) = G \cap \mathbb{A}_K^{n^2}$  using  $K$ -polynomials to generate the ideal  $I$  (*cf. op.cit.* Chapter XII, § 34.1, p. 218).

To a semi-simple, connected algebraic group  $G$  defined over  $K$  and a  $K$ -split, maximal torus  $T \subset G$ , one associates the group  $\text{Hom}(T, \mathbb{G}_m)$  ( $\mathbb{G}_m = GL_1(K)$ ): this is a free abelian group of rank equal to the dimension of  $T$ . The group  $\mathbb{R} \otimes_{\mathbb{Z}} \text{Hom}(T, \mathbb{G}_m)$  plays in this context, the role of the dual  $\mathfrak{h}^*$  of a Cartan Lie-algebra.

One shows that there exist sub-tori  $S \subset T$ ,  $\dim S = \dim T - 1$ , whose centralizers  $Z(S)$  are of dimension  $\dim T + 2$  and such that  $Z(S)/S$  is isomorphic either to  $SL_2(K)$  or  $PGL_2(K)$ . The study of these groups allows one to introduce pairs of elements  $\pm\alpha \in \text{Hom}(T, \mathbb{G}_m)$  and by varying  $S$  one defines a full set of roots  $\Phi$  (*cf. op.cit.* Chapter XII, § 34.5 and [4], § 25.7).

If  $N$  denotes the normalizer of the torus  $T \subset G$ , then the (finite) group  $W = N/T$  acts on  $\mathbb{R} \otimes_{\mathbb{Z}} \text{Hom}(T, \mathbb{G}_m)$  and is called the ( $K$ -) *Weyl group*.

The theory of Chevalley groups over a field  $K$  has been further extended in [3] and [14]. To every semi-simple, complex Lie group  $G$ , and more generally to an abstract root system, one associates canonically a group scheme  $\mathfrak{G}$  over  $\mathbb{Z}$ , such that  $G$  gets

identified with the group  $\mathfrak{G}(\mathbb{C})$  of complex points of  $\mathfrak{G}$ . We shall return to this construction in § 4.3.

In the next paragraphs we shall first review and then apply a construction due to M. Demazure and J. Tits (*cf.* [14], [27]) which associates to an algebraic reductive group  $G$  defined over  $K$  and a  $K$ -split maximal torus  $T \subset G$ , a *canonical* extension of the Weyl group  $W$ , obtained by considering the groups of  $K$ -rational points of  $T$  and of its normalizer. This construction makes explicit use of a suitable extension of the Weyl group (so called the extended Weyl group) whose definition is *independent* of the field  $K$  and is given only in terms of the root system of  $G$ .

The notion of extended Weyl group is related to that of extended Coxeter group  $V$  associated to a Coxeter matrix  $M$  and a given abstract root system  $\{L, \Phi, n_r\}$ . The group  $V$  is a certain extension of the Coxeter group of  $M$  by a free abelian group whose rank equals the cardinality of the set of the reflections associated to the roots.

**4.1. Root systems and Coxeter groups.** In this paragraph we follow § 4.1 and § 2.2 of [27] and we shall review several fundamental notions associated to the notion of a root system.

A root system  $\{L, \Phi, n_r\}$  is the data given by:

- a lattice  $L$ , *i.e.* a free abelian group of finite rank (the group of weights);
- a finite subset  $\Phi \subset L$  (the set of roots);
- for each  $r \in \Phi$ , a  $\mathbb{Z}$ -valued linear form  $n_r : L \rightarrow \mathbb{Z}$  (the co-root associated to  $r$ )

which satisfy the following axioms:

- (1)  $L \otimes \mathbb{Q}$  is generated, as  $\mathbb{Q}$ -vector space, by  $\Phi$  and by the intersection of the kernels of the  $n_r$ ;
- (2)  $n_r(r) = 2, \forall r \in \Phi$ ;
- (3) the relations  $r \in \Phi, ar \in \Phi, a \in \mathbb{Q}$  imply  $a = \pm 1$ ;
- (4) if  $r, s \in \Phi$ , then  $r - n_s(r)s \in \Phi$ .

For each  $r \in \Phi$ , the *reflection associated to  $r$*  is the map  $s_r : L \xrightarrow{\sim} L$  defined by  $s_r(x) = x - n_r(x) \cdot r$ . The following equality holds  $s_r = s_{-r}$ .

We recall (*cf.* [14], Exposé XXI) that the lattice  $L$  can be endowed with a total order which divides the set  $\Phi$  of the roots into two disjoint sets: positive and negative roots. The set of positive roots  $\Phi^+$  is a subset of  $\Phi$  satisfying the conditions:

- if  $r_1, r_2 \in \Phi^+$  then  $r_1 + r_2 \in \Phi^+$
- for each  $r \in \Phi$  exactly one of the conditions  $r \in \Phi^+, -r \in \Phi^+$  holds.

One then lets  $\Phi^o \subset \Phi^+$  be the set of simple (indecomposable) roots of  $\Phi^+$ . It is a collection  $\Phi^o = \{\rho_i, i \in \Pi\} \subset \Phi$  ( $\Pi =$  finite set) of linearly independent roots such that every root can be written as an integral linear combination of the  $\rho_i$ , with integer coefficients either all non-negative or all non-positive. Then, the square matrix  $M = (m_{ij}), i, j \in \Pi$ , with  $2m_{ij}$  equal to the number of roots which are a linear combination of the  $\rho_i$  and  $\rho_j$ , is a *Coxeter matrix i.e.* a symmetric square matrix with diagonal elements equal to 1 and off diagonal ones positive integers  $\geq 2$ .

The *Coxeter group*  $W = W(M)$  associated to  $M$  is defined by a system of generators  $\{r_i, i \in \Pi\}$  (the *fundamental reflections*) and relations

$$(r_i r_j)^{m_{ij}} = 1, \forall i, j \in \Pi. \quad (30)$$

The group generated by the reflections associated to the roots is canonically isomorphic to the Coxeter group  $W(M)$ : the isomorphism is defined by sending the reflection associated to a simple root to the corresponding generator, *i.e.*  $s_{\rho_i} \mapsto r_i$ . The root system  $\{L, \Phi, n_r\}$  is said to be *simply connected* if the co-roots  $n_r$  generate the dual lattice  $L' = \text{Hom}(L, \mathbb{Z})$  of  $L$ . In this case, the co-roots  $n_{\rho_i}$  determine a basis of  $L'$ .

Given a root system  $\{L, \Phi, n_r\}$ , there exists a simply connected root system  $\{\tilde{L}, \tilde{\Phi}, n_{\tilde{r}}\}$  and a homomorphism  $\varphi : L \rightarrow \tilde{L}$  uniquely determined, up-to isomorphism, by the following two conditions:

$$\varphi(\Phi) = \tilde{\Phi}, \quad n_r = n_{\varphi(r)} \circ \varphi, \quad \forall r \in \Phi. \quad (31)$$

The restriction of  $\varphi$  to  $\Phi$  determines a bijection of  $\Phi$  with  $\tilde{\Phi}$ .

The *Braid group*  $B = B(M)$  associated to a Coxeter matrix  $M$  is defined by generators  $\{q_i, i \in \Pi\}$  and the relations

$$\text{prod}(m_{ij}; q_i, q_j) := \underbrace{\cdots q_i q_j q_i q_j}_{m_{ij}} = \text{prod}(m_{ij}; q_j, q_i), \quad \forall i, j \in \Pi. \quad (32)$$

The *extended Coxeter group*  $V = V(M)$  associated to a Coxeter matrix  $M$  is the quotient of  $B(M)$  by the commutator subgroup of the kernel  $X(M)$  of the canonical surjective homomorphism  $B(M) \rightarrow W(M)$ . It is defined by generators and relations as follows. One lets  $S$  be the set of elements of  $W$  which are conjugate to one of the  $r_i$ , *i.e.* the set of reflections (*cf.* [27], §1.2). One considers two sets of generators:

$$\{q_i, i \in \Pi\}, \quad \{g(s), s \in S\}$$

The relations are given by (32) and the following:

- (1)  $q_i^2 = g(r_i), \quad \forall i \in \Pi,$
- (2)  $q_i \cdot g(s) \cdot q_i^{-1} = g(r_i(s)), \quad \forall s \in S, i \in \Pi.$
- (3)  $[g(s), g(s')] = 1, \quad \forall s, s' \in S.$

Tits shows in Théorème 2.5 of [27] that the subgroup  $U = U(M) \subset V$  generated by the  $g(s)$  ( $s \in S$ ) is a free, abelian, normal subgroup of  $V$  that coincides with the kernel of the natural surjective map  $f : V \rightarrow W$ ,  $f(q_i) = r_i$  (and  $f(g(s)) = 1$ ). The group  $U = U(M)$  is the quotient of  $X(M)$  by its commutator subgroup.

In the following paragraph we shall recall the construction of [27] of the extended Weyl group using the extended Coxeter group  $V$ .

**4.2. The group  $\mathcal{N}_{D, \epsilon}(L, \Phi)$ .** We keep the same notation as in the previous paragraph. In § 4.3 of [27] Tits introduces, for a given root system  $\{L, \Phi, n_r\}$ , a functor

$$(D, \epsilon) \rightarrow \{N, p, N_s; s \in S\} = \mathcal{N}$$

which associates to the pair  $(D, \epsilon)$  of an abelian group and an element  $\epsilon \in D$  with  $\epsilon^2 = 1$ , the data (*i.e.* an object  $\mathcal{N}$  of a suitable category) given by a group  $N = \mathcal{N}_{D, \epsilon}(L, \Phi)$ , a surjective homomorphism of groups  $p : N \rightarrow W = W(M)$  and for each reflection  $s \in S$ , a subgroup  $N_s \subset N$  satisfying the following conditions:

- (n1)  $\text{Ker}(p)$  is an abelian group;
- (n2) For  $s \in S$ ,  $n \in N$  and  $w = p(n)$ ,  $nN_s n^{-1} = N_{w(s)}$ ;
- (n3)  $p(N_s) = \{1, s\}, \quad \forall s \in S.$

A morphism connecting two objects  $\mathcal{N}$  and  $\mathcal{N}'$  is a homomorphism  $a : N \rightarrow N'$  such that  $p' \circ a = p$  and  $a(N_s) \subset N'_s$  for all  $s \in S$  (cf. § 3 of [27]).

Notice, in particular, that the data  $\{V, f, V_s; s \in S\}$ , characterizing the extended Coxeter group, where  $V_s \subset V$  is the subgroup generated by  $Q_s = \{v \in V, v^2 = g(s)\}$ , satisfy (n1)-(n3).

The object  $\mathcal{N}$  is obtained by the following canonical construction. One considers the abelian group  $T = \text{Hom}(L, D)$  endowed with the natural (left) action of  $W$  (denoted by  $t \mapsto w(t)$ , for  $w \in W, t \in T$ ) induced by the corresponding action on  $L$  (generated by the reflections associated to the roots). For each  $r \in \Phi$ , let  $s = s_r \in W$  be the reflection associated to the root  $r$ . One lets  $T_s$  be the subgroup of  $T$  made by homomorphisms of the form

$$L \ni x \mapsto a^{\nu(x)} \quad (33)$$

for some  $a \in D$  and where  $\nu : L \rightarrow \mathbb{Z}$  is a linear form proportional to  $n_r$ . Also, one defines (for  $s = s_r$ )

$$h_s(x) = \epsilon^{n_r(x)}, \quad x \in L. \quad (34)$$

(note that replacing  $r \rightarrow -r$  does not alter the result since  $\epsilon^2 = 1$ ). The formula (34) determines a map  $h : S \rightarrow T$ . Then, the data  $\{T, T_s, h_s; s \in S\}$  fulfill the following conditions  $\forall w \in W, s \in S$ :

- (1)  $w(T_s) = T_{w(s)}$ ;
- (2)  $w(h_s) = h_{w(s)}$ ;
- (3)  $h_s \in T_s$ ;
- (4)  $s(t) \cdot t^{-1} \in T_s, \quad \forall t \in T$ ;
- (5)  $s(t) = t^{-1}, \quad \forall t \in T_s$ .

One then obtains the object  $\mathcal{N}$  as follows (cf. Proposition 3.4 of [27]). One defines the group  $N = \mathcal{N}_{D, \epsilon}(L, \Phi)$  as the quotient of the semi-direct product group  $T \rtimes V$  ( $V =$  extended Coxeter group) by the graph of the homomorphism  $U \rightarrow T$  ( $U = U(M)$ ) which extends the map  $g(s) \mapsto h_s^{-1}$ . One identifies  $T$  with its canonical image in  $N$  and for each  $s \in S$ , one lets  $N_s$  be the subgroup of  $N$  generated by the canonical image of  $T_s \times Q_s$ , where  $T_s$  is as above and  $Q_s = \{v \in V, v^2 = g(s)\}$ . The surjective group homomorphism  $p : N \rightarrow W = W(M)$  is induced by  $id \times f$  where  $f$  is the canonical group homomorphism  $f : V \rightarrow W = W(M)$ . By construction one has a morphism connecting the data  $\{V, f, V_s; s \in S\}$  to  $\{N, p, N_s; s \in S\} = \mathcal{N}$ . It gives a homomorphism  $a : V \rightarrow N$  such that, in particular,

$$a(g(s)) = h_s, \quad \forall s \in S. \quad (35)$$

More precisely, one has the following result (cf. § 3.4 and § 4.3 of [27])

**Proposition 4.1.** *The data  $\mathcal{N} = \{N, p, N_s; s \in S\}$  satisfy the conditions (n1)-(n3). Moreover, every map*

$$\alpha : \{q_i, i \in \Pi\} \rightarrow N$$

*such that  $\alpha(q_i) \in N_{r_i} \setminus T_{r_i} = N_{r_i} \cap p^{-1}(r_i)$  extends to a homomorphism of groups  $V \rightarrow N$ .*

We collect together, for an easy reference, the main properties of the construction of [27] reviewed in this paragraph.



**Proposition 4.2.** *Let  $\{L, \Phi, n_r\}$  be a root system. To a pair  $(D, \epsilon)$  of an abelian group and an element  $\epsilon \in D$ ,  $\epsilon^2 = 1$ , corresponds a canonical extension  $\mathcal{N}_{D, \epsilon}(L, \Phi)$  of the Coxeter group  $W$  by  $T = \text{Hom}(L, D)$*

$$1 \rightarrow T \rightarrow N \xrightarrow{p} W \rightarrow 1 \quad (36)$$

and for each reflection  $s \in S$  a subgroup  $N_s \subset N$ . These data satisfy the following properties:

- $nN_s n^{-1} = N_{w(s)}$ , for  $s \in S$ ,  $n \in N$  and  $w = p(n)$ .
- $p(N_s) = \{1, s\}$ ,  $\forall s \in S$ .
- $N_s \cap T = T_s$ ,  $\forall s \in S$ .
- $a^2 = h_s \in T_s$ ,  $\forall a \in p^{-1}(s) = N_s \setminus T_s$ .
- For each pair  $i \neq j$  in  $\Pi$ , let  $m = m_{ij}$  be the order of  $r_i r_j \in W$ , then

$$\text{prod}(m; a_i, a_j) = \text{prod}(m; a_j, a_i), \quad \forall a_k \in N_{r_k}, \quad p(a_k) = r_k \neq 1. \quad (37)$$

The canonical extension  $\mathcal{N}_{D, \epsilon}(L, \Phi)$  of  $W$  by  $T$  is functorial in the pair  $(D, \epsilon)$ , with respect to morphisms  $t : D \rightarrow D'$  such that  $t(\epsilon) = \epsilon'$ .

The meaning of equation (37) is the following one. Once a choice of a section  $W \supset \Phi^\circ \ni s \mapsto \alpha(s) \in N_s$  of the map  $p$  has been made on the set of simple roots  $\Phi^\circ \subset W$ , this section admits a natural extension to all of  $W$  as follows. One writes  $w \in W$  as a word of minimal length  $w = \rho_1 \cdots \rho_k$ , in the generators  $\rho_j \in \Phi^\circ$ . Then (37) ensures that the corresponding product  $\alpha(w) = \alpha(\rho_1) \cdots \alpha(\rho_k) \in N$  is independent of the choice of the word of minimal length representing  $w$  (cf. [27] Proposition 2.1).

**4.3. Chevalley Schemes.** We keep the notation of § 4.1 and § 4.2. To a root system  $\{L, \Phi, n_r\}$  one associates, following [3] and [14], a reductive group scheme  $\mathfrak{G} = \mathfrak{G}\{L, \Phi, n_r\}$  over  $\mathbb{Z}$ : the *Chevalley scheme*. We denote by  $\mathcal{T}$  a maximal torus that is part of a split structure of  $\mathfrak{G}$  and by  $\mathcal{N}$  its normalizer.

To a reflection  $s \in S$  correspond naturally the following data: a one dimensional sub-torus  $\mathcal{T}_s \subset \mathcal{T}$ , a rank one semi-simple subgroup  $\mathfrak{G}_s \subset \mathfrak{G}$  containing  $\mathcal{T}_s$  as a maximal torus and a point  $h_s \in \mathcal{T}_s$  (belonging to the center of  $\mathfrak{G}_s$ ). One denotes by  $\mathcal{N}_s$  the normalizer of  $\mathcal{T}_s$  in  $\mathfrak{G}_s$ .

Let  $A$  be a commutative ring with unit and let  $A^*$  be its multiplicative group. We denote by  $\mathfrak{G}(A)$ ,  $\mathcal{T}(A)$ , *resp.*  $\mathcal{N}(A)$  the groups of points of  $\mathfrak{G}$ ,  $\mathcal{T}$ , *resp.*  $\mathcal{N}$  which are rational over  $A$ . The quotient  $\mathcal{N}(A)/\mathcal{T}(A)$  is canonically isomorphic to  $W = W(M)$ . More precisely, there exists a unique surjective homomorphism  $p_A : \mathcal{N}(A) \rightarrow W(M)$ , whose kernel is  $\mathcal{T}(A)$  so that the data  $\{\mathcal{N}(A), p_A, \mathcal{N}_s(A); s \in S\}$  satisfy the conditions (n1)-(n3) of § 4.2.

The goal of this paragraph is to review a fundamental result of [27] which describes the data above only in terms of  $A$  and the root system  $\{L, \Phi, n_r\}$ . To achieve this result one makes use of the following facts:

- The group  $\mathcal{T}(A)$  is canonically isomorphic to  $\text{Hom}_{\mathbb{Z}}(L, A^*)$  and the left action of  $W(M) \simeq \mathcal{N}(A)/\mathcal{T}(A)$  on  $\mathcal{T}(A)$  is induced from the natural action of  $W$  on  $L$ .
- If  $s = s_r$  is the reflection associated to a root  $r \in \Phi$ , then

$$\mathcal{N}_{s_r}(A) \cap \mathcal{T}(A) = \mathcal{T}_{s_r}(A) = \{\rho \in \text{Hom}(L, A^*) \mid \exists a \in A^*, \rho(x) = a^{\nu(x)}, \forall x \in L\}$$

where  $\nu : L \rightarrow \mathbb{Z}$  is a linear form proportional to  $n_r$  (cf. § 4.2). Taking  $a = -1$  and  $\nu = n_r$ , one gets the element  $h_s(A) \in \mathcal{T}_s(A)$ .

- The normalizer  $\mathcal{N}_s$  of  $\mathcal{T}_s$  in  $\mathfrak{G}_s$  is such that all elements of  $\mathcal{N}_s(A)$  which are not in  $\mathcal{T}_s(A)$  have a square equal to  $h_s(A) \in \mathcal{T}_s(A)$ .

We are now ready to state Theorem 4.4 of [27] which plays a key role in our construction.

**Theorem 4.3.** *The group extension*

$$1 \rightarrow \mathcal{T}(A) \rightarrow \mathcal{N}(A) \xrightarrow{p} W \rightarrow 1$$

*is canonically isomorphic to the group extension*

$$1 \rightarrow \text{Hom}(L, A^*) \rightarrow \mathcal{N}_{A^*, -1}(L, \Phi) \xrightarrow{p} W \rightarrow 1.$$

Here  $\mathcal{N}_{A^*, -1}(L, \Phi)$  refers to the functorial construction of Proposition 4.2 for the group  $D = A^*$  and  $\epsilon = -1 \in D$ . Note that the case  $A = \mathbb{Z}$  corresponds to  $D = \{\pm 1\}$ ,  $\epsilon = -1$ , and gives the extension  $\mathcal{N}(\mathbb{Z})$  of  $W$  by  $\text{Hom}(L, \{\pm 1\}) \simeq (\mathbb{Z}/2\mathbb{Z})^\ell$ . This particular case contains the essence of the general construction since for any pair  $(D, \epsilon)$  the group  $\mathcal{N}_{D, \epsilon}(L, \Phi)$  is the amalgamated semi-direct product

$$\text{Hom}(L, D) \rtimes_{\text{Hom}(L, \{\pm 1\})} \mathcal{N}(\mathbb{Z}).$$

**4.4. Bruhat decomposition.** We keep the notation as in the earlier paragraphs of this section. To each root  $r \in \Phi$  corresponds a root subgroup  $\mathfrak{X}_r \subset \mathfrak{G}$  defined as the range of an isomorphism  $x_r$  from the additive group  $\mathbb{G}_{a, \mathbb{Z}}$  to its image in  $\mathfrak{G}$  and fulfilling the equation

$$hx_r(\xi)h^{-1} = x_r(r(h)\xi), \quad \forall h \in \mathcal{T}. \quad (38)$$

We recall the following standard notation and well-known relations:

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t), \quad \forall t \in A^*, \quad n_r = n_r(1) \quad (39)$$

$$h_r(t) = n_r(t)n_r(-1), \quad \forall t \in A^*, \quad h_r = h_r(-1), \quad h_r(t_1)h_r(t_2) = h_r(t_1t_2). \quad (40)$$

Let  $r, s \in \Phi$  be linearly independent roots and  $t, u \in A$ . The *commutator* of  $x_s(u)$  and  $x_r(t)$  is defined as

$$[x_s(u), x_r(t)] = x_s(u)^{-1}x_r(t)^{-1}x_s(u)x_r(t). \quad (41)$$

The following formula, due to Chevalley, expresses the above commutator as a product of generators corresponding to roots of the form  $ir + js \in \Phi$ , for  $i, j > 0$

**Lemma 4.4.** *Let  $r, s \in \Phi$  be linearly independent roots. Then there exist integers  $C_{ijrs} \in \mathbb{Z}$  such that*

$$x_s(u)^{-1}x_r(t)x_s(u)x_r(t)^{-1} = \prod_{i,j} x_{ir+js}(t^i u^j)^{C_{ijrs}} \quad (42)$$

*where the product is applied to pairs  $(i, j)$  of strictly positive integers such that  $ir + js \in \Phi$ , and the terms are arranged in order of increasing  $i + j$ .*

The above formula holds in particular in the case that  $r, s$  belong to the subset  $\Phi^+ \subset \Phi$  of the positive roots (cf. 4.1 and [14], Exposé XXI), in which case the product is taken over all positive roots of the form  $ir + js$ ,  $i > 0$ ,  $j > 0$  in increasing order, for the chosen ordering of the lattice  $L$  (cf. [14], Exposé XXII, Lemme 5.5.6 p. 208).

Let  $\mathcal{U}(A)$  be the subgroup of  $\mathfrak{G}(A)$  generated by the elements  $x_r(t)$  for  $r \in \Phi^+$ ,  $t \in A$ . By construction, the subgroup  $\mathcal{U} \subset \mathfrak{G}$  is generated by the root subgroups  $\mathfrak{X}_r$

corresponding to the positive roots. For any  $w \in W$ , we let  $\Phi_w = \{r \in \Phi^+ | w(r) < 0\}$  and we denote by  $\mathcal{U}_w$  the subgroup of  $\mathfrak{G}$  generated by the root subgroups  $\mathfrak{X}_r$  for  $r \in \Phi_w$ .

Chevalley proved in [2], Théorème 2 the existence of a canonical form for the elements of the group  $\mathfrak{G}(K)$ , when  $K$  is a field. We recall this result

**Theorem 4.5.** *Let  $K$  be a field. The group  $\mathfrak{G}(K)$  is the disjoint union of the subsets (cells)*

$$C_w = \mathcal{U}(K) \mathcal{T}(K) n_w \mathcal{U}_w(K)$$

where for each  $w \in W$ ,  $n_w \in \mathcal{N}(K)$  is a chosen coset representative for  $w$ . The natural map

$$\varphi_w : \mathcal{U}(K) \times \mathcal{T}(K) \times \mathcal{U}_w(K) \rightarrow C_w, \quad \varphi_w(x, h, x') = x h n_w x' \quad (43)$$

is a bijection for any  $w \in W$ .

We refer to [14], Exposé XXI, Théorème 5.7.4 and Remarque 5.7.5.

**4.5. Chevalley group schemes as gadgets.** For the definition of the gadget over  $\mathbb{F}_1$  associated to a Chevalley group  $G$  and in particular for the construction of the natural transformation  $e_G$  (cf. § 2.2), one needs to choose a Chevalley basis of the (complex) Lie algebra of  $G$  and a total ordering of the lattice  $L$ . We keep the notation as in § 4.4 and in § 4.1 through § 4.3.

Chevalley proved (cf. [2], Lemma 6 and [3]) that, over any commutative ring  $A$ , each element of  $\mathcal{U}(A)$  is *uniquely* expressible in the form

$$\prod_{r \in \Phi^+} x_r(t_r), \quad t_r \in A, \quad \forall r \in \Phi^+$$

where the product is taken over all positive roots in increasing order. More precisely one has the following

**Lemma 4.6.** *The map*

$$t = (t_r)_{r \in \Phi^+} \mapsto \psi(t) = \prod_{r \in \Phi^+} x_r(t_r) \quad (44)$$

*establishes a bijection of the free  $A$ -module with basis the positive roots in  $\Phi^+$  with  $\mathcal{U}(A)$ .*

The proof of this lemma applies without change to give the following variant, where we let  $\Phi_w = \{r \in \Phi^+ | w(r) < 0, w \in W\}$  and we denote by  $\mathcal{U}_w = \prod_{r \in \Phi_w} \mathfrak{X}_r$ .

**Lemma 4.7.** *The map*

$$t = (t_r)_{r \in \Phi_w} \mapsto \psi_w(t) = \prod_{r \in \Phi_w} x_r(t_r) \quad (45)$$

*establishes a bijection of the free  $A$ -module with basis  $\Phi_w$  with  $\mathcal{U}_w(A)$ .*

The key identity in the proof of Lemmas 4.6 and 4.7 is the commutator relation of Lemma 4.4.

We are now ready to apply the theory reviewed in the previous paragraphs to construct the functor

$$\underline{G} : \mathcal{F}_{ab}^{(2)} \rightarrow \mathcal{S}ets$$

from the category  $\mathcal{F}_{ab}^{(2)}$  of pairs  $(D, \epsilon)$  of a finite abelian group and an element of square one, to the category of graded sets.

**Definition 4.8.** *The functor  $\underline{G} : \mathcal{F}_{ab}^{(2)} \rightarrow \text{Sets}$  is defined as the graded product*

$$\underline{G}(D, \epsilon) = \underline{\mathbb{A}}^{\Phi^+}(D) \times \prod_{w \in W} (p^{-1}(w) \times \underline{\mathbb{A}}^{\Phi_w}(D)) \quad (46)$$

where  $p$  is the projection  $\mathcal{N}_{D, \epsilon}(L, \Phi) \xrightarrow{p} W$  as in §§4.2 and 4.3. All elements of  $p^{-1}(w)$  have degree equal to the rank of  $\mathfrak{G}$ .

It follows immediately that there are no elements of degree less than the rank  $\ell$  of  $\mathfrak{G}$  and that the set of elements of degree  $\ell$  is canonically identified with  $\mathcal{N}_{D, \epsilon}(L, \Phi)$ .

We now move to the definition of the natural transformation

$$e_G : \underline{G} \rightarrow \text{Hom}(\text{Spec } \mathbb{C}[-], G_{\mathbb{C}}), \quad (D, \epsilon) \mapsto \text{Hom}(\text{Spec } \mathbb{C}[D, \epsilon], G_{\mathbb{C}}).$$

For this part, we make use of the natural transformations  $e_F$  of (26) for  $F = \Phi^+$  and  $F = \Phi_w$  and of Theorem 4.3 to obtain, for a given character  $\chi$ ,  $\chi(\epsilon) = -1$ , associated to a point in  $\text{Spec } \mathbb{C}[D, \epsilon]$ , maps

$$e_{\Phi^+} : \underline{\mathbb{A}}^{\Phi^+}(D) \rightarrow \mathbb{C}^{\Phi^+} \quad (47)$$

$$e_{\Phi_w} : \underline{\mathbb{A}}^{\Phi_w}(D) \rightarrow \mathbb{C}^{\Phi_w} \quad (48)$$

$$e_{\mathcal{N}} : \mathcal{N}_{D, \epsilon}(L, \Phi) \rightarrow \mathcal{N}(\mathbb{C}). \quad (49)$$

The last map is compatible with the projection  $p$ , thus maps  $p^{-1}(w)$  to  $p^{-1}(w)$  for any  $w \in W$ . We now make use of Lemmas 4.6 and 4.7 to obtain the natural transformation  $e_G$  defined as follows

$$e_G(a, n, b) = \psi(e_{\Phi^+}(a)) e_{\mathcal{N}}(n) \psi_w(e_{\Phi_w}(b)) \in \mathfrak{G}(\mathbb{C}) = G_{\mathbb{C}} \quad (50)$$

where  $\psi$  and  $\psi_w$  are defined as in Lemmas 4.6 and 4.7, for  $A = \mathbb{C}$ .

**4.6. Proof that  $G$  determines a variety over  $\mathbb{F}_{12}$ .** In this paragraph we shall prove that the gadget  $G = (\underline{G}, G_{\mathbb{C}}, e_G)$  associated to a Chevalley group  $G$  (or equivalently to its root system  $\{L, \Phi, n_r\}$ , cf. § 4.1) defines a variety over  $\mathbb{F}_{12}$ . We keep the earlier notation. We first recall the following important result of Chevalley (cf. [3], Proposition 1).

**Proposition 4.9.** *Let  $w_0 \in W$  be the unique element of the Weyl group such that  $w_0(\Phi^+) = -\Phi^+$  and let  $w'_0$  be a lift of  $w_0$  in  $G_{\mathbb{Z}}$ . Consider the following morphism, associated to the product in the group,*

$$\theta : \mathcal{U} \times p^{-1}(w_0) \times \mathcal{U} \rightarrow \mathfrak{G}, \quad \theta(u, n, v) = unv \quad (51)$$

Then  $\theta$  defines an isomorphism of  $\mathcal{U} \times p^{-1}(w_0) \times \mathcal{U}$  with an open affine (dense) subscheme  $\Omega$  of  $\mathfrak{G}$ , whose global algebra of coordinates is of the form

$$\mathcal{O}_{\Omega} = \mathcal{O}_{\mathfrak{G}}[d^{-1}] \quad (52)$$

where  $d \in \mathcal{O}_{\mathfrak{G}}$  takes the value 1 on  $w'_0$ .

We refer also to proposition 4.1.2 page 172 in [14] combined with proposition 4.1.5. The next theorem shows that the gadget  $G = (\underline{G}, G_{\mathbb{C}}, e_G)$  over  $\mathbb{F}_{12}$  fulfills the condition of Definition 2.8.

**Theorem 4.10.** *The gadget  $G = (\underline{G}, G_{\mathbb{C}}, e_G)$  defines a variety over  $\mathbb{F}_{12}$ .*

*Proof.* By construction  $G$  is a finite and graded gadget. It is easy to guess that the sought for scheme  $G_{\mathbb{Z}}$  over  $\mathbb{Z}$  is the Chevalley group scheme  $\mathfrak{G}$  associated to the root system  $\{L, \Phi, n_r\}$  (cf. eg [14], Corollary 1.2, Exposé XXV). One has by construction an immersion of gadgets  $G \hookrightarrow \mathcal{G}(\mathfrak{G})$ . The injectivity of the map of sets follows from two facts,

- The characters of  $\mathbb{C}[D, \epsilon]$  separate points of  $D$  (cf. (19)).
- The uniqueness of the Bruhat decomposition in  $G_{\mathbb{C}} = \mathfrak{G}(\mathbb{C})$ .

It remains to be checked that  $G$  fulfills the universal property of Definition 2.8. Let  $V = \text{Spec}(\mathcal{O}(V))$  be an affine variety of finite type over  $\mathbb{Z}$  and  $\phi : G \rightarrow \mathcal{G}(V)$ , be a morphism of gadgets. This means that we are given a pair  $(\underline{\phi}, \phi_{\mathbb{C}})$  where

$$\phi_{\mathbb{C}} : \mathcal{O}_{\mathbb{C}}(V) \rightarrow \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$$

is a homomorphism of  $\mathbb{C}$ -algebras, and  $\underline{\phi}$  is a natural transformation of functors from pairs  $(D, \epsilon)$  to sets

$$\underline{\phi}(D, \epsilon) : \underline{G}(D, \epsilon) \rightarrow \text{Hom}(\mathcal{O}(V), \beta(D, \epsilon))$$

which satisfies the following compatibility condition (cf. (17)): for any pair  $(D, \epsilon)$  the following diagram commutes

$$\begin{array}{ccc} \underline{G}(D, \epsilon) & \xrightarrow{\underline{\phi}(D, \epsilon)} & \text{Hom}(\mathcal{O}(V), \beta(D, \epsilon)) \\ \epsilon_G(D, \epsilon) \downarrow & & \downarrow e_{\mathcal{G}(V)}(D, \epsilon) = \mathbb{C} \\ \text{Hom}(\mathcal{O}_{\mathbb{C}}(\mathfrak{G}), \mathbb{C}[D, \epsilon]) & \xrightarrow{\phi_{\mathbb{C}}} & \text{Hom}(\mathcal{O}_{\mathbb{C}}(V), \mathbb{C}[D, \epsilon]). \end{array} \quad (53)$$

One needs to show that  $\phi_{\mathbb{C}}(\mathcal{O}(V)) \subset \mathcal{O}(\mathfrak{G})$ . Let  $h \in \mathcal{O}(V)$  and  $f = \phi_{\mathbb{C}}(h)$ . Then by construction  $f \in \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$ . By Proposition 4.9, the intersection  $\mathcal{O}_{\mathbb{C}}(\mathfrak{G}) \cap \mathcal{O}_{\Omega}$  coincides with  $\mathcal{O}(\mathfrak{G})$  since  $\mathcal{O}_{\Omega} = \mathcal{O}_{\mathfrak{G}}[d^{-1}]$  while elements of  $\mathcal{O}_{\mathbb{C}}(\mathfrak{G})$  have a trivial pole part in  $d^{-1}$ . Thus it is enough to show that the restriction of  $f$  to the open affine subscheme  $\Omega \subset \mathfrak{G}$  belongs to  $\mathcal{O}_{\Omega}$ , to conclude that  $f \in \mathcal{O}(\mathfrak{G})$ . Let us choose a lift  $w'_0$  of  $w_0$  in  $\mathcal{N}_{\mathbb{Z}}$ . In fact we can more precisely choose a lift  $w'_0$  of  $w_0$  in  $\mathcal{N}_{\mathbb{Z}/2\mathbb{Z}, \epsilon}(L, \Phi)$ , where  $\mathbb{Z}/2\mathbb{Z}$  is the group of order two generated by  $\epsilon$  and then take the image of  $w'_0$  under the map which sends  $\epsilon$  to  $-1$ . We have  $p^{-1}(w_0) = w'_0 \mathcal{T}$ . As in [3] (cf. § 4, Proposition 1), the algebra  $\mathcal{O}_{\Omega}$  is the tensor product of the following three algebras:

- $\mathcal{O}(\mathcal{U})$  which is the algebra of polynomials over  $\mathbb{Z}$  generated by the coordinates  $t_r$  of Lemma 4.6.
- $\mathcal{O}(\mathcal{T}) = \mathbb{Z}[L]$  the group ring of the abelian group  $L$ .
- Another copy of  $\mathcal{O}(\mathcal{U})$ .

We consider elements of  $\underline{G}(D, \epsilon)$  of the form

$$g \in C = \underline{\mathbb{A}}^{\Phi^+}(D) \times p^{-1}(w_0) \times \underline{\mathbb{A}}^{\Phi^+}(D) \quad (54)$$

and use the choice of  $w'_0 \in p^{-1}(w_0) \subset \mathcal{N}_{D, \epsilon}(L, \Phi)$  to identify the cosets  $p^{-1}(w_0) = \text{Hom}(L, D)w'_0$ . Then, we choose generators  $v_j$ ,  $1 \leq j \leq \ell$ , of the free abelian group  $L$  and use them to identify  $\text{Hom}(L, D)$  with the set of  $(y_j)_{j \in \{1, \dots, \ell\}}$ ,  $y_j \in D$ . Each

map  $y$  from  $Y = \Phi \cup \{1, \dots, \ell\}$  to  $D$  defines uniquely an element  $g(y) \in C$  by<sup>6</sup>

$$g(y) = (y_r)_{r \in \Phi^+} \times (y_j)_{j \in \{1, \dots, \ell\}} \times (y_{-r})_{r \in \Phi^+} \quad (55)$$

Then  $g(y) \in \underline{G}(D, \epsilon)$  and  $\underline{\phi}(D, \epsilon)(g(y)) \in \text{Hom}(\mathcal{O}(V), \beta(D, \epsilon))$  so that by evaluating on  $h \in \mathcal{O}(V)$  one gets

$$\underline{\phi}(D, \epsilon)(g(y))(h) \in \mathbb{Z}(D, \epsilon) \subset \mathbb{C}[D, \epsilon]. \quad (56)$$

By the commutativity of the diagram (53), this is the same as evaluating on  $f \in \mathcal{O}_{\mathbb{C}}(\mathfrak{G})$  the homomorphism  $e_G(D, \epsilon)(g(y))$ . We denote by  $k = f|_{\Omega}$  the restriction of  $f$  to  $\Omega$ , it is given by a polynomial with complex coefficients

$$k = P(t_r, u_j, u_j^{-1}) \in \mathbb{C}[t_r, u_j, u_j^{-1}], \quad r \in \Phi, \quad 1 \leq j \leq \ell. \quad (57)$$

Let  $n$  be an integer and  $D = (\mathbb{Z}/n\mathbb{Z})^Y \times \mathbb{Z}/2\mathbb{Z} = D_1 \times \mathbb{Z}/2\mathbb{Z}$  be the group  $D_1$  of maps from  $Y = \Phi \cup \{1, \dots, \ell\}$  to the cyclic group of order  $n$ , times the cyclic group of order two with generator  $\epsilon$  ( $\epsilon^2 = 1$ ). We denote by  $\xi$  the generator of  $\mathbb{Z}/n\mathbb{Z}$  and for  $s \in Y$  we let  $\xi_s \in D_1$  have all its components equal to  $0 \in \mathbb{Z}/n\mathbb{Z}$  except for the component at  $s$  which is  $\xi$ . One has a homomorphism of algebras

$$\theta_n : \mathbb{C}[t_r, u_j, u_j^{-1}] \rightarrow \mathbb{C}[D_1] \sim \mathbb{C}[D, \epsilon], \quad t_r \mapsto \xi_r, \quad u_j \mapsto \xi_j$$

Using (56), we know that for each  $n$ ,  $\theta_n(k) \in \mathbb{Z}[D_1]$ . Now for  $k \in \mathbb{C}[t_r, u_j, u_j^{-1}]$  one can compute the coefficients  $b_I$  of the polynomial as the Fourier coefficients

$$b_I = (2\pi)^{-d} \int_{(S^1)^d} k(e^{i\alpha_1}, \dots, e^{i\alpha_d}) e^{-iI \cdot \alpha} \prod d\alpha_j$$

and hence as the limit

$$b_I = \lim_{n \rightarrow \infty} n^{-d} \sum_{1 \leq a_j \leq n} k(e^{2\pi i \frac{a_1}{n}}, \dots, e^{2\pi i \frac{a_d}{n}}) e^{-iI \cdot \alpha}, \quad \alpha_j = 2\pi \frac{a_j}{n}$$

When  $k(x) = \prod t_r^{m_r} \prod u_j^{m_j}$  is a monomial, the sum

$$\sum_{1 \leq a_j \leq n} k(e^{2\pi i \frac{a_1}{n}}, \dots, e^{2\pi i \frac{a_d}{n}}) e^{-iI \cdot \alpha}$$

is either zero or  $n^d$  and the latter case only happens if all the components of the multi index  $m - I$  are divisible by  $n$ . Thus

$$n^{-d} \sum_{1 \leq a_j \leq n} k(e^{2\pi i \frac{a_1}{n}}, \dots, e^{2\pi i \frac{a_d}{n}}) e^{-iI \cdot \alpha}$$

only depends on  $\theta_n(k)$  and is a relative integer if  $\theta_n(k) \in \mathbb{Z}[D_1]$ . It follows that all the  $b_I$  are in  $\mathbb{Z}$  and hence  $k \in \mathcal{O}(\Omega)$ .  $\square$

**4.7. The distinction between  $G_k$  and  $\mathfrak{G}(k)$ .** Let  $\mathfrak{G}$  be the Chevalley group scheme associated to a root system as in §4.3 and let  $k$  be a field.

The subgroups  $\mathfrak{X}_r$  generate in the group  $\mathfrak{G}(k)$  of points that are rational over  $k$  a subgroup  $G_k$  which is the *commutator subgroup* of the group  $\mathfrak{G}(k)$ . The subgroup  $G_k \subset \mathfrak{G}(k)$  is often called a Chevalley group over  $k$  and is not in general an algebraic group. If  $\mathfrak{G}$  is the universal Chevalley group, then one knows that  $G_k = \mathfrak{G}(k)$ , so that the distinction between the commutator subgroup  $G_k$  and the group  $\mathfrak{G}(k)$  is irrelevant.

<sup>6</sup>not all elements of  $C$  are of this form

In the construction pursued in this paper of the gadget associated to a Chevalley group, one can take into account this subtlety between  $G_k$  and  $\mathfrak{G}(k)$  by constructing the following sub-gadget. Let  $(\tilde{L}, \tilde{\Phi}, n_{\tilde{r}})$  be the simply connected root system associated to  $(L, \Phi, n_r)$  (cf. § 4.1) and let  $\varphi : L \rightarrow \tilde{L}$  be the morphism connecting the two roots systems as follows

$$\varphi(\Phi) = \tilde{\Phi}, \quad n_r = n_{\varphi(r)} \circ \varphi, \quad \forall r \in \Phi. \quad (58)$$

One simply replaces the term  $\text{Hom}(L, D)$  in the construction of the functor  $\underline{G}(D)$  (cf. Definition 4.8) by the following subgroup

$$\{\chi \in \text{Hom}(L, D) \mid \exists \chi' \in \text{Hom}(\tilde{L}, D), \chi = \chi' \circ \varphi\} \quad (59)$$

which is the range of the restriction map from  $\text{Hom}(\tilde{L}, D)$  to  $\text{Hom}(L, D)$ .

Unlike for the group  $\mathfrak{G}(k)$  the function  $N(q)$  that counts the number of points of  $G_k$  for  $k = \mathbb{F}_q$  is not in general a polynomial function of  $q$ .

## 5. SCHEMES OVER $\mathbb{F}_1$

Except for the extra structure given by the grading, the definition of affine variety over  $\mathbb{F}_1$  that has been used in this paper is identical to the notion proposed by C. Soulé in [25] and the replacement of the category  $\mathcal{R}$  by the category  $\mathcal{F}_{ab}$  of finite abelian groups was his suggestion. Thus, Theorem 4.10 is a solution of the problem formulated in [25] (§ 5.4) about Chevalley group schemes.

Even though this early notion played an important role to get the theory started, it is also too loose inasmuch as the only requirement one imposes is to have enough points with cyclotomic coordinates, but there is no control on the exact size of this set.

In the following, we shall explain that our construction shows how to strengthen considerably the conditions required in [25] on a variety over  $\mathbb{F}_1$ , leading also to the definition of a natural notion of scheme over  $\mathbb{F}_1$  which reconciles the original point of view of Soulé with the one developed by Deitmar in [12] following Kurokawa, Ochiai, and Wakayama [20] and the log-geometry of monoids of K. Kato [19].

First, we point out that the category  $\mathcal{F}_{ab}$  of finite abelian groups that has been used in this paper in the definition of the functor  $\underline{X}$  can be replaced by the larger category  $\mathcal{M}_{ab}$  of abelian monoids. An abelian monoid is a commutative semi-group with a unit 1 and a 0-element. Morphisms in this category send 1 to 1 (and 0 to 0).

Moreover, the construction of the natural transformation  $e_G$  can be extended to yield, for any commutative ring  $A$  and for the monoid  $M = A$ , a map from the set  $\underline{G}(M)$  to the group  $\mathfrak{G}(A)$ . If  $A$  is a field, the resulting map is a bijection. More precisely we have the following result

**Theorem 5.1.** *Let  $\mathfrak{G}$  be the scheme over  $\mathbb{Z}$  associated to a Chevalley group  $G$ .*

- *The construction (46) of the functor  $\underline{G}$  extends to the category  $\mathcal{M}_{ab}^{(2)}$  of pairs  $(M, \epsilon)$  made by an abelian monoid  $M$  and an element  $\epsilon$  of square one.*
- *The construction (50) of the natural transformation  $e_G$  extends to arbitrary commutative rings  $A$  to yield a map*

$$e_{G,A} : \underline{G}(A, -1) \rightarrow \mathfrak{G}(A) \quad (60)$$

- *When  $A$  is a field the map  $e_{G,A}$  is a bijection.*

*Proof.* Let  $(M, \epsilon)$  be an object of  $\mathcal{M}_{ab}^{(2)}$ . In [27], § 4.3, the construction of  $\mathcal{N}_{D, \epsilon}(L, \Phi)$  is carried out for abelian groups  $D$  and it applies in particular to the multiplicative group  $M^*$  of a given monoid  $M$ . The functor  $\underline{G} : \mathcal{M}_{ab}^{(2)} \rightarrow \mathcal{E}ns$  is the product

$$\underline{G}(M, \epsilon) = M^{\Phi^+} \times \prod_{w \in W} (p^{-1}(w) \times M^{\Phi_w}) \quad (61)$$

where  $p$  is the projection  $\mathcal{N}_{M^*, \epsilon}(L, \Phi) \xrightarrow{p} W$ .

One defines the ring  $\beta(M, \epsilon) = \mathbb{Z}[M, \epsilon]$  in such a way that the following adjunction relation holds for any commutative ring  $A$ ,

$$\text{Hom}(\beta(M, \epsilon), A) \cong \text{Hom}((M, \epsilon), \beta^*(A)) \quad (62)$$

where  $\beta^*(A) = (A, -1)$  is the object of  $\mathcal{M}_{ab}^{(2)}$  given by the ring  $A$  viewed as a (multiplicative) monoid and the element  $-1 \in A$ . One has

$$\beta(M, \epsilon) = \mathbb{Z}[M, \epsilon] = \mathbb{Z}[M]/J, \quad J = (1 + \epsilon)\mathbb{Z}[M] \quad (63)$$

Note that this ring is not always flat over  $\mathbb{Z}$  (e.g. when  $\epsilon = 1$ ) but the natural morphism  $M \rightarrow \mathbb{Z}[M, \epsilon]$  is always an injection.

Note also that in (60) the natural transformation sends  $\underline{G} \circ \beta^*$  to  $\mathfrak{G}$  rather than the way it was formulated above in this paper *i.e.*

$$e^G : \underline{G} \rightarrow \mathfrak{G} \circ \beta, \quad e^G(M, \epsilon) : \underline{G}(M, \epsilon) \rightarrow \mathfrak{G}(\mathbb{Z}[M, \epsilon]).$$

However, the adjunction (62) shows that these two descriptions are equivalent. Now, we define the natural transformation  $e_{G, A}$ . One considers (for each  $w \in W$ ) the maps (44)

$$t = (t_r)_{r \in \Phi^+} \mapsto \psi(t) = \prod_{r \in \Phi^+} x_r(t_r) \in \mathcal{U}(A) \subset \mathfrak{G}(A)$$

and (45)

$$t = (t_r)_{r \in \Phi_w} \mapsto \psi_w(t) = \prod_{r \in \Phi_w} x_r(t_r) \in \mathcal{U}_w(A) \subset \mathfrak{G}(A).$$

For the normalizer one uses Tits' isomorphism (Theorem 4.3):

$$e_{\mathcal{N}} : \mathcal{N}_{A^*, -1}(L, \Phi) \rightarrow \mathcal{N}(A).$$

For all  $n \in p^{-1}(w)$ ,  $a \in M^{\Phi^+}$  and  $b \in M^{\Phi_w}$ , one lets

$$e_{G, A}(a, n, b) = \psi(a) e_{\mathcal{N}}(n) \psi_w(b).$$

This definition describes the natural transformation

$$e_{G, A} : \underline{G}(A, -1) \rightarrow \mathfrak{G}(A).$$

The last statement follows from Bruhat's Theorem in the form of Theorem 4.5.  $\square$

Thus Theorem 5.1 suggests to strengthen the conditions imposed on a variety over  $\mathbb{F}_1$  and to define a scheme over  $\mathbb{F}_1$  as follows (using the adjoint pair of functors  $\beta(M) = \mathbb{Z}[M]$  and  $\beta^*(A) = A$ )

**Definition 5.2.** *An  $\mathbb{F}_1$ -scheme is given by a covariant functor  $\underline{X}$  from  $\mathcal{M}_{ab}$  to the category of sets, a  $\mathbb{Z}$ -scheme  $X$  and a natural transformation from  $\underline{X} \circ \beta^*$  to  $X$  such that*

- $\underline{X}$  is locally representable.
- The natural transformation is bijective if  $A = \mathbb{K}$  is a field.



For arbitrary commutative rings the natural transformation yields a map

$$e_{X,A} : \underline{X}(A) \rightarrow X(A) \tag{64}$$

which is bijective when  $A$  is a field.

A nice feature of this definition is that it ensures that the counting of points gives the correct answer. Indeed, the above conditions ensure that the number of points over  $\mathbb{F}_{1^n}$  which is given by the cardinality of  $\underline{X}(D)$  for<sup>7</sup>  $D = \mathbb{Z}/n\mathbb{Z}$ , agrees with the cardinality of  $X(\mathbb{F}_q)$  when  $n = q - 1$  and  $q$  is a prime power.

In the forthcoming paper [6], we shall develop the general theory of  $\mathbb{F}_1$ -schemes and show in particular that under natural finiteness conditions the restriction of the functor  $\underline{X}$  to the full subcategory  $\mathcal{F}_{ab}$  of  $\mathcal{M}_{ab}$  of finite abelian groups is automatically a functor to finite graded sets. Moreover under a torsion free hypothesis the number of points over  $\mathbb{F}_{1^n}$  is a polynomial  $P(n)$ , where  $P(x)$  has positive integral coefficients. Unlike the theory developed in [12], our conditions do not imply that the varieties under study are necessarily toric.

What we have shown in this paper is that Chevalley group schemes yield varieties over  $\mathbb{F}_{1^2}$ , but *not* that the group operation  $\mu$  can be defined over  $\mathbb{F}_{1^2}$ . In fact, it is only the terms of lowest degree (equal to the rank of  $G$ ) that yield a group, namely the group  $\mathcal{N}_{D,\epsilon}(L, \Phi)$  of J. Tits. The structure of the terms of higher order is more mysterious.

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<sup>7</sup>One adjoins a zero element to get a monoid in the above sense

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