

SCHEMES OVER \mathbb{F}_1 AND ZETA FUNCTIONS

ALAIN CONNES AND CATERINA CONSANI

ABSTRACT. We develop a theory of schemes over the field of characteristic one which reconciles the previous attempts by Soulé and by Deitmar. Our construction fits with the geometry of monoids of Kato and is no longer limited to toric varieties. We compute the zeta function of an arbitrary Noetherian scheme (over the field of characteristic one) and prove that the torsion in the local geometric structure introduces ramification. Then we show that Soulé’s definition of the zeta function of an algebraic variety over \mathbb{F}_1 is equivalent to an integral formula. This result provides one with a way to extend the definition of such a function to the case of an arbitrary counting function with polynomial growth. We test this construction on elliptic curves over the rational numbers. Finally, we compare the above mentioned integral formula with the explicit formulae of number theory and we determine the counting function for the hypothetical curve $\text{Spec } \mathbb{Z}$ over the field of characteristic one.

CONTENTS

1. Introduction	2
2. Schemes as locally representable \mathbb{Z} -functors: a review.	5
2.1. Local \mathbb{Z} -functors	6
2.2. Open \mathbb{Z} -subfunctors	7
2.3. Covering by \mathbb{Z} -subfunctors	8
3. \mathfrak{Mo} -schemes.	9
3.1. Monoids: the category \mathfrak{Mo} .	9
3.2. Automatic locality.	10
3.3. Open \mathfrak{Mo} -subfunctors.	11
3.4. Open covering by \mathfrak{Mo} -subfunctors.	12
3.5. \mathfrak{Mo} -schemes.	12
3.6. Geometric realization.	14
3.7. Order structure of $\text{Spec}(M)$	16
3.8. Restriction to abelian groups.	18
3.9. Local dimension	20
3.10. Finiteness conditions	21
4. The category \mathfrak{MA}	21
4.1. Gluing two categories using adjoint functors	22
4.2. Extension of functors.	23
4.3. \mathbb{F}_1 -Functors.	25
5. \mathbb{F}_1 -schemes and their zeta functions	26
5.1. Torsion free Noetherian \mathbb{F}_1 -schemes	26
5.2. Extension of counting functions in the torsion case	28
5.3. An integral formula for $\frac{\partial_s \zeta_N(s)}{\zeta_N(s)}$	32

5.4. Analyticity of the integrals	33
5.5. Zeta function of Noetherian \mathbb{F}_1 -schemes	35
6. Beyond \mathbb{F}_1 -schemes	37
6.1. Modified zeta function	37
6.2. Elliptic curves	39
6.3. Counting functions and distributions	42
7. Appendix: Abelian groups and finite fields	47
8. Appendix: Primes of bad reduction of an elliptic curve	49
References	50

1. INTRODUCTION

In this paper we develop a theory of schemes over \mathbb{F}_1 and compute their zeta function. The notion of a scheme over \mathbb{F}_1 that we introduce in this article is an attempt at unifying the points of view developed on the one side by C. Soulé in [19] and in our paper [1] and on the other side by A. Deitmar in [5], [6] (following N. Kurokawa, H. Ochiai and M. Wakayama [15]), by K. Kato in [13] (with the geometry of logarithmic structures) and by B. Töen and M. Vaquié in [20]. In [1] we introduced a refinement of the original notion of an affine variety over \mathbb{F}_1 as in [19] and following this viewpoint we proved that Chevalley group schemes are examples of affine varieties over \mathbb{F}_{1^2} . While in the process of assembling this construction we realized that the functors (from finite abelian groups to graded sets) describing these affine schemes fulfill stronger properties than the ones required in the original definition of Soulé. In this paper we develop this approach by proving that the functors underlying the structure of the most common examples of schemes (of finite type) over \mathbb{F}_1 extend from (finite) abelian groups to a larger category obtained by gluing together the category $\mathfrak{M}\mathfrak{o}$ of commutative monoids (used in [15], [13], [5], [20]) with the category $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ of commutative rings. This process uses a natural pair of adjoint functors relating $\mathfrak{M}\mathfrak{o}$ to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ and follows an idea we learnt from P. Cartier. The resulting category $\mathfrak{M}\mathfrak{R}$ (*cf.* §4 for details) defines an ideal framework in which the above two approaches are combined together to determine a very natural notion of variety (and of scheme) over \mathbb{F}_1 . In particular, the conditions imposed in the original definition of a variety over \mathbb{F}_1 in [19] are now applied to a covariant functor from $\mathfrak{M}\mathfrak{R}$ to the category of sets. Such a functor determines a scheme (of finite type) over \mathbb{F}_1 if it also fulfills the following three properties:

- The restriction to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ is a scheme in the sense of [8].
- The restriction to $\mathfrak{M}\mathfrak{o}$ is locally representable.
- The natural transformation applied to a field yields a bijection (of sets).

The category $\mathfrak{A}\mathfrak{b}$ of abelian groups embeds as a full subcategory in $\mathfrak{M}\mathfrak{o}$. This fact allows one, in particular, to restrict a covariant functor from $\mathfrak{M}\mathfrak{o}$ to sets to the subcategory (isomorphic to) $\mathfrak{A}\mathfrak{b}$. In §3.8 we prove that if the $\mathfrak{M}\mathfrak{o}$ -functor is locally representable, then the restriction to $\mathfrak{A}\mathfrak{b}$ yields a functor to *graded* sets. This result shows that the grading that we assumed in [1] is now derived as a byproduct of this new refined structure.

In the first part of the paper (*cf.* §2) we shortly review the classical theory of \mathbb{Z} -schemes as covariant functors from \mathfrak{Ring} to sets which are *local* (*cf.* §2.1) and *locally representable* (*cf.* §2.2, §2.3). This is done in order to stress the parallel with the theory of \mathfrak{Mo} -schemes. In §3 we develop the notion of a \mathfrak{Mo} -scheme as a covariant functor from \mathfrak{Mo} to sets and show (§3.2) that for an \mathfrak{Mo} -functor, locality is automatically fulfilled. We then parallel (§3.3, §3.4) the classical theory of \mathbb{Z} -functors as in §2 to obtain the notion of a \mathfrak{Mo} -scheme (§3.5) which is also illustrated in a few examples. The geometric realization of an \mathfrak{Mo} scheme (§3.6) is a geometric scheme in the sense of K. Kato, who originally defined this geometry of monoids in [13] (*cf.* §9), and of Deitmar [5], [6]. In §3.7 we describe the natural order structure inherent to the topology of the geometric realization of an \mathfrak{Mo} -scheme and show that any (finite) simplicial complex appears as a geometric realization of such a scheme. We then show in §3.9 that the notion of local dimension at a point of an \mathfrak{Mo} -scheme yields a natural grading of the functor obtained by restriction of an \mathfrak{Mo} -scheme to \mathfrak{Ab} . This construction makes use of a finiteness condition for monoids which is the analogue of the Noetherian condition for rings: this fact is recalled for completeness in §3.10.

§4 is devoted to the construction of the category \mathfrak{MR} and to the notion of an \mathbb{F}_1 -functor.

The second part of the paper is dedicated to the computation of the zeta function of a Noetherian \mathbb{F}_1 -scheme X . Our first result is Theorem 5.2 which, in particular, extends Theorem 1 of [7] beyond the toric case, under a torsion free hypothesis

(a) There exists a polynomial $N(x+1)$ with positive integral coefficients such that

$$\# X(\mathbb{F}_{1^n}) = N(n+1) \quad \forall n \in \mathbb{N}.$$

(b) For each finite field \mathbb{F}_q , the cardinality of the set of points of the \mathbb{Z} -scheme X which are rational over \mathbb{F}_q is equal to $N(q)$.

(c) The zeta function of X in the sense of [19] is given by

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{\left(1 - \frac{1}{s}\right)^{\otimes n(x)}}$$

where the \otimes -product is the Kurokawa tensor product and $n(x)$ denotes the local dimension at the point x (*i.e.* the rank of \mathcal{O}_x^*).

We then remove the no-torsion hypothesis and compute in particular the zeta function of the extensions \mathbb{F}_{1^n} of \mathbb{F}_1 . The general result is stated in Theorem 5.12 and in its Corollary 5.13 which present the zeta function of a Noetherian \mathbb{F}_1 -scheme X as the product

$$\zeta(X, s) = e^{h(s)} \prod_{j=0}^n (s-j)^{\alpha_j}$$

of the exponential of an entire function by a finite product of fractional powers of simple monomials. The exponents α_j are rational numbers defined explicitly, in terms of the structure sheaf \mathcal{O} in monoids, as follows

$$\alpha_j = (-1)^{j+1} \sum_{x \in X} (-1)^{n(x)} \binom{n(x)}{j} \sum_d \frac{1}{d} \nu(d, \mathcal{O}_x^*)$$

where $\nu(d, \mathcal{O}_x^*)$ is the number of injective homomorphisms from $\mathbb{Z}/d\mathbb{Z}$ to the group \mathcal{O}_x^* of invertible elements in \mathcal{O}_x . In order to establish this result we need to consider

the case when the counting function $\# X(\mathbb{F}_{1^n}) = N(n+1)$ is no longer a polynomial in the integer $n \in \mathbb{N}$. In §5.3 we show that the limit formula used in [19] to define the zeta function of an algebraic variety over \mathbb{F}_1 , can be now replaced by an equivalent integral formula which determines the equation

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = - \int_1^\infty N(u) u^{-s-1} du \quad (1)$$

describing the logarithmic derivative of the zeta function $\zeta_N(s)$ associated to the counting function $N(q)$. To treat the case of the counting function of a Noetherian \mathbb{F}_1 -scheme, we use Nevanlinna theory to uniquely extend the counting function $N(n)$ to arbitrary complex arguments $z \in \mathbb{C}$ and then we compute the corresponding integrals.

In §6 we show that the replacement, in the above exact formula (1), of the integral by the discrete sum, *i.e.*

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = - \sum_1^\infty N(u) u^{-s-1}$$

only modifies the zeta function of a Noetherian \mathbb{F}_1 -scheme by an exponential factor of an entire function. This leads to the Definition 6.2 of a modified zeta function over \mathbb{F}_1 whose main advantage is that of applying to the case of an arbitrary counting function with polynomial growth. In §6.2 we test this definition with elliptic curves E over \mathbb{Q} , by computing the zeta function associated to a *specific* counting function on E . This specific counting function $N(q, E)$ is uniquely specified by the following two conditions:

- For any prime power $q = p^\ell$, the value of $N(q, E)$ is the number¹ of points of the reduction of E modulo p in the finite field \mathbb{F}_q .
- The function $t(n)$ occurring in the equation $N(n, E) = n+1-t(n)$ is multiplicative.

We show that the obtained zeta function $\zeta_N^{\text{disc}}(s)$ of E fulfills the equation

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = -\zeta(s+1) - \zeta(s) + \frac{L(s+1, E)}{\zeta(2s+1)M(s+1)}$$

where $L(s, E)$ is the L -function of the elliptic curve, $\zeta(s)$ is the Riemann zeta function and

$$M(s) = \prod_{p \in S} (1 - p^{1-2s})$$

where S is the set of primes at which E has bad reduction. In Example 6.6 we exhibit the singularities of $\zeta_E(s)$ in a concrete case.

In the last section §6.3 of the paper, we consider the central question formulated in [16] which originally motivated the development of the study of the arithmetic over \mathbb{F}_1 . Can one find a “curve” (in a suitable sense) $C = \overline{\text{Spec } \mathbb{Z}}$ over \mathbb{F}_1 whose zeta function $\zeta_C(s)$ is the completed Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$? We apply the integral formula (1) and show, by making a comparison with the Riemann-Weil explicit formulae, that it provides a precise expression for the counting function $N(q)$, $q \in [1, \infty)$ associated to the hypothetical curve C . This is given

¹including the singular point

by the following equation (Theorem 6.7)

$$N(q) = q - \frac{d}{dq} \left(\sum_{\rho \in Z} \text{order}(\rho) \frac{q^{\rho+1}}{\rho+1} \right) + 1$$

where Z is the set of non-trivial zeros of the Riemann zeta function and the derivative is taken in the sense of distributions. The distribution $N(q)$ is *positive* on $(1, \infty)$ as expected for a counting function. The value at 1 is $-\infty$, in agreement with the interpretation $N(1) = \chi(C)$ as the Euler characteristic of C (*cf.* Remark 6.8). The above formula provides also a strong indication that the hypothetical curve $C = \overline{\text{Spec}} \mathbb{Z}$ is related to the interpretation of the explicit formulae as a trace formula using the noncommutative geometric framework of the adèle class space as developed in [2], [3] and [4].

The paper ends with two appendices. §7 presents a very simple characterization of monoids coming from fields. In §8 we briefly recall, for the convenience of the readers not acquainted with the language of arithmetic geometry, some standard facts and notations inherent to the study of the reduction of elliptic curves at the primes of bad reduction.

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2. SCHEMES AS LOCALLY REPRESENTABLE \mathbb{Z} -FUNCTORS: A REVIEW.

In the following sections we shall review the basic notions of the theory of \mathbb{Z} -functors and \mathbb{Z} -schemes: we refer to [8] (Chapters I, III) for a detailed exposition.

We denote by \mathbf{Sets} , \mathbf{Mo} , \mathbf{Ab} , \mathbf{Ring} respectively the categories of sets, commutative monoids², abelian groups and commutative rings.

Definition 2.1. A \mathbb{Z} -functor \mathcal{F} is a covariant functor from \mathbf{Ring} to \mathbf{Sets} .

Morphisms in the category of \mathbb{Z} -functors are natural transformations (of functors). Schemes over \mathbb{Z} determine a full subcategory \mathbf{Sch} of the category of \mathbb{Z} -functors. In fact, a scheme X over $\text{Spec} \mathbb{Z}$ is *entirely* characterized by the \mathbb{Z} -functor

$$\underline{X} : \mathbf{Ring} \rightarrow \mathbf{Sets}, \quad \underline{X}(R) = \text{Hom}_{\mathbb{Z}}(\text{Spec } R, X). \quad (2)$$

To a ring homomorphism $\rho : R_1 \rightarrow R_2$ one associates the morphism of (affine) schemes $\rho^* : \text{Spec}(R_2) \rightarrow \text{Spec}(R_1)$, $\rho^*(\mathfrak{p}) = \rho^{-1}(\mathfrak{p})$ and the map of sets

$$\rho : \underline{X}(R_1) \rightarrow \underline{X}(R_2), \quad \varphi \rightarrow \varphi \circ \rho^*.$$

If $\psi : X \rightarrow Y$ is a morphism of schemes then one gets for every ring R a map of sets

$$\underline{X}(R) \rightarrow \underline{Y}(R), \quad \varphi \rightarrow \psi \circ \varphi.$$

The functors of the form \underline{X} , for X a scheme over $\text{Spec} \mathbb{Z}$, are *local* \mathbb{Z} -functors (in the sense we shall recall in § 2.1, Definition 2.2) and *locally representable* by commutative rings (*i.e.* they have an open cover by affine \mathbb{Z} -subfunctors in the sense explained in §§ 2.2, 2.3).

²with a unit and a zero element

2.1. Local \mathbb{Z} -functors.

For any commutative (unital) ring R , the geometric space $\text{Spec}(R)$ is the set of prime ideals $\mathfrak{p} \subset R$. The topology on $\text{Spec}(R)$ is the Jacobson topology *i.e.* the closed subsets are the sets $V(J) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supset J\}$, where $J \subset R$ runs through the collection of ideals of R . The open subsets of $\text{Spec}(R)$ are the complements of the $V(J)$'s *i.e.* the sets

$$D(J) = V(J)^c = \{\mathfrak{p} \in \text{Spec}(R) \mid \exists f \in J, f \notin \mathfrak{p}\}.$$

The open sets $D(f) = V(fR)^c$, for $f \in R$, form a base of the topology of $\text{Spec}(R)$. For any $f \in R$ one lets R_f be the ring of fractions with denominators a power of f . One has a natural ring homomorphism $R \rightarrow R_f$. Then, for any scheme X over $\text{Spec} \mathbb{Z}$, the associated functor \underline{X} as in (2) fulfills the following *locality* property. For any finite cover of $\text{Spec}(R)$ by open sets $D(f_i)$ the following sequence of maps of sets

$$\underline{X}(R) \xrightarrow{u} \prod_i \underline{X}(R_{f_i}) \xrightarrow[w]{v} \prod_{ij} \underline{X}(R_{f_i f_j}) \quad (3)$$

is exact. This means that u is injective and the range of u is characterized as the set $\{z \in \prod_i \underline{X}(R_{f_i}) \mid v(z) = w(z)\}$. The exactness is a consequence of the fact that a morphism of schemes is defined by local conditions. For \mathbb{Z} -functors we have the following definition

Definition 2.2. *A \mathbb{Z} -functor \mathcal{F} is local if for any object R of \mathfrak{Ring} and a partition of unity $\sum_{i \in I} h_i f_i = 1$ in R ($I = \text{finite set}$), the following sequence of sets is exact:*

$$\mathcal{F}(R) \xrightarrow{u} \prod_i \mathcal{F}(R_{f_i}) \xrightarrow[w]{v} \prod_{ij} \mathcal{F}(R_{f_i f_j}). \quad (4)$$

Example 2.3. This example shows how locality may fail. The Grassmannian $\text{Gr}(k, n)$ of k -dimensional spaces in an n -dimensional space is defined by the functor which associates to a ring R the set of all complemented submodules of rank k of the free (right) module R^n . Since any such complemented submodule is projective by construction we have

$$\text{Gr}(k, n) : \mathfrak{Ring} \rightarrow \mathfrak{Sets}, \quad \text{Gr}(k, n)(R) = \{E \subset R^n \mid E \text{ projective, } \text{rk}(E) = k\}.$$

Let $\rho : R_1 \rightarrow R_2$ be a homomorphism of rings, the corresponding map of sets is given as follows: for $E_1 \in \text{Gr}(k, n)(R_1)$, one lets $E_2 = E_1 \otimes_{R_1} R_2 \in \text{Gr}(k, n)(R_2)$. If one takes a naive definition of the rank, *i.e.* by just requiring that $E \cong R^k$ as an R -module, one does not obtain a local \mathbb{Z} -functor. In fact, let us consider the case $k = 1$ and $n = 2$ which defines the projective line $\mathbb{P}^1 = \text{Gr}(1, 2)$. To show that locality fails in this case, one takes the algebra $R = C(S^2)$ of continuous functions on the sphere S^2 and the partition of unity $f_1 + f_2 = 1$ subordinate to a covering of S^2 by two disks D_j ($j = 1, 2$), so that $\text{Supp}(f_j) \subset D_j$. One then considers the non-trivial line bundle L on S^2 arising from the identification $S^2 \simeq \mathbb{P}^1(\mathbb{C}) \subset M_2(\mathbb{C})$ which determines an idempotent $e \in M_2(C(S^2))$. The range of e defines a finite projective submodule $E \subset R^2$. The localized algebra R_{f_j} is the same as $C(\bar{D}_j)_{f_j}$ and thus the induced module E_{f_j} is free (of rank one). The modules $E_i = E \otimes_R R_{f_i}$ are free submodules of $R_{f_i}^2$ and the induced modules on $R_{f_i f_j}$ are the same. But since E is not free they do not belong to the image of u and the sequence (4) is not exact in this case.

To obtain a local \mathbb{Z} -functor one has to implement a more refined definition of the rank which is given by requiring that for any prime ideal \mathfrak{p} of R the induced module on the residue field of R at \mathfrak{p} is a vector space of dimension k .

2.2. Open \mathbb{Z} -subfunctors.

The \mathbb{Z} -functor associated to an affine scheme $\text{Spec } A$ is given by

$$\mathbf{spec } A(R) = \text{Hom}_{\mathbb{Z}}(\text{Spec } R, \text{Spec } A) \simeq \text{Hom}_{\mathbb{Z}}(A, R). \quad (5)$$

The open sets of an affine scheme are in general not affine and they provide interesting examples of schemes. The subfunctor

$$\text{Hom}_{\mathbb{Z}}(\text{Spec } R, D(J)) \subset \text{Hom}_{\mathbb{Z}}(\text{Spec } R, \text{Spec } A) \quad (6)$$

of $\mathbf{spec } A$ associated to the open set $D(J)$ (for a given ideal $J \subset A$) has the following explicit description:

$$\underline{D(J)} : \mathfrak{Ring} \rightarrow \mathfrak{Sets} \quad \underline{D(J)}(R) = \{\rho \in \text{Hom}_{\mathbb{Z}}(A, R) \mid \rho(J)R = R\} \quad (7)$$

in view of the fact that $\text{Spec}(\rho)^{-1}(D(J)) = D(\rho(J)R)$. In general, we say that \mathcal{U} is a subfunctor of \mathcal{F} if for each ring R , $\mathcal{U}(R)$ is a subset of $\mathcal{F}(R)$ (with the natural compatibility for the maps).

Definition 2.4. Let \mathcal{U} be a subfunctor of \mathcal{F} . One says that \mathcal{U} is open if for any ring A and any natural transformation $\phi : \mathbf{spec } A \rightarrow \mathcal{F}$, the subfunctor of $\mathbf{spec } A$, inverse image of \mathcal{U}

$$\phi^{-1}(\mathcal{U})(R) = \{x \in \mathbf{spec } A(R) \mid \phi(x) \in \mathcal{U}(R) \subset \mathcal{F}(R)\}$$

is of the form $\underline{D(J)}$, for some open set $D(J) \subset \text{Spec } (A)$.

Equivalently, using Yoneda's Lemma, the above definition can be expressed by saying that, given any ring A and an element $z \in \mathcal{F}(A)$, there exists an ideal $J \subset A$ such that, for any $\rho \in \text{Hom}(A, R)$

$$\mathcal{F}(\rho)(z) \in \mathcal{U}(R) \iff \rho(J)R = R. \quad (8)$$

For any open subset $Y \subset X$ of a scheme X the subfunctor

$$\text{Hom}_{\mathbb{Z}}(\text{Spec } R, Y) \subset \text{Hom}_{\mathbb{Z}}(\text{Spec } R, X)$$

is open, and all open subfunctors of \underline{X} arise in this way.

Example 2.5. We consider the projective line \mathbb{P}^1 identified with $X = \text{Gr}(1, 2)$. Let $\mathcal{U} \subset X$ be the subfunctor described, on a ring R , by the collection of all submodules of rank one of R^2 which are supplements of the submodule $P = \{(0, y) \mid y \in R\} \subset R^2$. Let p_1 be the projection on the first copy of R , then:

$$\mathcal{U}(R) = \{E \in \text{Gr}(1, 2) \mid p_{1|E} \text{ isomorphism}\}$$

Proving that \mathcal{U} is open is equivalent, using (8), to find for any ring A and $E \in X(A)$ an ideal $J \subset A$ such that for any ring R and $\rho : A \rightarrow R$ one has

$$p_{1|E \otimes_A R} \text{ isomorphism} \iff R = \rho(J)R. \quad (9)$$

It is easy to see that the ideal J given by the annihilator of the cokernel of $p_{1|E}$ satisfies (9) (cf. [8] Chapter I, Example 3.9).

2.3. Covering by \mathbb{Z} -subfunctors.

To motivate the definition of a covering of a \mathbb{Z} -functor, we start by describing the case of an affine scheme. Let X be the \mathbb{Z} -functor

$$R \rightarrow X(R) = \text{Hom}(A, R) \tag{10}$$

associated to the affine scheme $\text{Spec}(A)$. We have seen that the open subfunctors $\underline{D}(I) = X_I \subset X$ correspond to ideals $I \subset A$ with

$$X_I(R) = \underline{D}(I)(R) = \{\rho \in \text{Hom}(A, R) \mid \rho(I)R = R\}.$$

The condition that the open sets $D(I_\alpha)$ form a covering of $\text{Spec}(A)$ is expressed algebraically by the equality $\sum_\alpha I_\alpha = A$. We want to express this condition in terms of the subfunctors X_{I_α} .

Lemma 2.6. *Let X be as in (10). Then $\sum_\alpha I_\alpha = A$ if and only if for any field K one has*

$$X(K) = \cup_\alpha X_{I_\alpha}(K).$$

Proof. Assume first that $\sum I_\alpha = A$ (a finite sum) i.e. $1 = \sum a_\alpha$, with $a_\alpha \in I_\alpha$. Let K be a field, then for $\rho \in \text{Hom}(A, K)$, one has $\rho(a_\alpha) \neq 0$ for some α . Then $\rho(a_\alpha)K = K$, i.e. $\rho \in X_{I_\alpha}(K)$ so that the union of all $X_{I_\alpha}(K)$ is $X(K)$. Conversely, assume that $\sum I_\alpha \neq A$. Then there exists a prime ideal $\mathfrak{p} \in A$ containing all I_α . Then, let K be the field of fractions of A/\mathfrak{p} and let $\rho : A \rightarrow K$ be the natural homomorphism. One has $I_\alpha \subset \text{Ker}\rho$, thus $\rho \notin \cup_\alpha X_{I_\alpha}(K)$. \square

Notice that when R is neither a field or a local ring, the equality $X(R) = \cup_\alpha X_{I_\alpha}(R)$ cannot be expected. In fact the range of a morphism $\rho \in \text{Hom}(\text{Spec}(R), \text{Spec}A) = X(R)$ may not be contained in a single open set of the covering of $\text{Spec}(A)$ by the $D(I_\alpha)$ so that ρ belongs to none of the $X_{I_\alpha}(R) = \text{Hom}(\text{Spec}(R), D(I_\alpha))$.

Definition 2.7. *Let X be a \mathbb{Z} -functor. Let $\{X_\alpha\}$ be a family of open subfunctors of X . Then the $\{X_\alpha\}_{\alpha \in S}$ form a covering of X if for any field K one has $X(K) = \cup_\alpha X_\alpha(K)$.*

In the case of an affine scheme one recovers the usual notion of open cover. In fact, in that case, any open cover admits a finite subcover. Indeed, the condition is $\sum_\alpha I_\alpha = A$ and if it holds one gets $1 = \sum_{\alpha \in F} a_\alpha$ for some finite subset of indices. For an arbitrary scheme this finiteness condition may not hold. However, since a scheme is always “locally affine”, one can say, calling “quasi-compact” the above finiteness condition, that any scheme is locally quasi-compact.

To conclude this short review of the basic properties of schemes viewed as \mathbb{Z} -functors, we quote the main theorem which allows one to consider a scheme as a local and locally representable \mathbb{Z} -functor.

Theorem 2.8. *The \mathbb{Z} -functors of the form \underline{X} for X a scheme over $\text{Spec}\mathbb{Z}$ are local and admit an open cover by representable subfunctors.*

Proof. cf. [8] Chapter I, § 1, 4.4 \square

3. $\mathfrak{M}\mathfrak{o}$ -SCHEMES.

In this section we shall describe, following the same functorial approach as in the previous section, a generalization of the theory of \mathbb{Z} -functors and \mathbb{Z} -schemes obtained by replacing the category of rings with that of commutative monoids. In the second part of the paper (§ 5) we shall derive, as byproduct of these two approaches, a new notion of \mathbb{F}_1 -schemes and their associated zeta functions. Our construction has an evident link with the theory of logarithmic structures developed by K. Kato in [13], with the arithmetic theory over \mathbb{F}_1 described by N. Kurokawa, H. Ochiai, M. Wakayama in [15], with the algebro-topological approach followed by B. Töen and M. Vaquié in [20] and with the notion of schemes over \mathbb{F}_1 developed by A. Deitmar in [5], [6].

3.1. Monoids: the category $\mathfrak{M}\mathfrak{o}$.

We denote by $\mathfrak{M}\mathfrak{o}$ the category whose objects are commutative monoids M denoted multiplicatively, with a neutral element 1 (*i.e.* a unit) and an absorbing element 0 (*i.e.* $0 \cdot x = x \cdot 0 = 0$, $\forall x \in M$).

A homomorphism $\varphi : M \rightarrow N$ of monoids in $\mathfrak{M}\mathfrak{o}$ is unital (*i.e.* $\varphi(1) = 1$) and satisfying $\varphi(0) = 0$.

Given a commutative group H , we denote by

$$\mathbb{F}_1[H] = H \cup \{0\} \quad (0 \cdot h = h \cdot 0 = 0 \quad \forall h \in H)$$

the monoid obtained by adding $\{0\}$ to H . Using the analogy with the category of rings, one sees that in $\mathfrak{M}\mathfrak{o}$ the monoids of the form $\mathbb{F}_1[H]$ play the role of the *fields* in $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$. They form a full subcategory of $\mathfrak{M}\mathfrak{o}$ isomorphic to the category of abelian groups.

Definition 3.1. *An $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} is a covariant functor from the category $\mathfrak{M}\mathfrak{o}$ to the category of sets.*

To every object M in $\mathfrak{M}\mathfrak{o}$ one associates the covariant functor $X = \mathbf{spec} M$:

$$\mathbf{spec} M : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s} \quad N \mapsto \mathbf{spec} M(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N). \quad (11)$$

Notice that by applying Yoneda's lemma, a morphism of functors (natural transformation) φ from $\mathbf{spec} M$ to any functor $\mathcal{F} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ is completely determined by the element $\varphi(id_M) \in \mathcal{F}(M)$ and that any such element gives rise to a morphism $\mathbf{spec} M \rightarrow \mathcal{F}$. By applying this remark to the functor $\mathcal{F} = \mathbf{spec} N$, for monoids N , we obtain an *inclusion* of $\mathfrak{M}\mathfrak{o}$ as a full subcategory of the category of $\mathfrak{M}\mathfrak{o}$ -functors.

An ideal I of a monoid M is a subset $I \subset M$ such that $0 \in I$ and the following condition is satisfied (*cf.* [9]):

$$x \in I \implies xy \in I, \quad \forall y \in M.$$

As in the case of rings, an ideal $I \subset M$ defines an interesting subfunctor $\underline{D(I)} \subset \mathbf{spec} M$:

$$\underline{D(I)} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad \underline{D(I)}(N) = \{\rho \in \text{Spec}(M)(N) \mid \rho(I)N = N\}. \quad (12)$$

The notion of prime ideal easily adapts from $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ to $\mathfrak{M}\mathfrak{o}$ (*cf.* [9]): an ideal $\mathfrak{p} \subset M$ is *prime* iff its complement \mathfrak{p}^c is a multiplicative subset *i.e.*

$$x \notin \mathfrak{p}, y \notin \mathfrak{p} \implies xy \notin \mathfrak{p}.$$

One lets, for any ideal $I \subset M$, $D(I)$ be the set of prime ideals \mathfrak{p} which do not contain I . The inverse image of a prime ideal by a morphism of monoids is a prime ideal. Moreover the complement $\mathfrak{p}_M = M^{*c}$ of the set of invertible elements in a monoid M is a prime ideal of M which contains all other prime ideals.

Proposition 3.2. *Let $\rho : M \rightarrow N$ be a morphism in the category $\mathfrak{M}\mathfrak{o}$ and let $I \subset M$ be an ideal. Then, the following conditions are equivalent:*

- (1) $\rho(I)N = N$
- (2) $1 \in \rho(I)N$
- (3) $\rho^{-1}((N^*)^c)$ is a prime ideal belonging to $D(I)$
- (4) $\rho^{-1}(\mathfrak{p}) \in D(I)$, for any prime ideal $\mathfrak{p} \subset N$.

Proof. One has (1) \Leftrightarrow (2). Moreover, $\rho^{-1}((N^*)^c) \not\subset I$ if and only if $\rho(I) \cap N^* \neq \emptyset$ which is equivalent to (1). Thus (2) \Leftrightarrow (3).

If an ideal $J \subset M$ does not contain I , then the same holds obviously for all the sub-ideals of J . Then (3) \Rightarrow (4) since $\mathfrak{p}_N = (N^*)^c$ contains all the prime ideals of N . Taking $\mathfrak{p} = \mathfrak{p}_N$ one gets (4) \Rightarrow (3). \square

3.2. Automatic locality.

For \mathbb{Z} -functors the ‘‘locality’’ property was defined using coverings of affine schemes $\text{Spec}(R)$ by open sets $D(f_i)$ and requiring the exactness of sequences such as (3) and (4). In the framework of $\mathfrak{M}\mathfrak{o}$ -functors one is already in a local set-up since if one defines a topology on the set $\text{Spec}(M)$ of prime ideals of a monoid M by using the $D(I)$ ’s as open sets, one easily checks that the smallest ideal containing a collection of ideals $\{I_\alpha\}$ is just $I = \cup_\alpha I_\alpha$ and that $D(\cup_\alpha I_\alpha) = \cup_\alpha D(I_\alpha)$. More precisely one has the following result (cf. [5])

Lemma 3.3. *Let M be an object in $\mathfrak{M}\mathfrak{o}$ and let $\{W_\alpha\}$ be an open cover of the topological space $X = \text{Spec}(M)$. Then $W_\alpha = \text{Spec}(M)$, for some index α .*

Proof. The point $\mathfrak{p}_M = (M^*)^c \in \text{Spec}(M)$ must be contained in some W_α , hence $\mathfrak{p}_M \in D(I_\alpha)$ for some index α and this mean $I_\alpha \cap M^* \neq \emptyset$, hence $I_\alpha = M$. \square

Let M be an object of $\mathfrak{M}\mathfrak{o}$. For $f \in M$ one defines the *monoid* M_f as the quotient of the set made by the expressions $\frac{a}{f^n}$, for $a \in M$ and $n \in \mathbb{Z}_{\geq 0}$, by the following equivalence relation

$$\frac{a}{f^n} \sim \frac{b}{f^m} \Leftrightarrow \exists k \in \mathbb{Z}_{\geq 0}, \quad f^k f^m a = f^k f^n b.$$

It is straightforward to check that the product

$$\frac{a}{f^n} \cdot \frac{b}{f^m} = \frac{ab}{f^{n+m}}$$

is well-defined on the quotient M_f . For any $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} and any monoid M one has a sequence of maps of sets

$$\mathcal{F}(M) \xrightarrow{u} \prod_i \mathcal{F}(M_{f_i}) \xrightleftharpoons[w]{v} \prod_{ij} \mathcal{F}(M_{f_i f_j}) \quad (13)$$

that is obtained by using the open covering of $\text{Spec}(M)$ by open sets $D(f_i M)$ and the natural morphisms $M \rightarrow M_{f_i}$.

Lemma 3.4. *For any $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} the sequence (13) is exact.*

Proof. By Lemma 3.3, there exists an index i such that $f_i \in M^*$. We may assume that $i = 1$. Then the map $M \rightarrow M_{f_1}$ is invertible thus u is injective. Let $(x_i) \in \prod_i \mathcal{F}(M_{f_i})$ be a family, with $x_i \in \mathcal{F}(M_{f_i})$ such that $(x_i)_{f_j} = (x_j)_{f_i}$, for all i, j . This gives in particular the equality between the image of $x_i \in \mathcal{F}(M_{f_i})$ under the isomorphism $\mathcal{F}(\rho_{i1}) : \mathcal{F}(M_{f_i}) \rightarrow \mathcal{F}(M_{f_i f_1})$ and $\mathcal{F}(\rho_{i1})(x_1) \in \mathcal{F}(M_{f_i f_1})$. By writing $x_1 = \rho_1(x)$ one finds that $u(x)$ is equal to the family (x_i) . \square

The above lemma shows that locality is automatically verified for any $\mathfrak{M}\mathfrak{o}$ -functor.

3.3. Open $\mathfrak{M}\mathfrak{o}$ -subfunctors.

In perfect analogy with the case of \mathbb{Z} -schemes reviewed in § 2.2 we can now define open subfunctors as follows:

Definition 3.5. *A subfunctor $Y \subset X$ of an $\mathfrak{M}\mathfrak{o}$ -functor X is open if for any object M of $\mathfrak{M}\mathfrak{o}$ and any morphism of $\mathfrak{M}\mathfrak{o}$ -functors $\varphi : \mathbf{spec} M \rightarrow X$ there exists an ideal $I \subset M$ satisfying the following property*

For any object N of $\mathfrak{M}\mathfrak{o}$ and for any $\rho \in \mathbf{spec} M(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N)$ one has:

$$\varphi(\rho) \in Y(N) \subset X(N) \Leftrightarrow \rho(I)N = N. \quad (14)$$

In order to clarify the meaning of the above definition we consider a few examples.

Example 3.6. The functor

$$Y : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad N \rightarrow Y(N) = N^*$$

is an open subfunctor of the (identity) functor $X = \mathcal{D}^1$

$$X : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad N \rightarrow X(N) = N$$

For a monoid M , a morphism of functors $\varphi : \mathbf{spec} M \rightarrow X$ is simply given by an element $z \in X(M) = M$ (Yoneda's lemma). For any monoid N and $\rho \in \text{Hom}(M, N)$ one has $\varphi(\rho) = \rho(z) \in X(N) = N$, thus the condition $\varphi(\rho) \in Y(N) = N^*$ means that $\rho(z) \in N^*$. One takes for I the ideal generated by z in M : $I = zM$. Then it is straightforward to check that (14) is fulfilled.

Example 3.7. We start with a monoid M and an ideal $I \subset M$ and we define an $\mathfrak{M}\mathfrak{o}$ -functor X_I as the following subfunctor of $\mathbf{spec}(M)$:

$$X_I : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad X_I(N) = \{\rho \in \text{Hom}(M, N) \mid \rho(I)N = N\}.$$

This means that for all prime ideals $\mathfrak{p} \subset N$, one has $\rho^{-1}(\mathfrak{p}) \not\supset I$ (cf. Proposition 3.2). Let us show that this defines an open subfunctor of $\mathbf{spec} M$. Indeed, for any object A of $\mathfrak{M}\mathfrak{o}$ and $\varphi : \mathbf{spec} A \rightarrow \mathbf{spec} M$ one has $\varphi(\text{id}_A) = \eta \in \mathbf{spec} M(A) = \text{Hom}(M, A)$ and we can take in A the ideal $J = \eta(I)A$. This ideal fulfills the condition (14) for any object N of $\mathfrak{M}\mathfrak{o}$ and $\rho \in \text{Hom}(A, N)$. In fact, one has $\varphi(\rho) = \rho \circ \eta \in \text{Hom}(M, N)$ and $\varphi(\rho) \in X_I(N)$ means that $\rho(\eta(I))N = N$. This holds if and only if $\rho(J)N = N$.

3.4. Open covering by $\mathfrak{M}\mathfrak{o}$ -subfunctors.

Our next task is to generalize the notion of *open covers* to the category of $\mathfrak{M}\mathfrak{o}$ -functors. We recall that the category \mathfrak{Ab} of abelian groups is a full subcategory of $\mathfrak{M}\mathfrak{o}$ using the functor $H \rightarrow \mathbb{F}_1[H]$.

Definition 3.8. *Let X be an $\mathfrak{M}\mathfrak{o}$ -functor and let $\{X_\alpha\}$ be a family of open subfunctors. Then $\{X_\alpha\}$ is an open cover of X if*

$$X(H) = \bigcup_{\alpha} X_{\alpha}(H), \quad \forall H \in \text{obj}(\mathfrak{Ab}). \quad (15)$$

Since commutative groups replace fields in $\mathfrak{M}\mathfrak{o}$, the above definition is the generalization to $\mathfrak{M}\mathfrak{o}$ -functors of Definition 2.7. In Proposition 3.19 we shall prove that an open covering of an $\mathfrak{M}\mathfrak{o}$ -scheme is characterized by the equality (15) where groups H get substituted by monoids M .

For $X = \mathbf{spec} M$ and the X_α open subfunctors corresponding to ideals $I_\alpha \subset M$ the covering condition in the above definition is equivalent to state that $\exists \alpha, I_\alpha = M$. In fact, one takes $H = M^*$, $\epsilon : M \rightarrow \mathbb{F}_1[M^*] = \kappa$. Then, $\epsilon \in X(\mathbb{F}_1[H])$ and $\exists \alpha, \epsilon \in X_\alpha(\mathbb{F}_1[H])$ and thus $I_\alpha \cap M^* \neq \emptyset$ hence $I_\alpha = M$.

3.5. $\mathfrak{M}\mathfrak{o}$ -schemes.

Because locality is automatic for $\mathfrak{M}\mathfrak{o}$ -functors, the definition of an $\mathfrak{M}\mathfrak{o}$ -scheme simply involves the local representability.

Definition 3.9. *An $\mathfrak{M}\mathfrak{o}$ -scheme is an $\mathfrak{M}\mathfrak{o}$ -functor which admits an open cover by representable subfunctors.*

We shall consider the following elementary examples.

Example 3.10. The affine spaces \mathcal{D}^n . For a fixed $n \in \mathbb{N}$, let us consider the following $\mathfrak{M}\mathfrak{o}$ -functor:

$$\mathcal{D}^n : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{Sets}, \quad \mathcal{D}^n(M) = M^n$$

This functor is representable since it is described by

$$\mathcal{D}^n(M) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[T_1, \dots, T_n], M).$$

where the monoid $\mathbb{F}_1[T_1, \dots, T_n]$ is defined by

$$\mathbb{F}_1[T_1, \dots, T_n] := \{0\} \cup \{T_1^{a_1} \cdots T_n^{a_n} \mid a_j \in \mathbb{Z}_{\geq 0}\}, \quad (16)$$

i.e. the union of $\{0\}$ with the semi-group generated by the T_j .

Example 3.11. The projective line \mathbb{P}^1 . We consider the $\mathfrak{M}\mathfrak{o}$ -functor \mathbb{P}^1 which to any object M of $\mathfrak{M}\mathfrak{o}$ associates the set $\mathbb{P}^1(M)$ of complemented submodules E of rank one in M^2 , where the rank is defined locally. Such a submodule is the range of a matrix $e \in M_2(M)$ with each line having at most one non-zero entry, and such that $e^2 = e$. To a morphism $\rho : M \rightarrow N$ one associates the following map $\mathbb{P}^1(\rho)$

$$E \rightarrow N \otimes_M E \subset N^2$$

which replaces $e \in M_2(M)$ by $\rho(e) \in M_2(N)$. Let us now compare this $\mathfrak{M}\mathfrak{o}$ -functor with the $\mathfrak{M}\mathfrak{o}$ -functor X defined by

$$X(M) = M \cup_{M^*} M \quad (17)$$

with the gluing map given by $x \rightarrow x^{-1}$. In other words, we define on the disjoint union $M \cup M$ an equivalence relation given by (using the identification $M \times \{1, 2\} = M \cup M$)

$$(x, 1) \sim (x^{-1}, 2) \quad \forall x \in M^*.$$

We define a natural transformation e from X to $\underline{\mathbb{P}}^1$ by observing that with

$$e_1(a) = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}, \quad e_2(b) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in M$$

one obtains projections ($e^2 = e$) and the ranges fulfill

$$\text{Im } e_1(a) = \text{Im } e_2(b) \iff ab = 1$$

Lemma 3.12. *The natural transformation e is an isomorphism*

$$\underline{\mathbb{P}}^1(M) \cong M \cup_{M^*} M$$

The two copies of M define an open cover of $\underline{\mathbb{P}}^1$ by representable sub-functors \mathcal{D}^1 .

Proof. Let us show that a matrix $e \in M_2(M)$ with each line having at most one non-zero entry, such that $e^2 = e$ and of rank one is of the form $e_j(a)$ for some $j \in \{1, 2\}$. One first shows that one of the matrix elements of

$$e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible. Indeed, otherwise by localizing relative to the prime ideal $\mathfrak{p}_M = M^{*c}$ one would obtain the matrix zero which contradicts the hypothesis of rank = 1. Then one has the following cases

- If $a \in M^*$, then $b = 0$, $a^2 = a$, $a = 1$, and $d = 0$ since if $d \neq 0$ one has $d^2 = d$ and with $\mathfrak{p} \in D(d)$ i.e. $d \notin \mathfrak{p}$ one has $d \in M_{\mathfrak{p}}^*$ and the rank of e is 2 at \mathfrak{p} .
- If $b \in M^*$, then $a = 0$, $bd = b$ so that $d = 1$ and $c = 0$.
- If $c \in M^*$, then $d = 0$, $bc = 0$ and $b = 0$.
- If $d \in M^*$, then $c = 0$, $d^2 = d$, $d = 1$, and $a = 0$ since if $a \neq 0$ one has $a^2 = a$ and the rank of e is 2 at \mathfrak{p} , $a \notin \mathfrak{p}$.

The functor X admits by construction two sub-functors \mathcal{D}^1 embedded in it. We need to show that these two sub-functors are open in $\underline{\mathbb{P}}^1$, we do it for the first copy of \mathcal{D}^1 . Let N be an object of $\mathfrak{M}\mathfrak{o}$. A morphism $\mathbf{spec}(N) \rightarrow \mathbb{P}^1$ (in the category of $\mathfrak{M}\mathfrak{o}$ -functors) is determined by an element $z \in \underline{\mathbb{P}}^1(N)$. If z belongs to the first copy of N , it follows that for any $\rho \in \text{Hom}(N, M)$, $\rho(z)$ is in the first copy of $\mathcal{D}^1(M)$. In this case one can take $I = N$. Otherwise, z belongs to the second copy of N and in this case, likewise in the above Example 3.6, one takes $I = zN$. The local representability follows since \mathcal{D}^1 is representable.

Example 3.13. Let M be a monoid and $I \subset M$ be an ideal. Consider the $\mathfrak{M}\mathfrak{o}$ -functor X_I of Example 3.7. The next proposition shows that it is an $\mathfrak{M}\mathfrak{o}$ -scheme.

Proposition 3.14. *1) Let $f \in M$ and $I = fM$. Then the subfunctor $X_I \subset \mathbf{spec} M$ is represented by M_f .*

2) For any ideal I , the $\mathfrak{M}\mathfrak{o}$ -functor X_I is an $\mathfrak{M}\mathfrak{o}$ -scheme.

Proof. 1) For any monoid N one has $X_I(N) = \{\rho \in \text{Hom}(M, N) \mid \rho(f) \in N^*\}$. This condition means precisely that ρ extends to a morphism $\tilde{\rho} \in \text{Hom}(M_f, N)$, with

$$\tilde{\rho}\left(\frac{m}{f^n}\right) = \rho(m)\rho(f)^{-n} \in N.$$

Thus one has a canonical and functorial isomorphism $X_I(N) = \text{Hom}(M_f, N)$.

2) For each $f \in I$, the ideal $fM \subset I$ defines a subfunctor of X_I . This sub-functor is open because it is already open in $\mathbf{spec} M$ and there are less morphisms of type $\text{Spec}(A) \xrightarrow{\varphi} X_I$ than those of type $\text{Spec}(A) \xrightarrow{\varphi} \text{Spec}(M)$, as $X_I(A) \subset \mathbf{spec} M(A)$. Moreover, by the above, this subfunctor is affine. \square

3.6. Geometric realization.

In this section we shall review the construction of the geometric realization of a \mathfrak{Mo} -scheme. This theory generalizes for commutative monoids the geometric realization of a \mathbb{Z} -scheme for rings and it has been already implemented in the geometric constructions of [13] and [5], [6].

For M an object in \mathfrak{Mo} , we let $\text{Spec}(M)$ be the set of prime ideals $\mathfrak{p} \subset M$ endowed with the natural topology whose closed subsets are

$$V(I) = \{\mathfrak{p} \subset M \mid \mathfrak{p} \supset I\}$$

as I varies among the collection of all ideals of M . As for rings, the subset $V(I) \subset \text{Spec}(M)$ depends only upon the radical of I and one has (cf. [9])

Lemma 3.15. *Given an ideal $I \subset M$, the intersection of the prime ideals $\mathfrak{p} \subset M$, with $\mathfrak{p} \supset I$ coincides with the radical of I*

$$\bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I} := \{x \in M \mid \exists n \in \mathbb{N}, x^n \in I\}.$$

A geometric space is given by

- A topological space X .
- A sheaf of monoids \mathcal{O}_X on X .

Note that unlike the theory of geometric realization as in [8] (Chapter I, § 1 Definition 1.1), there is no need to impose that the stalks of a geometric space are “local” since a monoid is *automatically* a local structure. A morphism $\varphi : X \rightarrow Y$ between two geometric spaces is given by a pair $(\varphi, \varphi^\sharp)$ of a continuous map φ and a morphism of sheaves of monoids

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\varphi^\sharp} \Gamma(\varphi^{-1}(V), \mathcal{O}_X)$$

which fulfills the crucial property of being local, *i.e.* the map of stalks $\mathcal{O}_{\varphi(x)} \xrightarrow{\varphi^\sharp} \mathcal{O}_x$ fulfills the following definition (cf. [5])

Definition 3.16. *A morphism $\rho : M_1 \rightarrow M_2$ of monoids is local if the following property holds*

$$\rho^{-1}(M_2^*) = M_1^*.$$

Notice that the above condition can be formulated equivalently by either one of the following statement

$$(1) \rho^{-1}((M_2^*)^c) = (M_1^*)^c$$

$$(2) \rho((M_1^*)^c) \subset (M_2^*)^c.$$

The equivalence with (1) is clear since $\rho^{-1}(A^c) = \rho^{-1}(A)^c$. The equivalence with (2) follows by noticing that the condition in the definition implies (2). Conversely, if (2) holds one has $\rho^{-1}((M_2^*)^c) \supset (M_1^*)^c$, but $\rho^{-1}(M_2^*) \supset M_1^*$ and one gets (1).

The sheaf of monoids associated to the *prime spectrum* $\text{Spec}(M)$ is such that:

- The stalk at $\mathfrak{p} \in \text{Spec}(M)$ is $\mathcal{O}_{\mathfrak{p}} = S^{-1}M$, with $S = \mathfrak{p}^c$.
- The following map $\varphi : M_f \rightarrow \Gamma(D(fM), \mathcal{O})$ is an isomorphism

$$\varphi(x)(\mathfrak{p}) = \frac{a}{f^n} \in \mathcal{O}_{\mathfrak{p}} \quad \forall \mathfrak{p} \in D(fM), \quad \forall x = \frac{a}{f^n} \in M_f.$$

- Let U be an open set of $\text{Spec}(M)$, then a section $s \in \Gamma(U, \mathcal{O})$ is an element of $\prod_{\mathfrak{p} \in U} \mathcal{O}_{\mathfrak{p}}$ such that on any $D(f) \subset U$ it agrees with an element of M_f .

For any geometric space X one has a canonical morphism $\psi_X : X \rightarrow \text{Spec}(\mathcal{O}(X))$.

Definition 3.17. *A geometric space X is a prime spectrum if and only if the morphism ψ_X is an isomorphism. It is a geometric $\mathfrak{M}\mathfrak{o}$ -scheme if it admits an open covering by prime spectra.*

The terminology is justified since the $\mathfrak{M}\mathfrak{o}$ -functor $\underline{X}(M) = \text{Hom}(\text{Spec}M, X)$ associated to a geometric $\mathfrak{M}\mathfrak{o}$ -scheme X is an $\mathfrak{M}\mathfrak{o}$ -scheme in the sense of Definition 3.9. The theory then proceeds exactly in parallel with the original model described in [8]. By following *op.cit.* it can be proved that one obtains all $\mathfrak{M}\mathfrak{o}$ -schemes in the form

$$\underline{X}(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\text{Spec}(N), X), \quad (18)$$

where X is the associated geometric $\mathfrak{M}\mathfrak{o}$ -scheme. We shall not spell out the translation except to explain what replaces the notions of the residue field $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$ and the natural local surjection $\epsilon_x : \mathcal{O}_x \rightarrow \kappa(x)$. Once again we recall that in the category $\mathfrak{M}\mathfrak{o}$ the role of fields is played by monoids of the form $\mathbb{F}_1[H]$ where H is an abelian group.

Lemma 3.18. *Let M be a monoid and $\mathfrak{p} \subset M$ a prime ideal. Then*

- $\mathfrak{p}M_{\mathfrak{p}} \cap M_{\mathfrak{p}}^* = \emptyset$.
- There exists a unique homomorphism $\epsilon_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow \mathbb{F}_1[M_{\mathfrak{p}}^*]$ such that
 - $\epsilon_{\mathfrak{p}}(y) = 0, \forall y \in \mathfrak{p}$
 - $\epsilon_{\mathfrak{p}}(y) = y, \forall y \in M_{\mathfrak{p}}^*$.

Proof. a) If $j \in \mathfrak{p}$ then the image of j in $M_{\mathfrak{p}}$ cannot be invertible, since this would imply an equality of the form $sja = sf$ for some $s \notin \mathfrak{p}, f \notin \mathfrak{p}$ and hence a contradiction.

b) We define, for any monoid N , a natural homomorphism

$$N \xrightarrow{\epsilon} \mathbb{F}_1[N^*], \quad \epsilon(y) = 0, \quad \forall y \notin N^*, \quad \epsilon(y) = y, \quad \forall y \in N^*.$$

The non-invertible elements of N form a prime ideal and thus ϵ is multiplicative. For $N = M_{\mathfrak{p}}$, the first statement a) shows that the corresponding map $\epsilon_{\mathfrak{p}}$ fulfills the two conditions. To check uniqueness, note that the two conditions suffice to determine $\epsilon_{\mathfrak{p}}(y)$ for any $y = a/f$ with $a \in M$ and $f \notin \mathfrak{p}$. \square

Note that by construction the morphism $\epsilon_{\mathfrak{p}}$ is *local*. Thus it is natural to consider, for any geometric space (and in particular any $\mathfrak{M}\mathfrak{o}$ -scheme) the analogue of the residue field at a point x to be $\kappa(x) = \mathbb{F}_1[\mathcal{O}_x^*]$. The evaluation map is

$$\epsilon_x : \mathcal{O}_x \rightarrow \kappa(x) = \mathbb{F}_1[\mathcal{O}_x^*] \quad (19)$$

and it satisfies the properties as in b) of Lemma 3.18.

Proposition 3.19. *Let X be a geometric space and let $U_\alpha \subset X$ be an open subset. One introduces the following $\mathfrak{M}\mathfrak{o}$ -functors*

$$\underline{X}_\alpha(M) := \text{Hom}(\text{Spec}(M), U_\alpha) \subset \text{Hom}(\text{Spec}(M), X) =: \underline{X}(M).$$

The following conditions are equivalent

- (1) The subfunctors \underline{X}_α cover \underline{X} .
- (2) The union of the U_α is X .
- (3) For any monoid M : $\underline{X}(M) = \cup_\alpha \underline{X}_\alpha(M)$.

Proof. 1) \implies 2). Assume that 2) fails and let $x \notin \cup_\alpha U_\alpha$. Then the local map $\epsilon_x : \mathcal{O}_x \rightarrow \kappa(x)$ of (19) gives a morphism from $\text{Spec}(\kappa(x))$ to X and a corresponding element $\epsilon \in \underline{X}(M) = \text{Hom}(\text{Spec}(M), X)$ for $M = \kappa(x)$. By Definition 3.8, there exists an index α such that $\epsilon \in \underline{X}_\alpha(M)$ and this shows that $x \in U_\alpha$ so that the U_α 's cover X .

2) \implies 3). Let $\phi \in \underline{X}(M) = \text{Hom}(\text{Spec}(M), X)$. Then with \mathfrak{p}_M the maximal ideal of M , one has $\phi(\mathfrak{p}_M) \in X = \cup_\alpha U_\alpha$ hence there exists an index α such that $\phi(\mathfrak{p}_M) \in U_\alpha$. It follows that $\phi^{-1}(U_\alpha) \ni \mathfrak{p}_M$ is $\text{Spec}(M)$, and one gets $\phi \in \underline{X}_\alpha(M) = \text{Hom}(\text{Spec}(M), U_\alpha)$.

The implication 3) \implies 1) is straightforward. □

Proposition 3.20. *Let X be a geometric space.*

1) *For any monoid M there exists a canonical map*

$$\pi_M : \underline{X}(M) \rightarrow X \quad (20)$$

such that

$$\pi_M(\phi) = \phi(\mathfrak{p}_M), \quad \forall \phi \in \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\text{Spec}(M), X) \quad (21)$$

2) *Let U be an open subset of X and \underline{U} the associated subfunctor of \underline{X} , then*

$$\underline{U}(M) = \pi_M^{-1}(U) \subset \underline{X}(M) \quad (22)$$

Proof. 1) is a definition of the map π_M .

2) holds since for $\phi \in \underline{X}(M) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\text{Spec}(M), X)$ one has

$$\phi(\mathfrak{p}_M) \in U \iff \phi^{-1}(U) = \text{Spec}(M).$$

□

3.7. Order structure of $\text{Spec}(M)$.

In this section we analyze the structure of a natural (partial) ordering among prime ideals in a monoid relating inclusion with the topological notion of specialization.

Proposition 3.21. 1) Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec}(M)$. The following conditions are equivalent:

- (1) $\mathfrak{p}_1 \subset \mathfrak{p}_2$
- (2) $\mathfrak{p}_2 \in \overline{\mathfrak{p}_1}$.

2) The above (equivalent) conditions define a partial ordering on $\text{Spec}(M)$ in which any family \mathfrak{p}_α of elements of $\text{Spec}(M)$ has a least upper-bound.

Proof. 1) One has (1) \Rightarrow (2) since any closed set $V(I) \subset \text{Spec}(M)$ containing \mathfrak{p}_1 is such that $\mathfrak{p}_2 \supset I$. Conversely, if (2) holds one has $\mathfrak{p}_2 \in V(\mathfrak{p}_1)$ i.e. (1).

2) The condition (1) is clearly a partial ordering. Any family \mathfrak{p}_α of elements in $\text{Spec}(M)$ admits a least upper-bound which is given by $\mathfrak{p} = \cup_\alpha \mathfrak{p}_\alpha$. This is still a prime ideal since $\mathfrak{p}^c = \cap_\alpha \mathfrak{p}_\alpha^c$ is a multiplicative subset of M . \square

Proposition 3.22. 1) Let X be a geometric $\mathfrak{M}\mathfrak{o}$ -scheme. The relation

$$\mathfrak{p}_1 \leq \mathfrak{p}_2 \iff \mathfrak{p}_2 \in \overline{\mathfrak{p}_1}$$

defines a partial ordering on X .

- 2) The interval $I_{\mathfrak{p}} = \{\mathfrak{q} | \mathfrak{q} \leq \mathfrak{p}\}$ is the intersection of all open sets containing \mathfrak{p} .
- 3) Let $W \subset X$ be an open subset. The ordering on W , viewed as a geometric $\mathfrak{M}\mathfrak{o}$ -scheme, is induced by restricting the ordering \leq on X .

Proof. 1) The condition $\mathfrak{p}_2 \in \overline{\mathfrak{p}_1}$ means that for any closed subset $F \subset X$: $\mathfrak{p}_1 \in F \Rightarrow \mathfrak{p}_2 \in F$, thus \leq is transitive.

2) One has, as in any topological space,

$$\mathfrak{p}_2 \in \overline{\mathfrak{p}_1} \iff \mathfrak{p}_1 \in \bigcap_{\mathfrak{p}_2 \in W} W, \quad W \text{ open.}$$

3) Follows from 2). \square

Notice thus that the structure of an ‘‘interval’’ such as $I_{\mathfrak{p}} = \{\mathfrak{q} | \mathfrak{q} \leq \mathfrak{p}\}$ is local i.e. for $\mathfrak{p} \in W$ with $W \subset X$ open, one has $I_{\mathfrak{p}} \subset W$. Thus, the study of these intervals can be restricted, without loss of generality, to the affine case $X = \text{Spec}(M)$.

Remark 3.23. Let us assume that \mathfrak{p} and \mathfrak{q} belong to the same affine open subset $W \subset X$ ($X = \text{geometric } \mathfrak{M}\mathfrak{o}\text{-scheme}$), then by Proposition 3.21 we know that there exists a least upper bound $\mathfrak{p} \cup_W \mathfrak{q}$ of \mathfrak{p} and \mathfrak{q} in W . But in general this upper-bound depends upon the affine open set W . In fact, it is enough to take two prime ideals \mathfrak{p} and \mathfrak{q} in a monoid M such that they are both distinct from $\mathfrak{p}_M = (M^*)^c$ but their union is $(M^*)^c$. Then, one considers the geometric $\mathfrak{M}\mathfrak{o}$ -scheme X obtained by gluing two copies of $\text{Spec}(M)$ along the open set W defined as the complement of the closed point \mathfrak{p}_M . By construction X is covered by two open affine open subsets $W_j \cong \text{Spec}(M)$ ($j = 1, 2$) and the upper-bound $\mathfrak{p} \cup_{W_j} \mathfrak{q}$ gives the distinct points x_j of X corresponding to \mathfrak{p}_M .

Let us consider the monoid $M = \mathbb{F}_1[T_1, \dots, T_n]$:

$$\mathbb{F}_1[T_1, \dots, T_n] := \{0\} \cup \{T_1^{a_1} \cdots T_n^{a_n} | a_j \in \mathbb{Z}_{\geq 0}\}. \quad (23)$$

Lemma 3.24. The map

$$\sigma : \text{Spec}(\mathbb{F}_1[T_1, \dots, T_n]) \rightarrow \{J \subset \{1, \dots, n\}\}, \quad \sigma(\mathfrak{p}) = \{j \in \{1, \dots, n\} | T_j \in \mathfrak{p}\}$$

determines a bijection between $\text{Spec}(\mathbb{F}_1[T_1, \dots, T_n])$ and the set of all subsets of $\{1, \dots, n\}$.

Proof. One has $0 \in \mathfrak{p}$, $1 \notin \mathfrak{p}$. The element $x = T_1^{a_1} \cdots T_n^{a_n}$ belongs to \mathfrak{p} if and only if for one of the non-trivial monomials $T_j^{a_j}$, $a_j > 0$, one has $T_j \in \mathfrak{p}$ (since \mathfrak{p} is a prime ideal), *i.e.* if and only if x belongs to the ideal generated by the $T_j \in \mathfrak{p}$. Thus σ is injective. It is straightforward to check that σ is also surjective. Given a subset $J \subset \{1, \dots, n\}$ the corresponding prime ideal of M is

$$\mathfrak{p}_J = \bigcup_{j \in J} T_j M.$$

□

Thus $X = \text{Spec}(\mathbb{F}_1[T_1, \dots, T_n])$ is the set $\{J \subset \{1, \dots, n\}\}$ of subsets of $\{1, \dots, n\}$ and the ordering of Proposition 3.21 is given by inclusion of subsets.

The basis of the topology of X is given by the open sets $D(fM)$ which are described for $f = \prod_{j \in S} T_j^{a_j}$ ($a_j > 0$), by

$$\mathfrak{p}_J \in D(f) \Leftrightarrow J \cap S = \emptyset \Leftrightarrow J \subset S^c = \{1, \dots, n\} \setminus S.$$

Thus the open sets in the basis correspond to the intervals $\{\mathfrak{q} | \mathfrak{q} \leq \mathfrak{p}\}$ for the ordering. It follows that the open sets W are the hereditary subsets:

$$\mathfrak{p} \in W \Rightarrow \mathfrak{q} \in W \quad \forall \mathfrak{q} \leq \mathfrak{p}. \quad (24)$$

Example 3.25. We show how to associate to a finite simplicial complex Δ an $\mathfrak{M}\mathfrak{o}$ -scheme which encodes Δ *faithfully*.

We let S be a finite set and Δ a set of subsets of S such that

$$\{x\} \in \Delta, \forall x \in S; \quad A \in \Delta \Rightarrow B \in \Delta, \forall B \subset A.$$

To S we associate the monoid $M_S = \mathbb{F}_1[T_j, j \in S]$ (which can be viewed as the monoid of subsets of S with multiplicity with the operation of union). It follows from Lemma 3.24 that $X = \text{Spec}(M_S)$ gets identified with the set of all subsets of S with topology given by hereditary subsets as open sets *cf.* (24). Thus Δ gives by construction an open subset $W_\Delta \subset \text{Spec}(M_S) = X$. By Proposition 3.22, the ordering on W_Δ , viewed as a geometric $\mathfrak{M}\mathfrak{o}$ -scheme, is the same as the restriction of the ordering of X . This shows that the finite simplicial complex Δ is recovered from the partially ordered set of points of the $\mathfrak{M}\mathfrak{o}$ -scheme W_Δ .

3.8. Restriction to abelian groups.

In this section we describe the functor obtained by restricting $\mathfrak{M}\mathfrak{o}$ -schemes to the category \mathfrak{Ab} of abelian groups. We first recall the definition of the natural functor-inclusion of \mathfrak{Ab} in $\mathfrak{M}\mathfrak{o}$.

Proposition 3.26. *The covariant functor*

$$\mathbb{F}_1[\cdot] : \mathfrak{Ab} \rightarrow \mathfrak{M}\mathfrak{o} \quad H \mapsto \mathbb{F}_1[H]$$

embeds the category of abelian groups as a full subcategory of the category of commutative monoids.

Proof. We show that the group homomorphism

$$\mathrm{Hom}_{\mathfrak{Ab}}(H, K) \longrightarrow \mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], \mathbb{F}_1[K]) \quad \phi \mapsto \mathbb{F}_1[\phi]$$

is bijective. It is injective by restriction to $H \subset \mathbb{F}_1[H]$. Moreover, any unital monoid homomorphism in $\mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], \mathbb{F}_1[K])$ preserves the absorbing elements and sends invertible elements to invertible elements since it is unital. Thus it arises from a group homomorphism. \square

We shall identify \mathfrak{Ab} to this full subcategory of $\mathfrak{M}\mathfrak{o}$. Any functor $\underline{X} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ restricts to \mathfrak{Ab} and gives rise to a functor taking values into sets. The advantage of working with the category $\mathfrak{M}\mathfrak{o}$ (rather than with \mathfrak{Ab}) is to have more objects at disposal and hence more representable functors.

One has a pair of adjoint functors: $H \mapsto \mathbb{F}_1[H]$ from \mathfrak{Ab} to $\mathfrak{M}\mathfrak{o}$ and $M \mapsto M^*$ from $\mathfrak{M}\mathfrak{o}$ to \mathfrak{Ab} *i.e.* one has

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], M) \cong \mathrm{Hom}_{\mathfrak{Ab}}(H, M^*).$$

Let M be a monoid, we consider the *Weil restriction* of the functor $\mathfrak{spec} M$, it is given by

$$\mathfrak{Ab} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad H \mapsto \mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(M, \mathbb{F}_1[H]). \quad (25)$$

The first interesting example of an object in $\mathfrak{M}\mathfrak{o}$ is the monoid $\mathbb{F}_1[T]$ (T indeterminate):

$$\mathbb{F}_1[T] := \{0\} \cup \{T^n \mid n \in \mathbb{Z}_{\geq 0}\}, \quad (26)$$

which represents the affine line \mathcal{D}^1 : *cf.* Example 3.10. Given two monoids M_j ($j = 1, 2$), we let $M_1 \times_0 M_2$ be their smash product. We collapse all elements of $\{0\} \times M_2$ and of $M_1 \times \{0\}$ to $\{0\}$. The product in $M_1 \times_0 M_2$ is given by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

The map $x \mapsto (x, 1)$ sends 0 to 0 and is a morphism in $\mathfrak{M}\mathfrak{o}$. Notice that the restriction of this construction to the subcategory \mathfrak{Ab} of $\mathfrak{M}\mathfrak{o}$ gives: $\mathbb{F}_1[H] \times_0 \mathbb{F}_1[K] = \mathbb{F}_1[H \times K]$. The first statement of the next proposition shows that the restriction to \mathfrak{Ab} of the (functor) affine line \mathcal{D}^1 is the same as the affine line described in [19] and in [1]. The second statement shows that the class of functors obtained as restrictions to \mathfrak{Ab} of representable $\mathfrak{M}\mathfrak{o}$ -functors is stable under products.

Proposition 3.27. *1) For H an object in \mathfrak{Ab} , the following identification holds*

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[T], \mathbb{F}_1[H]) = H \cup \{0\} = \mathbb{F}_1[H]. \quad (27)$$

2) Let M_j ($j = 1, 2$) be two objects in $\mathfrak{M}\mathfrak{o}$ and denote by $M_1 \times_0 M_2$ the smash product monoid. Then one has an equivalence of functors

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(M_1 \times_0 M_2, \mathbb{F}_1[H]) = \mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(M_1, \mathbb{F}_1[H]) \times \mathrm{Hom}_{\mathfrak{M}\mathfrak{o}}(M_2, \mathbb{F}_1[H]). \quad (28)$$

Proof. 1) is obvious. 2) follows from the fact that a morphism ϕ from $M_1 \times_0 M_2$ to any monoid N is determined by a (arbitrary) pair of morphisms $\phi_j : M_j \rightarrow N$. \square

The next proposition shows that the restriction to \mathfrak{Ab} of a representable $\mathfrak{M}\mathfrak{o}$ -functor is a direct sum of representable functors.

Proposition 3.28. *The functor*

$$\underline{X} : \mathfrak{Ab} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad \underline{X}(H) = \mathrm{Hom}(M, \mathbb{F}_1[H])$$

is the direct sum of representable functors $\underline{X}_{\mathfrak{p}} : \mathfrak{Ab} \rightarrow \mathfrak{Sets}, \forall \mathfrak{p} \in \text{Spec}(M)$

$$\underline{X}(H) = \cup_{\mathfrak{p}} \underline{X}_{\mathfrak{p}}(H), \quad \underline{X}_{\mathfrak{p}}(H) = \text{Hom}_{\mathfrak{Ab}}((M_{\mathfrak{p}})^*, H).$$

Proof. Let $\rho \in \text{Hom}(M, \mathbb{F}_1[H])$. Then, the ideal $\mathfrak{p} = \rho^{-1}(0)$ is prime in M . For $f \in M \setminus \mathfrak{p}, \rho(f) \in H$ invertible. Thus, one can extend ρ to $M_{\mathfrak{p}}$ by $\tilde{\rho} : M_{\mathfrak{p}} \rightarrow \mathbb{F}_1[H], \tilde{\rho}(\frac{a}{f}) = \rho(a)\rho(f)^{-1}$. This defines a group homomorphism $h = \tilde{\rho} \in \text{Hom}(M_{\mathfrak{p}}^*, H)$. It follows that ρ corresponds to a pair (\mathfrak{p}, h) . The map $\rho \rightarrow (\mathfrak{p}, h)$ is injective. Indeed, one can reconstruct ρ by setting $\rho(x) = 0, \forall x \in \mathfrak{p}$ and $\rho(x) = h(x)$ for $x \notin \mathfrak{p}$, since in that case the image of x in $M_{\mathfrak{p}}$ is invertible. Moreover, it is easy to check that any pair (\mathfrak{p}, h) occurs and uniquely. \square

The above proposition can be extended to arbitrary \mathfrak{Mo} -schemes, in fact we have seen that for any such a scheme X one has a geometric realization and for $x \in X$ $\mathcal{O}_{\mathfrak{p}}^*$ is an abelian group. Then, one obtains the following

Proposition 3.29. *Let X be an \mathfrak{Mo} -scheme. Then the functor*

$$\underline{X} : \mathfrak{Ab} \rightarrow \mathfrak{Sets}, \quad H \mapsto \text{Hom}(\text{Spec}\mathbb{F}_1[H], X) = \underline{X}(H)$$

is the disjoint union

$$\underline{X}(H) = \cup_x \underline{X}_x(H), \quad \underline{X}_x(H) = \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, H)$$

Proof. Let $\varphi \in \text{Hom}(\text{Spec}\mathbb{F}_1[H], X)$. The unique point $\mathfrak{p} \in \text{Spec}\mathbb{F}_1[H]$ corresponds to the ideal $\{0\}$. Let $\varphi(\mathfrak{p}) = x \in X$ be its image; there is a corresponding map of the stalks

$$\varphi^{\#} : \mathcal{O}_{\varphi(\mathfrak{p})} \rightarrow \mathcal{O}_{\mathfrak{p}} = \mathbb{F}_1[H].$$

This homomorphism is local by hypothesis: this means that the inverse image of $\{0\}$ by $\varphi^{\#}$ is the maximal ideal of $\mathcal{O}_{\varphi(\mathfrak{p})} = \mathcal{O}_x$. Therefore, the map $\varphi^{\#}$ is entirely determined by the group homomorphism ρ from \mathcal{O}_x^* to H obtained as the restriction of $\varphi^{\#}$. Thus $\varphi \in \text{Hom}(\text{Spec}\mathbb{F}_1[H], X)$ is entirely specified by a point $x \in X$ and a group homomorphism $\rho \in \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, H)$. \square

3.9. Local dimension.

In algebraic geometry given an algebraic variety X one knows that the degree of transcendence of the residue field $\kappa(x)$ of a point $x \in X$ measures the *dimension* of the closure $\overline{\{x\}} \subset X$. For an \mathfrak{Mo} -scheme one has the following corresponding (local) notion

Definition 3.30. *Let X be an \mathfrak{Mo} -scheme and $x \in X$ a point. The local dimension $n(x)$ of x is the rank of the abelian group \mathcal{O}_x^* .*

Example 3.31. Let $M = \mathbb{F}_1[T_1, \dots, T_n]$ (cf. (23)). A prime ideal of M is of the form $\mathfrak{p} = \cup_{i \in J} T_i M$, where J is a subset of $\{1, \dots, n\}$. When $J = \emptyset, \mathfrak{p} = \{0\}$ and in $\mathcal{O}_{\mathfrak{p}}$ all the T_j 's become invertible, thus $\mathcal{O}_{\mathfrak{p}}^* \simeq \mathbb{Z}^n$ as abelian groups. When $J = \{1, \dots, n\}$ the point \mathfrak{p} is closed and in this case $\mathcal{O}_{\mathfrak{p}} = M$, and M^* is the trivial group. In general, one gets $\mathcal{O}_{\mathfrak{p}}^* \simeq \mathbb{Z}^{J^c}$, generated by the T_j 's, with $j \notin J$.

Example 3.32. Let $M = \mathbb{F}_1[H]$, where H is an abelian group. In this case one has a single closed point $\mathfrak{p} \in X = \text{Spec}(M)$ and the rank of $\mathcal{O}_{\mathfrak{p}}^*$ is equal to the rank of H . This shows in particular that the local dimension of a closed point of a \mathfrak{Mo} -scheme can be greater than zero.

Example 3.33. Let $M = \prod_J K_j$ where K_j are fields and J is a finite set of indices. Alternatively and more in general, let $M = \prod_J \mathbb{F}_1[H_j]$, where H_j are abelian groups. Any prime ideal in M is of the form $\mathfrak{p}_A = \{x \in M \mid \exists j \in A, x_j = 0\}$, with $\emptyset \neq A \subset J$. For any such ideal \mathfrak{p} one has $M_{\mathfrak{p}} = \prod_{j \in A} K_j$ and $M_{\mathfrak{p}}^* = \prod_{j \in A} K_j^*$. Thus the local dimension of \mathfrak{p} is equal to $\sum_{j \in A} r_j$, where $r_j = \text{rk } K_j^*$. Notice the difference with the Example 3.31 due to the equivalence relation in the definition of $M_{\mathfrak{p}}$ which plays though no role in the integral case.

3.10. Finiteness conditions.

In this short section we recall briefly some classical results on finitely generated monoids: we refer to [9] for the details. A congruence on a monoid M is an equivalence relation which is compatible with the semigroup operation. A monoid is Noetherian when any strictly increasing sequence of congruences is finite (*cf.* [9] page 30).

Proposition 3.34. *Let M be a monoid. The following conditions are equivalent:*

- (1) M is Noetherian.
- (2) M is finitely generated.
- (3) $\mathbb{Z}[M]$ is a Noetherian ring.

Proof. *cf.* [9]: Theorems 7.7, 7.8 and 5.10. □

It is proven in [9] Theorem 5.1 that if M is Noetherian, then for any prime ideal \mathfrak{p} the localized monoid $M_{\mathfrak{p}}$ is also Noetherian (the semi-group \mathfrak{p}^c is finitely generated and thus $M_{\mathfrak{p}}$ is also finitely generated). The same theorem also shows that the abelian group $M_{\mathfrak{p}}^*$ is finitely generated.

Definition 3.35. *An $\mathfrak{M}\mathfrak{o}$ -scheme is Noetherian if it admits a finite open cover by $\text{Spec}(M)$ with M Noetherian.*

4. THE CATEGORY $\mathfrak{M}\mathfrak{X}$

As we already remarked in [1], the construction of the (affine) varieties over \mathbb{F}_1 in the case of Chevalley groups consists of functors to the category of sets, fulfilling much stronger properties than those required originally (for affine varieties) in [19]. The domain of these functors contains both the category of commutative rings and that of monoids and one also requires the existence of a suitable natural transformation. In this section we develop the details of this construction. Rather than working separately with the two functors from $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ and $\mathfrak{M}\mathfrak{o}$ to sets, we follow an idea we learnt from P. Cartier and construct, using a pair of adjoint functors relating these two categories, a larger category which is a gluing of $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ and $\mathfrak{M}\mathfrak{o}$. We start by developing in §4.1 some generalities on the gluing of categories, using a pair of adjoint functors. In §4.2, we also treat in this generality the extension of functors. The specific case of interest is covered in §4.3.

4.1. Gluing two categories using adjoint functors.

We consider categories \mathcal{C} and \mathcal{C}' and a pair of adjoint functors $\beta : \mathcal{C} \rightarrow \mathcal{C}'$ and $\beta^* : \mathcal{C}' \rightarrow \mathcal{C}$. Thus one has a canonical identification

$$\mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \stackrel{\Phi}{\cong} \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)). \quad (29)$$

The naturality of Φ is expressed by the commutativity of the following diagram where the vertical arrows are given by composition:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)) \\ \mathrm{Hom}(\beta(f), h) \downarrow & & \downarrow \mathrm{Hom}(f, \beta^*(h)) \\ \mathrm{Hom}_{\mathcal{C}'}(\beta(G), S) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{C}}(G, \beta^*(S)) \end{array} \quad (30)$$

and where $f \in \mathrm{Hom}_{\mathcal{C}}(G, H)$, $h \in \mathrm{Hom}_{\mathcal{C}'}(R, S)$.

We shall now define a category obtained by gluing \mathcal{C} and \mathcal{C}' . The collection³ of objects of $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ is obtained as the disjoint union of the collection of objects of \mathcal{C} and \mathcal{C}' . One defines $\mathrm{Hom}_{\mathcal{C}''}(R, H)$ to be empty, while

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \cong \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)). \quad (31)$$

The morphisms between objects contained in a same category are unchanged. The composition of morphisms in \mathcal{C}'' is defined as follows. For $\phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(H', H)$, one defines $\phi \circ \psi \in \mathrm{Hom}_{\mathcal{C}''}(H', R)$ as the composite

$$\phi \circ \beta(\psi) \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H'), R) = \mathrm{Hom}_{\mathcal{C}''}(H', R). \quad (32)$$

Using (30) one gets that

$$\Phi(\phi \circ \beta(\psi)) = \Phi(\phi) \circ \psi \in \mathrm{Hom}_{\mathcal{C}}(H', \beta^*(R)). \quad (33)$$

Similarly, for $\theta \in \mathrm{Hom}_{\mathcal{C}'}(R, R')$ one defines $\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R')$ as the composition

$$\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R') = \mathrm{Hom}_{\mathcal{C}''}(H, R') \quad (34)$$

and using (30) one obtains that

$$\Phi(\theta \circ \phi) = \beta^*(\theta) \circ \Phi(\phi) \in \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R')). \quad (35)$$

Moreover, one also obtains specific morphisms α_H and α_R as follows

$$\alpha_H = \mathrm{id}_{\beta(H)} \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), \beta(H)) = \mathrm{Hom}_{\mathcal{C}''}(H, \beta(H)) \quad (36)$$

$$\alpha_R = \Phi^{-1}(\mathrm{id}_{\beta^*(R)}) \in \Phi^{-1}(\mathrm{Hom}_{\mathcal{C}}(\beta^*(R), \beta^*(R))) = \mathrm{Hom}_{\mathcal{C}''}(\beta^*(R), R). \quad (37)$$

One has by construction

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \{g \circ \alpha_H \mid g \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R)\} \quad (38)$$

and for any morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(H, K)$

$$\alpha_K \circ \rho = \beta(\rho) \circ \alpha_H. \quad (39)$$

Similarly

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \{\alpha_R \circ f \mid f \in \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R))\} \quad (40)$$

with the equality

$$\alpha_S \circ \beta^*(\rho) = \rho \circ \alpha_R, \quad \forall \rho \in \mathrm{Hom}_{\mathcal{C}'}(R, S) \quad (41)$$

³It is not a set: we refer for details to the discussion contained in the preliminaries of [8]

and

$$g \circ \alpha_H = \alpha_R \circ \Phi(g), \quad \forall g \in \text{Hom}_{\mathcal{C}'}(\beta(H), R). \quad (42)$$

Proposition 4.1. *With the above definition, $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ is a category which contains \mathcal{C} and \mathcal{C}' as full subcategories. Moreover, for any objects H of \mathcal{C} and R of \mathcal{C}' , one has*

$$\text{Hom}_{\mathcal{C}''}(H, R) = \text{Hom}_{\mathcal{C}'}(\beta(H), R) \cong \text{Hom}_{\mathcal{C}}(H, \beta^*(R)).$$

Proof. One needs to check that the composition of morphisms is associative *i.e.* that $h \circ (g \circ f) = (h \circ g) \circ f$. The only case to check is when the image of f is an object H of \mathcal{C} and the image of g is an object R of \mathcal{C}' . Then $f(G) = H$, with G an object of \mathcal{C} and $h(R) = S$ is an object of \mathcal{C}' . One has $g \in \text{Hom}_{\mathcal{C}'}(\beta(H), R)$ and

$$g \circ'' f = g \circ \beta(f), \quad h \circ'' (g \circ'' f) = h \circ (g \circ \beta(f)) = (h \circ g) \circ \beta(f) = (h \circ'' g) \circ'' f$$

□

4.2. Extension of functors.

In the above context of pairs of adjoint functors, let $\mathcal{F}, \mathcal{F}'$ be functors from \mathcal{C} and \mathcal{C}' to a category \mathcal{T} . One can check directly that giving a natural transformation from \mathcal{F} to $\mathcal{F}' \circ \beta$ is equivalent to giving a natural transformation from $\mathcal{F} \circ \beta^*$ to \mathcal{F}' . By implementing the category $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ one has in fact:

Proposition 4.2. *A covariant functor \mathcal{F} from $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ to a category \mathcal{T} is given by the pair of functors $\mathcal{F}, \mathcal{F}'$ from \mathcal{C} and \mathcal{C}' to \mathcal{T} defined by restriction, and in an equivalent manner:*

- 1) *The natural transformation from \mathcal{F} to $\mathcal{F}' \circ \beta$ given by $\mathcal{F}(\alpha_H)$.*
- 2) *The natural transformation from $\mathcal{F} \circ \beta^*$ to \mathcal{F}' given by $\mathcal{F}(\alpha_R)$.*

Proof. 1) Let \mathcal{F} be a covariant functor from \mathcal{C}'' to a category \mathcal{T} . Then (39) shows that $\mathcal{F}(\alpha_H)$ defines a natural transformation from $\mathcal{F}|_{\mathcal{C}}$ to $\mathcal{F}|_{\mathcal{C}'} \circ \beta$. Conversely a natural transformation from \mathcal{F} to $\mathcal{F}' \circ \beta$ determines, by (38), the extension from $\mathcal{C} \cup \mathcal{C}'$ to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$.

2) This proof is similar to the proof of 1). □

We now assume to be given a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{T}$, where say $\mathcal{T} = \mathbf{Sets}$ and we investigate under which conditions \mathcal{F} admits an extension to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$. Notice first that if \mathcal{F} is representable, then it admits a unique representable extension to \mathcal{C}'' . Indeed, there exists an object G of \mathcal{C} such that

$$\mathcal{F}(H) = \text{Hom}_{\mathcal{C}}(G, H).$$

If the extension of \mathcal{F} is represented by an object of \mathcal{C}'' , it must be an object of \mathcal{C} since there is no morphism of \mathcal{C}'' from an object of \mathcal{C}' to an object of \mathcal{C} . Moreover by restriction to \mathcal{C} one gets the uniqueness. Thus the extension to \mathcal{C}'' is unique (as a representable functor) and is given on \mathcal{C}' by

$$\mathcal{F}'(R) = \text{Hom}_{\mathcal{C}'}(\beta(G), R).$$

The natural transformation $\mathcal{F} \rightarrow \mathcal{F}' \circ \beta$ is simply given by the restriction of the functor β

$$\beta : \text{Hom}_{\mathcal{C}}(G, H) \rightarrow \text{Hom}_{\mathcal{C}'}(\beta(G), \beta(H)).$$

Similarly, the natural transformation $\mathcal{F} \circ \beta^* \rightarrow \mathcal{F}'$ is given by the identity map

$$\mathrm{Hom}_{\mathcal{C}}(G, \beta^*(R)) \rightarrow \mathcal{F}'(R) = \mathrm{Hom}_{\mathcal{C}' }(\beta(G), R) \cong \mathrm{Hom}_{\mathcal{C}}(G, \beta^*(R)).$$

Proposition 4.3. *Let $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{Sets}$ be a representable functor.*

1) *There exists a unique extension $\tilde{\mathcal{F}}$ of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ as a representable functor.*

2) *Let \mathcal{G} be any extension of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$, then there exists a unique morphism of functors from $\tilde{\mathcal{F}}$ to \mathcal{G} which is the identity on \mathcal{C} .*

Proof. 1) The object of \mathcal{C} representing \mathcal{F} is unique up to isomorphism and it represents $\tilde{\mathcal{F}}$.

2) A morphism ϕ from the representable functor $\tilde{\mathcal{F}}$ (to sets) to the functor \mathcal{G} is of the form $\phi = \rho^\#$ where $\rho = \phi(A)(\mathrm{id}_A) \in \mathcal{G}(A)$ where A is an object representing $\tilde{\mathcal{F}}$. Here A is an object of \mathcal{C} . But since the restriction of ϕ to \mathcal{C} is the identity map from $\mathcal{F}(A)$ to $\mathcal{G}(A) = \mathcal{F}(A)$ one obtains the required uniqueness. \square

The following corollary shows that even though the extension $\tilde{\mathcal{F}}$ of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ is not unique it is universal.

Corollary 4.4. *With the above notations, let \mathcal{G}' be a functor from \mathcal{C}' to sets and $\phi : \mathcal{F} \rightarrow \mathcal{G}' \circ \beta$ be a natural transformation. Then there exists a unique morphism of functors $\psi : \mathcal{F}' \rightarrow \mathcal{G}'$ such that*

$$\phi_H = \psi_{\beta(H)} \circ \tilde{\mathcal{F}}(\alpha_H), \quad \forall H \in \mathrm{obj}(\mathcal{C}). \quad (43)$$

Proof. Given ϕ and \mathcal{G}' , there exists by Proposition 4.2 a unique extension \mathcal{G} of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ which restricts to \mathcal{G}' on \mathcal{C}' and is such that

$$\phi_H = \mathcal{G}(\alpha_H), \quad \forall H \in \mathrm{obj}(\mathcal{C}).$$

A morphism of functors from $\tilde{\mathcal{F}}$ to \mathcal{G} extending the identity on \mathcal{C} is entirely specified by its restriction to \mathcal{C}' which is a morphism of functors ψ from \mathcal{F}' to \mathcal{G}' and must be compatible with the morphisms α_H . This compatibility is given by (43). Thus the existence and uniqueness of ψ follows from Proposition 4.3. \square

The next proposition states a similar, but simpler, result for extensions of functors from \mathcal{C}' to the larger category $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$.

Proposition 4.5. *Let $\mathcal{F}' : \mathcal{C}' \rightarrow \mathfrak{Sets}$ be a functor.*

1) *There exists a unique extension $\tilde{\mathcal{F}}$ of \mathcal{F}' to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ given by $\mathcal{F}' \circ \beta$ on \mathcal{C} and such that $\tilde{\mathcal{F}}(\alpha_H) = \mathrm{id}_{\beta(H)}$ for all objects H of \mathcal{C} .*

2) *Let \mathcal{G} be any extension of \mathcal{F}' to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$, then there exists a unique morphism of functors from \mathcal{G} to $\tilde{\mathcal{F}}$ which is the identity on \mathcal{C}' .*

Proof. The first statement follows from Proposition 4.2 2), using the identity as a natural transformation. Similarly, for the second statement, Proposition 4.2 2) gives a unique morphism of functors ϕ given by $\phi(H) = \mathcal{G}(\alpha_H)$ from the restriction

of \mathcal{G} to \mathcal{C} to $\mathcal{G} \circ \beta = \mathcal{F}' \circ \beta$. We extend ϕ as the identity on \mathcal{C}' .

$$\begin{array}{ccc} \mathcal{G}(H) & \xrightarrow{\mathcal{G}(\alpha_H)} & \mathcal{G}(\beta(H)) \\ \downarrow \phi_H & & \downarrow \\ \tilde{\mathcal{F}}(H) = \mathcal{F}'(\beta(H)) & \longrightarrow & \tilde{\mathcal{F}}(\beta(H)) \end{array} \quad (44)$$

The commutative diagram (44) shows that one obtains a morphism of functors from \mathcal{G} to $\tilde{\mathcal{F}}$. The same diagram also gives the uniqueness of ϕ . \square

4.3. \mathbb{F}_1 -Functors.

We now apply the above construction to the pair of adjoint covariant functors β and β^* which are defined as follows:

$$\beta : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}, \quad H \mapsto \beta(H) = \mathbb{Z}[M] \quad (45)$$

associates to a monoid M the convolution ring $\mathbb{Z}[M]$ (the 0 element of M is sent to 0). The adjoint functor β^*

$$\beta^* : \mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g} \rightarrow \mathfrak{M}\mathfrak{o} \quad R \mapsto \beta^*(R) = R \quad (46)$$

associates to a ring R the ring itself viewed as a multiplicative monoid. The adjunction relation means that

$$\mathrm{Hom}(\beta(H), R) \cong \mathrm{Hom}(H, \beta^*(R)). \quad (47)$$

We now apply Proposition 4.1 to construct the category $\mathfrak{M}\mathfrak{R} = \mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g} \cup_{\beta, \beta^*} \mathfrak{M}\mathfrak{o}$. Thus, one has for every object R of $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$, a morphism

$$\alpha_R \in \mathrm{Hom}(\beta^*(R), R) \quad (48)$$

and the following relation between the morphisms of $\mathfrak{M}\mathfrak{R}$:

$$f \circ \alpha_R = \alpha_S \circ \beta^*(f), \quad \forall f \in \mathrm{Hom}(R, S). \quad (49)$$

Similarly, for every monoid M one has a morphism

$$\alpha_M \in \mathrm{Hom}(M, \beta(M)) \quad (50)$$

with the relation

$$\beta(f) \circ \alpha_M = \alpha_K \circ f, \quad \forall f \in \mathrm{Hom}(M, K). \quad (51)$$

Definition 4.6. *An \mathbb{F}_1 -functor is a covariant functor from the category $\mathfrak{M}\mathfrak{R} = \mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g} \cup_{\beta, \beta^*} \mathfrak{M}\mathfrak{o}$ to the category of sets.*

It follows from Proposition 4.2 that an \mathbb{F}_1 -functor $X : \mathfrak{M}\mathfrak{R} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ is given by:

- An $\mathfrak{M}\mathfrak{o}$ -functor \underline{X} .
- A \mathbb{Z} -functor X .
- A natural transformation $e : \underline{X} \rightarrow X \circ \beta$.

The third condition can be equivalently replaced by the assignment of a natural transformation $\underline{X} \circ \beta^* \rightarrow X$.

5. \mathbb{F}_1 -SCHEMES AND THEIR ZETA FUNCTIONS

Now that we have at our disposal the category \mathfrak{MR} obtained by gluing \mathfrak{Mo} and \mathfrak{Ring} we introduce our notion of an \mathbb{F}_1 -scheme.

Definition 5.1. *An \mathbb{F}_1 -scheme is an \mathbb{F}_1 -functor $X : \mathfrak{MR} \rightarrow \mathfrak{Sets}$, such that:*

- *The restriction of X to \mathfrak{Ring} is a \mathbb{Z} -scheme.*
- *The restriction of X to \mathfrak{Mo} is an \mathfrak{Mo} -scheme.*
- *The natural transformation e associated to a field is a bijection (of sets).*

Notice that even though the restriction \underline{X} of X to \mathfrak{Mo} is an \mathfrak{Mo} -scheme, the composition $\underline{X} \circ \beta^*$ is in general not a \mathbb{Z} -scheme since it is not a local \mathbb{Z} -functor. As an example, we may consider the case of $X = \mathbb{P}^1$: here, the above composition determines only a smaller portion of the projective line as a scheme. However, one can associate to $\underline{X} \circ \beta^*$ a *unique* \mathbb{Z} -scheme $Y = (\underline{X})_{\mathbb{Z}}$ that is defined by assigning to a ring R the set $Y(R)$ of solutions of (3), using $\underline{X} \circ \beta^*$ and an arbitrary partition of unity in R . Proposition 4.3 describes a canonical morphism of \mathbb{Z} -schemes from $(\underline{X})_{\mathbb{Z}}$ to X . However, this morphism is not in general an isomorphism: we refer, for instance, to the case of Chevalley groups of [1]. Proposition 4.5 describes the natural morphism from the \mathfrak{Mo} -scheme \underline{X} to the “gadget” (cf. [1] Definition 2.5) associated to the \mathbb{Z} -scheme X .

5.1. Torsion free Noetherian \mathbb{F}_1 -schemes.

The first theorem stated in this section shows that for any Noetherian, torsion-free \mathbb{F}_1 -scheme the function counting the number of rational points (on the associated \mathbb{Z} -scheme) over a finite field \mathbb{F}_q is *automatically* polynomial. The same theorem also provides a nice description of the zeta function introduced (upside-down) in [19]. By following *op.cit.* (§ 6 Lemme 1) and implementing the correction concerning the inversion in the formula of the zeta function, the definition of the zeta-function of an algebraic variety $X = (\underline{X}, X_{\mathbb{C}}, e_X)$ over \mathbb{F}_1 such that $\#X_{\mathbb{Z}}(\mathbb{F}_q) = N(q)$, with $N(x)$ a polynomial function and $\#\underline{X}(\mathbb{F}_{1^n}) = N(q)$ if $n = q - 1$, is given as the limit

$$\zeta_X(s) := \lim_{q \rightarrow 1} Z(X, q^{-s})(q-1)^{N(1)}. \quad (52)$$

Here, the function $Z(X, q^{-s})$ (*i.e.* the Hasse-Weil zeta function) is defined by

$$Z(X, T) := \exp \left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r} \right). \quad (53)$$

One obtains

$$N(x) = \sum_0^d a_k x^k \implies \zeta_X(s) = \prod_0^d (s-k)^{-a_k}. \quad (54)$$

For instance, in the case of the projective line \mathbb{P}^1 one has $\zeta_{\mathbb{P}^1}(s) = \frac{1}{s(s-1)}$.

We say that an \mathbb{F}_1 -scheme is Noetherian if the associated \mathfrak{Mo} and \mathbb{Z} -schemes are Noetherian (cf. Definition 3.35). An \mathbb{F}_1 -scheme is said to be torsion free if the groups \mathcal{O}_x^* of invertible elements of the monoids \mathcal{O}_x are torsion free. The following result is related to Theorem 1 of [7], but applies to non-toric varieties.

Theorem 5.2. *Let X be a torsion free Noetherian \mathbb{F}_1 -scheme. Then*

- (1) *There exists a polynomial $N(x+1)$ with positive integral coefficients such that*

$$\# X(\mathbb{F}_{1^n}) = N(n+1), \quad \forall n \in \mathbb{N}.$$

- (2) *For each finite field \mathbb{F}_q the cardinality of the set of points of the \mathbb{Z} -scheme X which are rational over \mathbb{F}_q is equal to $N(q)$.*
(3) *The zeta function of X has the following description*

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{\left(1 - \frac{1}{s}\right)^{\otimes n(x)}}, \quad (55)$$

where \otimes denotes Kurokawa's tensor product and $n(x)$ is the local dimension of X at the point x .

In (55), we use the convention that when $n(x) = 0$ the expression

$$\left(1 - \frac{1}{s}\right)^{\otimes n(x)} = s.$$

We refer to [14] and [16] for the details of the definition of Kurokawa's tensor products and zeta functions. When X is the projective line \mathbb{P}^1 , one has three points: two are of dimension zero and one is of dimension one. Thus (55) gives

$$\prod_{x \in X} \frac{1}{\left(1 - \frac{1}{s}\right)^{\otimes n(x)}} = \frac{1}{s^2} \frac{1}{1 - \frac{1}{s}} = \frac{1}{s(s-1)}.$$

The formula (55) continues to make sense in the torsion case and corresponds to the treatment of torsion given in [7].

Proof. 1) By construction, $X(\mathbb{F}_{1^n})$ is the set obtained by evaluating the restriction of X (from the subcategory $\mathfrak{M}\mathfrak{o}$ of $\mathfrak{M}\mathfrak{A}$) on $\mathfrak{A}\mathfrak{b}$, at the cyclic group $H = \mathbb{Z}/n\mathbb{Z}$. By Proposition 3.29 one has

$$X(H) = \cup X_x(H), \quad X_x(H) = \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_x^*, H). \quad (56)$$

Since X is Noetherian it is finite and for each $x \in X$ the abelian group \mathcal{O}_x^* is finitely generated and torsion free by hypothesis. The rank of \mathcal{O}_x^* is $n(x)$ and thus the set $X_x(H) = \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_x^*, H)$ has cardinality $n^{n(x)}$. It results that

$$P(y) = \sum_{x \in X} y^{n(x)}$$

and the first statement follows with $N(x+1) = P(x)$.

2) follows from 1) and the fact that the natural transformation associated to any field is a bijection. In the case of a finite field \mathbb{F}_q , the corresponding monoid is $\mathbb{F}_1[H]$ for the cyclic group $H = \mathbb{Z}/n\mathbb{Z}$ of order $n = q - 1$.

To prove 3), we start by computing explicitly Kurokawa's tensor product (for $n > 0$) as

$$\left(1 - \frac{1}{s}\right)^{\otimes n} = \prod_{j \text{ even}} (s - n + j)^{\binom{n}{j}} / \prod_{j \text{ odd}} (s - n + j)^{\binom{n}{j}}. \quad (57)$$

This follows from the definition of Kurokawa's tensor product since the divisor of zeros of $1 - \frac{1}{s}$ is $\{1\} - \{0\}$ and its n -th power is given by the binomial formula

$$(\{1\} - \{0\})^n = \sum_{k=0}^n (-1)^k \binom{n}{k} \{n-k\}.$$

To obtain (55) we use (56) to express $\zeta_X(s)$ as a product over the points of X and apply (54) and (57) to show that the zeta function for the polynomial $(q-1)^n$ is the inverse of $(1 - \frac{1}{s})^{\otimes n}$. \square

5.2. Extension of counting functions in the torsion case.

The remaining part of this section is dedicated to the computation of the zeta function of arbitrary Noetherian \mathbb{F}_1 -schemes. In this case we shall prove that the counting function

$$\# X(\mathbb{F}_{1^n}) = N_X(n+1), \quad \forall n \in \mathbb{N}$$

is no longer a polynomial function of n and its description involves some periodic functions. Our first result will be that of showing that there is a *canonical* extension of the function $N_X(n)$ to the complex domain, as an entire function $N_X(z)$, whose growth is well controlled. Then, we will explain how to compute the zeta function (52) using $N_X(q)$ for arbitrary *real* values of $q \geq 1$. The simplest example of a Noetherian \mathbb{F}_1 -scheme is $X = \text{Spec}(\mathbb{F}_{1^m})$ and the number of points of X over \mathbb{F}_{1^n} is the cardinality of the set of group homomorphisms $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, *i.e.*

$$\# X(\mathbb{F}_{1^n}) = \# \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \text{gcd}(n, m). \quad (58)$$

This is a *periodic* function of n . More generally the following statement holds

Lemma 5.3. *Let X be a Noetherian \mathbb{F}_1 -scheme. The counting function is a finite sum of monomials of the form*

$$\# X(\mathbb{F}_{1^n}) = \sum_{x \in X} n^{n(x)} \prod \text{gcd}(n, m_j(x)) \quad (59)$$

where $n(x)$ is the local dimension at $x \in X$ and the $m_j(x)$ are the orders of the finite cyclic groups which compose the residue “field” $\kappa(x) = \mathbb{F}_1[\mathcal{O}_x^*]$ *i.e.*

$$\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \prod_j \mathbb{Z}/m_j(x)\mathbb{Z}.$$

Proof. It follows from the proof of the first part of Theorem 5.2 that

$$\# X(\mathbb{F}_{1^n}) = \sum_{x \in X} \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, \mathbb{Z}/n\mathbb{Z})$$

and the result follows from (58). \square

We now proceed by explaining how to extend a function such as (59) to an entire function in a canonical manner.

5.2.1. Extension of functions from \mathbb{Z} to \mathbb{C} .

Let $f(z)$ be an entire function of $z \in \mathbb{C}$ and $T(R, f)$ its *characteristic function*:

$$T(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\theta})| d\theta. \quad (60)$$

This function may be interpreted as the average magnitude of $\log^+ |f(z)|$. One sets

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

and one denotes by

$$N(a, R) = \sum_{\substack{f(z)=a \\ |z|<R}} \log \left| \frac{R}{z} \right| \quad (61)$$

the sum over the zeros of $f - a$ inside the disk of radius R . The First Fundamental Theorem of Nevanlinna theory states as follows

Theorem 5.4. (First Fundamental Theorem) *If $a \in \mathbb{C}$ then*

$$N(a, R) \leq T(R, f) - \log |f(0) - a| + \epsilon(a, R)$$

where

$$|\epsilon(a, R)| \leq \log^+ |a| + \log 2.$$

One deduces from this theorem that if an entire function f vanishes on $\mathbb{Z} \subset \mathbb{C}$, then one has that

$$N(0, R) \geq \sum_{\substack{z \in \mathbb{Z} \setminus 0 \\ |z| < R}} \log \left| \frac{R}{z} \right| \sim 2R$$

and hence that

$$\underline{\lim} \frac{T(R, f)}{R} \geq 2. \quad (62)$$

We shall now use this inequality to prove the following result

Proposition 5.5. *Let $N(n)$ be a real function on \mathbb{Z} of the form*

$$N(n) = \sum_{j \geq 0} T_j(n) n^j \quad (63)$$

where the $T_j(n)$ are periodic functions. Then, there exists a unique polynomial $Q(n)$ and a unique entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

- $\overline{f(\bar{z})} = f(z)$.
- $\overline{\lim} \frac{T(R, f)}{R} < 2$.
- $N(n) - (-1)^n Q(n) = f(n) \quad \forall n \in \mathbb{Z}$.

Proof. Choose $L \in \mathbb{N}$ such that all T_j fulfill $T_j(m + L) = T_j(m)$, for all $m \in \mathbb{Z}$. For each j , one has a Fourier expansion of the form

$$T_j(m) = \sum_{\alpha \in I_L} t_{j, \alpha} e^{2\pi i \alpha m} + s_j (-1)^m, \quad I_L = \left\{ \frac{k}{L} \mid |k| < L/2 \right\}.$$

We define $Q(n) = \sum_j s_j n^j$ and the function $f(z)$ by

$$f(z) = \sum_j \sum_{\alpha \in I_L} t_{j, \alpha} e^{2\pi i \alpha z} z^j.$$

Since $|\alpha| \leq c < \frac{1}{2}$, $\forall \alpha \in I_L$, one gets that

$$\log^+ |f(Re^{i\theta})| \leq 2\pi R c |\sin \theta| + o(R)$$

which in turn implies $\overline{\lim} \frac{T(R, f)}{R} < 2$. The equality $N(n) - (-1)^n Q(n) = f(n)$ for all $n \in \mathbb{Z}$ holds by construction and it remains to show the uniqueness. This amounts to show that a non-zero entire function $f(z)$ such that $\overline{f(\bar{z})} = f(z)$ and $f(n) = (-1)^n Q(n)$ for all $n \in \mathbb{Z}$ for some polynomial $Q(n)$ automatically fulfills (62). For $x \in \mathbb{R}$ one has $f(x) \in \mathbb{R}$. Let $s_k n^k$ be the leading term of $Q(n) = \sum s_j n^j$, then

outside a finite set of indices n the sign of $Q(n)$ is independent of n for $n > 0$ and for $n < 0$. It follows that the function f changes sign in each interval $[n, n + 1]$ and hence admits at least one zero in such intervals. Moreover, $N(0, R) = \sum_{\substack{f(z)=0 \\ |z| < R}} \log \left| \frac{R}{z} \right|$

fulfills

$$\liminf \frac{N(0, R)}{R} \geq 2$$

and hence, by Theorem 5.4 one gets

$$\liminf \frac{T(R, f)}{R} \geq 2$$

which gives the required uniqueness. \square

Definition 5.6. Let $N(n)$ be a real function on \mathbb{Z} of the form (63), then the canonical extension $N(z)$, $z \in \mathbb{C}$ is given by

$$N(z) = f(z) + \cos(\pi z)Q(z) \quad (64)$$

where f and Q are uniquely determined by Proposition 5.5.

We shall explicitly compute this extension in the case of the function $N_X(q)$ for $X = \text{Spec}(\mathbb{F}_1[H])$.

5.2.2. *The counting function for $X = \text{Spec}(\mathbb{F}_1[H])$.*

In this subsection we compute the counting function $N_X(n)$ for $X = \text{Spec}(\mathbb{F}_1[H])$, with H a finitely generated abelian group. We start by considering the cyclic case $H = \mathbb{Z}/m\mathbb{Z}$.

Lemma 5.7. Let $X = \text{Spec}(\mathbb{F}_1^m)$. Then, the canonical extension of the counting function $N_X(n)$ is given by

$$N(z) = \sum_{d|m} \frac{\varphi(d)}{d} \left(\sum_{|k| < d/2} e^{2i\pi \frac{(z-1)k}{d}} + \epsilon_d \cos(\pi(z-1)) \right), \quad z \in \mathbb{C}, \quad (65)$$

where $\epsilon_d = 1$ if d is even and $\epsilon_d = 0$ otherwise.

Proof. It follows from (58) that $N(n) = \gcd(n-1, m)$. Moreover one knows that

$$\gcd(n, m) = \sum_{\substack{d|n \\ d|m}} \varphi(d)$$

where φ is the Euler totient function. Thus

$$\gcd(n, m) = \sum_{d|m} \frac{\varphi(d)}{d} \sum e^{2i\pi \frac{nk}{d}} \quad (66)$$

where the sum $\sum e^{2i\pi \frac{nk}{d}}$ is taken over all characters of the additive group $\mathbb{Z}/d\mathbb{Z}$ evaluated on n modulo d . This sum can be written as

$$\sum_{|k| < d/2} e^{2i\pi \frac{nk}{d}} = \sum_{|k| < d/2} e^{2i\pi \frac{nk}{d}} + \epsilon_d \cos(\pi n)$$

which gives the canonical extension of $N(n)$ after a shift of one in the variable. \square

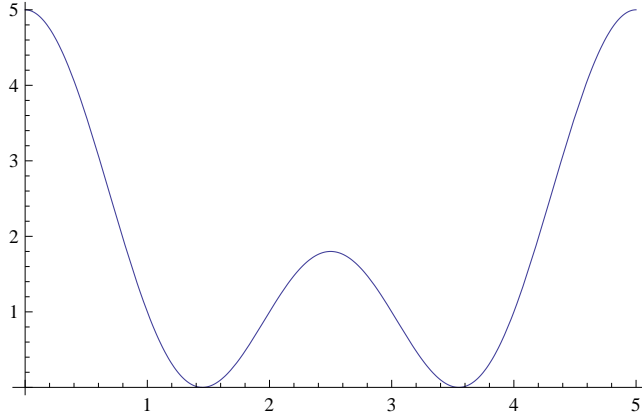


FIGURE 1. The graph of the canonical extension $N(q)$ for $H = \mathbb{Z}/5\mathbb{Z}$.

We now provide a conceptual explanation for the expansion (65). Let C_d be the d -th cyclotomic polynomial

$$C_d(x) = \prod_{r|d} (x^{d/r} - 1)^{\mu(r)}.$$

One has the decomposition into irreducible factors over \mathbb{Q}

$$x^m - 1 = \prod_{d|m} C_d(x)$$

which determines the decomposition

$$\mathbb{Q}[\mathbb{Z}/m\mathbb{Z}] = \bigoplus_{d|m} \mathbb{K}_d$$

where $\mathbb{K}_d = \mathbb{Q}(\xi_d)$ is the cyclotomic field extension of \mathbb{Q} by a primitive d -th root of 1. This decomposition corresponds to a factorization of the zeta function

$$\zeta_X(s) = \prod_p \zeta_{X/\mathbb{F}_p}(s)$$

as a product

$$\zeta_X(s) = \prod_{d|m} \zeta_{X/\mathbb{K}_d}(s)$$

where each term is in turns a product of L -functions

$$\zeta_{X/\mathbb{K}_d}(s) = \prod_{\chi \in \hat{G}(d)} L(s, \chi), \quad G(d) = (\mathbb{Z}/d\mathbb{Z})^*$$

and the order of $G(d)$ is $\varphi(d)$. The decomposition

$$\zeta_X(s) = \prod_{d|m} \prod_{\chi \in \hat{G}(d)} L(s, \chi)$$

corresponds precisely to the decomposition (66).

Let now $H = \prod_j \mathbb{Z}/m_j\mathbb{Z}$ be a finite abelian group. Then, the extension of the group ring $\mathbb{Z}[H]$ over \mathbb{Q} can be understood in the same way as the tensor product

$$\mathbb{Q}[H] = \bigotimes_j \left(\bigoplus_{d|m_j} \mathbb{K}_d \right).$$

In fact, by applying the Galois correspondence, the decomposition of $H = \prod_j \mathbb{Z}/m_j\mathbb{Z}$ corresponds to the disjoint union of orbits for the action of the Galois group $\hat{\mathbb{Z}}^*$ on the cyclotomic extension $\mathbb{Q}^{\text{ab}}/\mathbb{Q}$ (i.e. the maximal abelian extension of \mathbb{Q}). If we denote the group H additively, two elements $x, y \in H$ are on the same Galois orbit if and only if $\{nx \mid n \in \mathbb{Z}\} = \{ny \mid n \in \mathbb{Z}\}$ i.e. if and only if they generate the same subgroup of H . Thus the set Z of orbits is the set of cyclic subgroups of H .

For a cyclic group C of order d we let, as in (65)

$$N(q, C) = \frac{\varphi(d)}{d} \left(\sum_{|k| < d/2} e^{2i\pi \frac{(q-1)k}{d}} + \epsilon_d \cos(\pi(q-1)) \right). \quad (67)$$

Then, we obtain the following

Lemma 5.8. *Let $X = \text{Spec}(\mathbb{F}_1[\Gamma])$ where $\Gamma = \mathbb{Z}^k \times H$ is a finitely generated abelian group (i.e. H is a finite abelian torsion group). Then, the canonical extension of the counting function $N_X(q)$ is given by*

$$N(q) = (q-1)^k \sum_{C \subset H} N(q, C) \quad (68)$$

where the sum is over the cyclic subgroups $C \subset H$.

Proof. It is enough to show that the function $N(n+1)$ as in (68) gives the number of homomorphisms from Γ to $\mathbb{Z}/n\mathbb{Z}$. Indeed, as a function of q it is already in the canonical form of Definition 5.6. Then, by duality we just need to compare the number $\iota(n, H)$ of homomorphisms ρ from $\mathbb{Z}/n\mathbb{Z}$ to H with $\sum_{C \subset H} N(q, C)$. The cyclic subgroup $C = \text{Im } \rho$ is uniquely determined, thus

$$\iota(n, H) = \sum_{C \subset H} \mu(n, C)$$

where $\mu(n, C)$ is the number of surjective homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to C . If d denotes the order of C , this number is 0 unless $d|n$ and in the latter case it is $\varphi(d)$, thus using (67) it coincides in all cases with $N(n+1, C)$. \square

5.3. An integral formula for $\frac{\partial_s \zeta_N(s)}{\zeta_N(s)}$.

Let $N(q)$ be a real continuous function on $[1, \infty)$ such that $|N(q)| \leq Cq^k$ for some finite positive integer k and a fixed positive integer constant C . Then the associated generating function is

$$Z(q, T) = \exp \left(\sum_{r \geq 1} N(q^r) T^r / r \right)$$

and one knows that the power series $Z(q, q^{-s})$ converges for $\Re(s) > k$. The definition of the associated zeta function over \mathbb{F}_1

$$\zeta_N(s) = \lim_{q \rightarrow 1} (q-1)^\chi Z(q, q^{-s}), \quad \chi = N(1)$$

requires some care for assuring its convergence. One can eliminate the ambiguity in the extraction of the finite part by working instead with the function

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = - \lim_{q \rightarrow 1} F(q, s) \quad (69)$$

where

$$F(q, s) = -\partial_s \sum_{r \geq 1} N(q^r) \frac{q^{-rs}}{r}. \quad (70)$$

Lemma 5.9. *With the above notations and for $\Re(s) > k$*

$$\lim_{q \rightarrow 1} F(q, s) = \int_1^\infty N(u) u^{-s} d^*u, \quad d^*u = du/u \quad (71)$$

and

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = - \int_1^\infty N(u) u^{-s} d^*u. \quad (72)$$

Proof. The proof follows immediately by noticing that

$$F(q, s) = \sum_{r \geq 1} N(q^r) q^{-rs} \log q$$

is the Riemann sum for the integral $\int_1^\infty N(u) u^{-s} d^*u$. \square

Thus, assuming that $N(1) = 0$ (which is achieved by subtracting a constant) we get the following expression by integrating in s

$$\log(\zeta_N(s)) = \int_1^\infty \frac{N(u)}{\log u} u^{-s} d^*u. \quad (73)$$

When $N(1) \neq 0$ one has to choose a principal value in the expression (73) near $u = 1$, since the term $\frac{N(u)}{\log u}$ is singular. To match the normalization used in [19], we take the principal value as

$$\log(\zeta_N(s)) = \lim_{\epsilon \rightarrow 0} \left(\int_{1+\epsilon}^\infty \frac{N(u)}{\log u} u^{-s} d^*u + N(1) \log \epsilon \right). \quad (74)$$

This choice does not alter (72) which we shall use to investigate the analytic nature of the function $\zeta_N(s)$.

5.4. Analyticity of the integrals.

The statement of Lemma 5.9 and the explicit form of the function $N(q)$ as in Lemma 5.7 clearly indicate that we need to analyze basic integrals of the form

$$f(s, a) = \int_1^\infty e^{iau} u^{-s} d^*u. \quad (75)$$

Lemma 5.10. *For $a > 0$*

$$f(s, a) = a^s \int_a^\infty e^{iu} u^{-s} d^*u \quad (76)$$

defines an entire function of $s \in \mathbb{C}$.

Proof. For $\Re(s) > 0$, the integral in (76) is absolutely convergent. Integrating by parts one obtains

$$\int_a^\infty e^{iu} u^{-s-1} du + \int_a^\infty \frac{1}{i} e^{iu} (-s-1) u^{-s-2} du = \left[\frac{1}{i} e^{iu} u^{-s-1} \right]_a^\infty$$

which supplies the equation

$$af(s, a) = -i(s+1)f(s+1, a) + ie^{ia}. \quad (77)$$

This shows that f is an entire function of s , since after iterating (77) the variable s can be moved to the domain of absolute convergence. \square

The function $f(s, a)$ in (75) can be expressed in terms of hypergeometric functions in the following way

$$f(s, a) = e^{-\frac{1}{2}i\pi s} a^s \Gamma(-s) + i \frac{a}{s-1} H\left[\frac{1-s}{2}, \left(\frac{3}{2}, \frac{3-s}{2}\right), -\frac{a^2}{4}\right] + \frac{1}{s} H\left[-\frac{s}{2}, \left(\frac{1}{2}, \frac{2-s}{2}\right), -\frac{a^2}{4}\right]$$

where the hypergeometric function H is defined as

$$H(\alpha, (\beta, \gamma), z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k (\gamma)_k} \frac{z^k}{k!}$$

and one sets $(x)_k = \prod_{j=0}^{k-1} (x+j)$. Note that the formulas simplify as follows

$$\frac{1}{s-1} H\left[\frac{1-s}{2}, \left(\frac{3}{2}, \frac{3-s}{2}\right), -\frac{a^2}{4}\right] = \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k+1)!(s-(2k+1))}$$

and

$$\frac{1}{s} H\left[-\frac{s}{2}, \left(\frac{1}{2}, \frac{2-s}{2}\right), -\frac{a^2}{4}\right] = \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k)!(s-2k)}.$$

By using (75) and expanding the exponential in powers of a in the integral

$$\int_0^1 e^{iau} u^{-s} d^*u = \sum_{n=0}^{\infty} \int_0^1 \frac{(iau)^n}{n!} u^{-s} d^*u$$

the resulting expression for $f(s, a)$ in terms of hypergeometric functions becomes

$$f(s, a) = e^{-\frac{1}{2}i\pi s} a^s \Gamma(-s) + \sum_{n=0}^{\infty} \frac{(ia)^n}{n!(s-n)}. \quad (78)$$

Thus one gets

Lemma 5.11. *For $a > 0$, the following identity holds*

$$f(s, a) = e^{-\frac{1}{2}i\pi s} a^s \Gamma(-s) - \sum_{\text{poles}} \frac{\text{residue}}{s-n}. \quad (79)$$

Proof. The last term of the expression (78) is a sum of the form $\sum_{\text{poles } g} \frac{\text{residue}(g)}{s-n}$ where $g(s) = e^{-\frac{1}{2}i\pi s} a^s \Gamma(-s)$. This suffices to conclude the proof since one knows from lemma 5.10 that $f(s, a)$ is an entire function of s . One can also check directly that the expression for the residues as in (78) is correct by using *e.g.* the formula of complements. \square

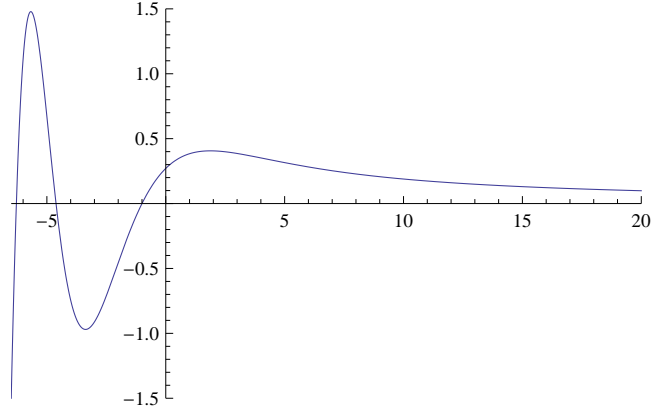


FIGURE 2. Graph of the function $\partial_s H_d(s)$ for $d = 3$.

5.5. Zeta function of Noetherian \mathbb{F}_1 -schemes.

The above discussion has shows that the elementary constituents of the zeta function of a Noetherian \mathbb{F}_1 -scheme are of the following type

$$\xi_d(s) = s^{-\frac{\varphi(d)}{d}} e^{-\frac{\varphi(d)}{d} H_d(s)} \quad (80)$$

where $H_d(s)$ is the primitive, vanishing for $\Re(s) \rightarrow \infty$, of the entire function

$$\partial_s H_d(s) = \sum_{\substack{|k| \leq d/2 \\ k \neq 0}} e^{-i \frac{2\pi k}{d}} f(s, \frac{2\pi k}{d}). \quad (81)$$

When d is even, one divides by 2 the contribution of $k = \frac{d}{2}$. Note that, by Lemma 5.11, the function $\partial_s H_d(s)$ is obtained by applying the transformation

$$F \mapsto F - \sum_{\text{poles}} \frac{\text{residue}}{s - n}$$

to the function

$$F_d(s) = \Gamma(-s) \sum_{\substack{|k| \leq d/2 \\ k \neq 0}} \cos\left(\frac{2\pi k}{d} + \frac{\pi s}{2}\right) (2\pi \frac{k}{d})^s.$$

Theorem 5.12. *Let X be a Noetherian \mathbb{F}_1 -scheme. Then the zeta function of X is a finite product*

$$\zeta_X(s) = \prod_{x \in X} \zeta_{X_x}(s) \quad (82)$$

where X_x denotes the constituent of X over the residue field $\kappa(x) = \mathbb{F}_1[\mathcal{O}_x^*]$ and $\mathcal{O}_x^* = \Gamma$ is a finitely generated abelian group $\Gamma = H \times \mathbb{Z}^n$ so that

$$\zeta_{X_x}(s) = \prod_{k \geq 0} \zeta(\mathbb{F}_1[H], s - n + k)^{(-1)^k \binom{n}{k}}. \quad (83)$$

$\zeta(\mathbb{F}_1[H], s)$ is the product

$$\zeta(\mathbb{F}_1[H], s) = \prod_{d||H} \xi_d(s)^{\gamma(H,d)} \quad (84)$$

where $\gamma(H, d)$ denotes the number of cyclic subgroups $C \subset H$ of order d .

Proof. The proof of (82) follows the same lines as that given in Theorem 5.2. The expression (83) for $\zeta_{X_x}(s)$ in terms of $\zeta(\mathbb{F}_1[H], s)$ follows from (68) and the simple translation in s that the multiplication of the counting function by a power of q generates using Lemma 5.9. By (68) and (67) it remains to show that the zeta function $Z_d(s)$ associated to the counting function (67) coincides with $\xi_d(s)$ defined in (80). By Lemma 5.9 and (72) the logarithmic derivative of $Z_d(s)$ is

$$\frac{\partial_s Z_d(s)}{Z_d(s)} = - \int_1^\infty \frac{\varphi(d)}{d} \left(\sum_{|k| < d/2} e^{2i\pi \frac{(u-1)k}{d}} + \epsilon_d \cos(\pi(u-1)) \right) u^{-s} d^* u.$$

The term coming from $k = 0$ contributes with $-\frac{\varphi(d)}{d} \frac{1}{s}$ and thus corresponds to the fractional power $s^{-\frac{\varphi(d)}{d}}$ in (80). The other terms contribute with integrals of the form

$$I(s, \alpha) = \int_1^\infty e^{2\pi i \alpha (u-1)} u^{-s} d^* u.$$

One has

$$I(s, \alpha) = e^{-2\pi i \alpha} \int_1^\infty e^{2\pi i \alpha u} u^{-s} d^* u = e^{-2\pi i \alpha} (2\pi \alpha)^s \int_{2\pi \alpha}^\infty e^{iv} v^{-s} d^* v.$$

Thus with the notation (76) one obtains

$$I(s, \alpha) = e^{-ia} f(s, a), \quad a = 2\pi \alpha. \quad (85)$$

The equality $Z_d(s) = \xi_d(s)$ follows using (81). \square

Corollary 5.13. *Let X be a Noetherian \mathbb{F}_1 -scheme. Then the zeta function of X is the product*

$$\zeta_X(s) = e^{h(s)} \prod_{j=0}^n (s-j)^{\alpha_j} \quad (86)$$

of an exponential of an entire function by a finite product of fractional powers of simple monomials. The exponents α_j are rational numbers given explicitly as

$$\alpha_j = (-1)^{j+1} \sum_{x \in X} (-1)^{n(x)} \binom{n(x)}{j} \sum_d \frac{1}{d} \nu(d, \mathcal{O}_x^*) \quad (87)$$

where $n(x)$ is the local dimension (i.e. the rank of \mathcal{O}_x^*), and $\nu(d, \mathcal{O}_x^*)$ is the number of injective homomorphisms from $\mathbb{Z}/d\mathbb{Z}$ to \mathcal{O}_x^* .

Proof. By (80) one has

$$\zeta(\mathbb{F}_1[H], s) = \prod_{d||H} \xi_d(s)^{\gamma(H,d)} = s^{-\epsilon_H} e^{-\sum_{d||H} \frac{\varphi(d)\gamma(H,d)}{d} H_d(s)}$$

where

$$\epsilon_H = \sum_{d||H} \frac{\varphi(d)\gamma(H,d)}{d}. \quad (88)$$

Next, by (83) one has

$$\zeta(\mathbb{F}_1[\Gamma], s) = \prod_{k \geq 0} \zeta(\mathbb{F}_1[H], s - n + k)^{(-1)^k \binom{n}{k}}$$

which is a product of the form

$$\zeta(\mathbb{F}_1[\Gamma], s) = \prod_{k \geq 0} (s - j)^{-(-1)^{n-j} \binom{n}{j} \epsilon_H} e^{h(s)}$$

where $h(s)$ is an entire function. Thus one gets that the exponent α_j of $(s - j)$ is

$$\alpha_j = (-1)^{j+1} \sum_{x \in X} (-1)^{n(x)} \binom{n(x)}{j} \sum_{d \mid |H_x|} \frac{\varphi(d) \gamma(H_x, d)}{d}$$

where $n(x)$ is the local dimension and H_x is the finite group in the decomposition of \mathcal{O}_x^* . The above expression gives (87) since $\varphi(d) \gamma(H_x, d)$ is the number of injective homomorphisms from $\mathbb{Z}/d\mathbb{Z}$ to \mathcal{O}_x^* . \square

6. BEYOND \mathbb{F}_1 -SCHEMES

One naturally wonders if the method of the computation of the zeta function over \mathbb{F}_1 that we have described in the above section can in some way be extended to a variety X which is of more complicated type than that appearing as the \mathbb{Z} -scheme part of a Noetherian \mathbb{F}_1 -scheme. In this section we shall first show that as long as we are only interested in the singularities of $\zeta_X(s)$, one may replace the integral which appears in the computation of $\frac{\partial_s \zeta_N(s)}{\zeta_N(s)}$ in (72) by the corresponding infinite sum *i.e.*

$$-\int_1^\infty N(u) u^{-s} d^*u \rightarrow -\sum_{n \geq 1} N(n) n^{-s-1}. \quad (89)$$

This substitution provides one with a natural definition of a modified zeta function defined by

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = -\sum_1^\infty N(n) n^{-s-1}. \quad (90)$$

We shall then compute this function for a *particular* choice of the extension of the counting function $N(q)$, in the case of an elliptic curve over \mathbb{Q} .

Finally, in §6.3 we will determine the *uniquely* defined distribution $N(u)$ on $[1, \infty)$ which describes the counting function for the hypothetical curve $C = \overline{\text{Spec } \mathbb{Z}}$ over \mathbb{F}_1 whose zeta function $\zeta_C(s)$ (over \mathbb{F}_1) coincides with the completed Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ over \mathbb{C} .

6.1. Modified zeta function.

In the context of Noetherian \mathbb{F}_1 -schemes one has

Proposition 6.1. *If one performs the replacement (89) in the definition of $\frac{\partial_s \zeta_N(s)}{\zeta_N(s)}$ in (72), one does not alter the singularities of $\zeta_X(s)$ for a Noetherian \mathbb{F}_1 -scheme X . In other words, the ratio of the two zeta functions is the exponential of an entire function.*

Proof. It is enough to treat the case when the function $N(u)$ is of the form

$$N(u) = \frac{\varphi(d)}{d} u^\ell \left(\sum_{|k| < d/2} e^{2i\pi \frac{(u-1)k}{d}} + \epsilon_d \cos(\pi(u-1)) \right) \quad (91)$$

and show that the following defines an entire function of s

$$h(s) = \int_1^\infty N(u) u^{-s} d^*u - \sum_1^\infty N(n) n^{-s-1}.$$

The term u^ℓ generates a shift in s and thus we can assume that $\ell = 0$. Let $G(d) = (\mathbb{Z}/d\mathbb{Z})^*$ be the multiplicative group of residues modulo d which are prime to d . By extending the characters $\chi \in \hat{G}(d)$ by 0 on residues modulo d which are not prime to d , one obtains the equality

$$\sum_{\chi \in \hat{G}(d)} \chi(n) = \begin{cases} \varphi(d) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

Thus the restriction of $N(u)$ to the integers agrees with this sum

$$N(n) = \sum_{\chi \in \hat{G}(d)} \chi(n)$$

and one gets

$$\sum_{n \geq 1} N(n) n^{-s-1} = \sum_{\chi \in \hat{G}(d)} L(\chi, s+1).$$

When $\chi \neq 1$ the function $L(\chi, s+1)$ is entire, thus the only singularity arises from the function $L(1, s+1)$ which is known to have a unique pole at $s = 0$ with residue $\prod_{p|d} (1 - \frac{1}{p})$. The only singularity of the integral $\int_1^\infty N(u) u^{-s} d^*u$ is due to the contribution of the constant term *i.e.*

$$\int_1^\infty \frac{\varphi(d)}{d} u^{-s} d^*u = \frac{\varphi(d)}{d} \frac{1}{s}.$$

Thus the equality $\prod_{p|d} (1 - \frac{1}{p}) = \frac{\varphi(d)}{d}$ shows that the function $h(s)$ is an entire function. \square

We can thus adopt the following definition

Definition 6.2. Let X be an \mathbb{F}_1 -scheme. The modified zeta function $\zeta_X^{\text{disc}}(s)$ is defined by

$$\frac{\partial_s \zeta_X^{\text{disc}}(s)}{\zeta_X^{\text{disc}}(s)} = - \sum_{n \geq 1} N(n) n^{-s-1} \quad (92)$$

where $N(n+1) = \#X(\mathbb{F}_{1^n})$.

By Proposition 6.1 the singularities of this modified zeta function are the same as the singularities of $\zeta_X(s)$: *cf.* Corollary 5.13. The advantage of this modified definition is that it requires no choice of interpolating function. This means that one can define $\zeta_X^{\text{disc}}(s)$ up to a multiplicative normalization factor by just having some polynomial control on the size of growth of the finite set $X(\mathbb{F}_{1^n})$, without adding any further hypothesis.

6.2. Elliptic curves.

Definition 6.2 applies each time one has a reasonable guess of the definition of an extension of the counting of points from the case $q = p^\ell$ of prime powers to the case of arbitrary positive integers. For elliptic curves E over \mathbb{Q} it is tempting to use the associated modular form

$$F_E(\tau) = \sum_{n=1}^{\infty} a(n)q^n, \quad q = e^{2\pi i\tau} \quad (93)$$

since the sequence $a(n)$ of the coefficients of the above series is defined for all values of n and also fulfills the equation

$$N(p, E) = \#E(\mathbb{F}_p) = p + 1 - a(p) \quad (94)$$

at each prime number p of good reduction for E . This is however, a bit too naive since (94) does not continue to hold at prime powers.

We define the function $t(n)$ as the only multiplicative function which agrees with $a(p)$ at each prime and for which (94) continues to hold for prime powers, then we have the following

Lemma 6.3. *Let $t(m)$ be the only multiplicative function such that $t(1) = 1$ and*

$$N(q, E) = q + 1 - t(q) \quad (95)$$

for any prime power q . Then the generating function

$$R(s, E) = \sum_{n \geq 0} t(n) n^{-s} \quad (96)$$

has the following description

$$R(s, E) = \frac{L(s, E)}{\zeta(2s-1)M(s)} \quad (97)$$

where $L(s, E)$ is the L -function of the elliptic curve, $\zeta(s)$ is the Riemann zeta function, and

$$M(s) = \prod_{p \in S} (1 - p^{1-2s}) \quad (98)$$

where S is the set of primes of bad reduction for E .

Proof. One has an Euler product for the L -function $L(s, E) = \sum a(n)n^{-s}$ of the elliptic curve, of the form

$$L(s, E) = \prod_p L_p(s, E) \quad (99)$$

where for almost all primes, *i.e.* for $p \notin S$ with S a finite set

$$L_p(s, E) = ((1 - \alpha_p p^{-s})(1 - \bar{\alpha}_p p^{-s}))^{-1} = \sum_{n=0}^{\infty} a(p^n) p^{-ns} \quad (100)$$

where $\alpha_p = \sqrt{p} e^{i\theta_p}$. Since the function $t(n)$ is multiplicative one has an Euler product

$$R(s, E) = \sum_{n \geq 0} t(n) n^{-s} = \prod_p \left(\sum_{n \geq 0} t(p^n) p^{-ns} \right) = \prod_p R_p(s, E). \quad (101)$$

Let $p \notin S$ be a prime of good reduction. Then for $q = p^\ell$, one has

$$N(q, E) = q + 1 - \alpha_p^\ell - \bar{\alpha}_p^\ell. \quad (102)$$

We now show that

$$(1 - p^{1-2s}) \sum_{n=0}^{\infty} a(p^n) p^{-ns} = \sum_{n=0}^{\infty} t(p^n) p^{-ns}. \quad (103)$$

This relation follows from (100) and the equality

$$\frac{(1 - px^2)}{(1 - \alpha_p x)(1 - \bar{\alpha}_p x)} = \frac{1}{(1 - \alpha_p x)} + \frac{1}{(1 - \bar{\alpha}_p x)} - 1$$

for $x = p^{-s}$. The above formula is familiar from the expression of the Poisson kernel (*cf.* [17] §§ 5.24, 11.5)

$$P_r(\theta) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Using (103) one gets

$$R_p(s, E) = (1 - p^{1-2s}) \sum a(p^n) p^{-ns} \quad (104)$$

where the right hand side is the local factor $L_p(s, E)$ of the L -function (*cf.* [18] Appendix C §16). When $p \in S$ is a prime of bad reduction for E , the local factor $L_p(s, E)$ of the L -function is described as follows (*cf. op.cit.*)

- $L_p(s, E) = \sum p^{-ns}$ when E has split multiplicative reduction at p .
- $L_p(s, E) = \sum (-1)^n p^{-ns}$ when E has non-split multiplicative reduction.
- $L_p(s, E) = 1$ when E has additive reduction.

Let us show now that in all the above cases one has

$$R_p(s, E) = L_p(s, E). \quad (105)$$

One knows that in the singular case, the elliptic curve has exactly one singular point. Note that this point is already defined over \mathbb{F}_p since if a cubic equation has a double root this root is rational (*cf.* Appendix 8). Thus, for any power $q = p^\ell$ one has

$$N(q, E) = 1 + N_{\text{ns}}(q, E) \quad (106)$$

where the cardinality $N_{\text{ns}}(q, E)$ of the set of the non-singular points is given for any power $q = p^\ell$ as follows (*cf.* [18] exercise 3.5 page 104)

- $N_{\text{ns}}(q, E) = q - 1$ when E has split multiplicative reduction over \mathbb{F}_q .
- $N_{\text{ns}}(q, E) = q + 1$ when E has non-split multiplicative reduction over \mathbb{F}_q .
- $N_{\text{ns}}(q, E) = q$ when E has additive reduction over \mathbb{F}_q .

Thus one gets the following three cases

- $t(p^\ell) = 1$ for all ℓ when E has split multiplicative reduction at p .
- $t(p^\ell) = (-1)^\ell$ when E has non-split multiplicative reduction at p .
- $t(p^\ell) = 0$ when E has additive reduction at p .

The second case follows since the split property of E depends upon the parity of ℓ . We thus check the equality (105) for all primes of bad reduction $p \in S$. Together with (104) and (101), this implies (97). \square

Remark 6.4. Note that the function $N(n) = n+1-t(n)$ is always positive. Indeed, it is enough to show that for all n one has $|t(n)| \leq n$. Since $t(n)$ is multiplicative, it is enough to show the above inequality when $n = p^k$ is a prime power. In the case of good reduction one has $t(p^k) = \alpha_p^k + \bar{\alpha}_p^k$ and since the modulus of α_p is \sqrt{p} one gets

$$|t(q)| \leq 2\sqrt{q}, \quad q = p^k.$$

This proves that $|t(q)| \leq q$ for all q satisfying $2\sqrt{q} \leq q$ i.e. when $q \geq 4$. For $q = 2$ one has $2\sqrt{2} \cong 2.828 < 3$ and since $|t(2)|$ is an integer one obtains $|t(2)| \leq 2$. Similarly, for $q = 3$ one has $2\sqrt{3} \cong 3.4641 < 4$ and thus $|t(3)| \leq 3$. In the case of bad reduction one has always $|t(q)| \leq 1$.

Definition 6.2 gives the following equation for the modified zeta function $\zeta_N^{\text{disc}}(s)$ associated to the counting function $N(q, E)$

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = - \sum_1^\infty (n+1-t(n)) n^{-s-1} = -\zeta(s+1) - \zeta(s) + R(s+1, E).$$

Using Lemma 6.3 we then obtain

Theorem 6.5. *Let E be an elliptic curve over \mathbb{Q} and let $L(s, E)$ be the associated L -function. Then the modified zeta function $\zeta_N^{\text{disc}}(s)$ of E fulfills the following equality*

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = -\zeta(s+1) - \zeta(s) + \frac{L(s+1, E)}{\zeta(2s+1)M(s+1)} \quad (107)$$

where $\zeta(s)$ is the Riemann zeta function.

The Riemann zeta function has trivial zeros at the points $-2n$ for $n \geq 1$. They generate singularities of $\zeta_E(s)$ at the points $s = -n - \frac{1}{2}$, i.e. at $s = -\frac{3}{2}, -\frac{5}{2}, \dots$. The non-trivial zeros of the Riemann zeta function generate singularities at points which, if RH holds, have real part $-\frac{1}{4}$. Note that the poles of the archimedean local factor $N_E^{s/2} (2\pi)^{-s} \Gamma(s)$ ($N_E =$ conductor) of the L -function of E determine trivial zeros. These zeros occur for $s \in -\mathbb{N}$ and do not cancel the above singularities. Finally, we have all the zeros of $M(s+1)$. Each factor $(1-p^{-(2s-1)})$ contributes by an arithmetic progression with real part $-\frac{1}{2}$

$$s \in -\frac{1}{2} + \bigcup_{p \in S} \frac{\pi i}{\log p} \mathbb{Z}.$$

Note that the pole at $-\frac{1}{2}$ has order the cardinality of S .

For a better understanding of the role played by the primes of bad reduction we consider the following example.

Example 6.6. Let consider the elliptic curve in $\mathbb{P}^2(\mathbb{Q})$ given by the zeros of the homogeneous equation

$$E : Y^2 Z + Y Z^2 = X^3 - X^2 Z - 10 X Z^2 - 20 Z^3.$$

Then the sequence $a(n)$ is given by the coefficients of the modular form

$$F_E(q) = \sum_{n=1}^{\infty} a(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2.$$

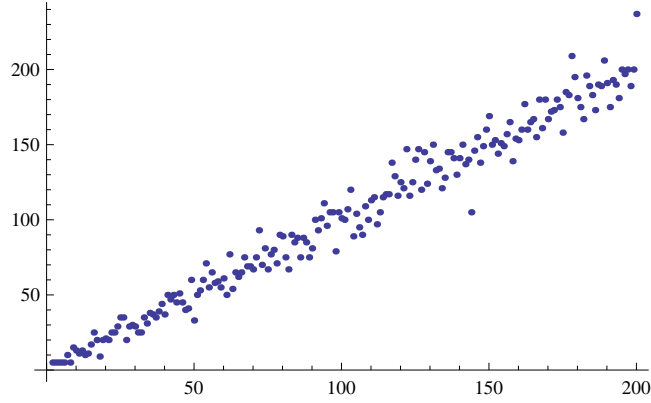


FIGURE 3. Graph of the function $N(n) = n + 1 - t(n)$ for $n \leq 200$, for the elliptic curve of the Example 6.6.

These coefficients are easy to compute for small values of n using the Euler formula

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (-1)^n (q^{(3n^2-n)/2} + q^{(3n^2+n)/2}).$$

The first values are

$$a(0) = 0, a(1) = 1, a(2) = -2, a(3) = -1, a(4) = 2, a(5) = 1, a(6) = 2, a(7) = -2, \dots$$

The function $N(n)$ is given by $N(n) = n + 1 - t(n)$: it is important to check that it takes positive values. It gives the graph of Figure 3. In this example, the only prime of bad reduction is $p = 11$ and the singularities of the modified zeta function $\zeta_N^{\text{disc}}(s)$ are described in Figure 4.

The truly challenging question is that of defining a natural functor \underline{E} to finite sets, such that $\#\underline{E}(\mathbb{F}_{1^n}) = N(n + 1)$ for all n .

6.3. Counting functions and distributions.

Following [16], one naturally wonders on the existence of a “curve” $C = \overline{\text{Spec } \mathbb{Z}}$ defined in a suitable sense over \mathbb{F}_1 , whose zeta function $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$. In the following we shall show that the integral formula (72) gives a precise information on the counting function $N(q)$ of C . By (72) one gets

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = - \int_1^{\infty} N(u) u^{-s} d^*u. \quad (108)$$

This integral formula appears typically in the Riemann-Weil explicit formulae: in this case one gets (with $\Re(s) > 1$)

$$- \frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} + \int_1^{\infty} \kappa(u) u^{-s} d^*u, \quad (109)$$

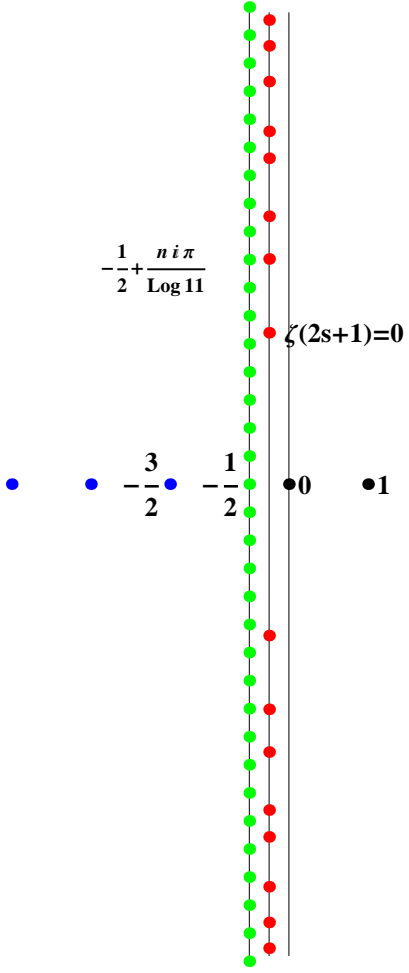


FIGURE 4. Singularities of $\zeta_N^{\text{disc}}(s)$.

where $\Lambda(n)$ is the von-Mangoldt function⁴ and $\kappa(u)$ is the distribution

$$\kappa(u) = \frac{u^2}{u^2 - 1} \quad \forall u > 1$$

defined using a principal value to eliminate the divergence at $u = 1$. More specifically the distribution $\kappa(u)$ is given, for any test function f , by

$$\int_1^\infty \kappa(u) f(u) d^*u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1} d^*u + c f(1), \quad c = \frac{1}{2}(\log \pi + \gamma)$$

where $\gamma = -\Gamma'(1)$ is the Euler constant. Thus one sees that one should think of the counting function $N(q)$ of the hypothetical “curve” C over \mathbb{F}_1 , as a distribution which is the sum of $\kappa(q)$ with the discrete term given by $n \Lambda(n)$. This discrete term

⁴with value $\log p$ for powers p^ℓ of primes and zero otherwise

is equal to the derivative $\frac{d}{du}\varphi(u)$ in the sense of distributions of the function⁵

$$\varphi(u) = \sum_{n < u} n \Lambda(n) \quad (110)$$

so that one can write (109) as

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^{\infty} \left(\frac{d}{du} \varphi(u) + \kappa(u) \right) u^{-s} d^*u, \quad (111)$$

which, when combined with (108) gives us the following formula for $N(u)$

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u) \quad (112)$$

The above expression encloses in a very subtle and intrinsic form a fundamental information on the description of the counting function as geometric “trace type” formula. To substantiate this statement, we recall the well-known equation (*cf.* [11], Chapter IV, Theorems 28 and 29, and use $\varphi(u) = u\psi_0(u) - \psi_1(u)$) valid for $u > 1$ (and not a prime power)

$$\varphi(u) = \frac{u^2}{2} - \sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} + a(u) \quad (113)$$

where

$$a(u) = \text{ArcTanh}\left(\frac{1}{u}\right) - \frac{\zeta'(-1)}{\zeta(-1)}$$

and Z denotes the set of non-trivial zeros of the Riemann zeta function while the sum over Z in the above formula is taken in a symmetric manner to ensure convergence. When one differentiates (113) in a formal manner the term in $a(u)$ gives

$$\frac{d}{du} a(u) = \frac{1}{1-u^2}$$

so that at the formal level, *i.e.* disregarding the principal value, we get

$$\frac{d}{du} a(u) + \kappa(u) = 1$$

Thus when one differentiates (113) at the formal level one gets

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u) \sim u - \sum_{\rho \in Z} \text{order}(\rho) u^{\rho} + 1 \quad (114)$$

This formula for the counting function is now entirely similar to the formula for the counting function of the number of points of a curve C over \mathbb{F}_p in the form

$$\#C(\mathbb{F}_q) = N(q) = q - \sum_{\alpha \in Z} \alpha^{\ell} + 1, \quad \forall q = p^{\ell}$$

where the α 's are the eigenvalues of the Frobenius. We have neglected, in the above formal computation, the principal value for the distribution $\kappa(u)$ and we get more precisely,

⁵the value at the points of discontinuity does not affect the distribution

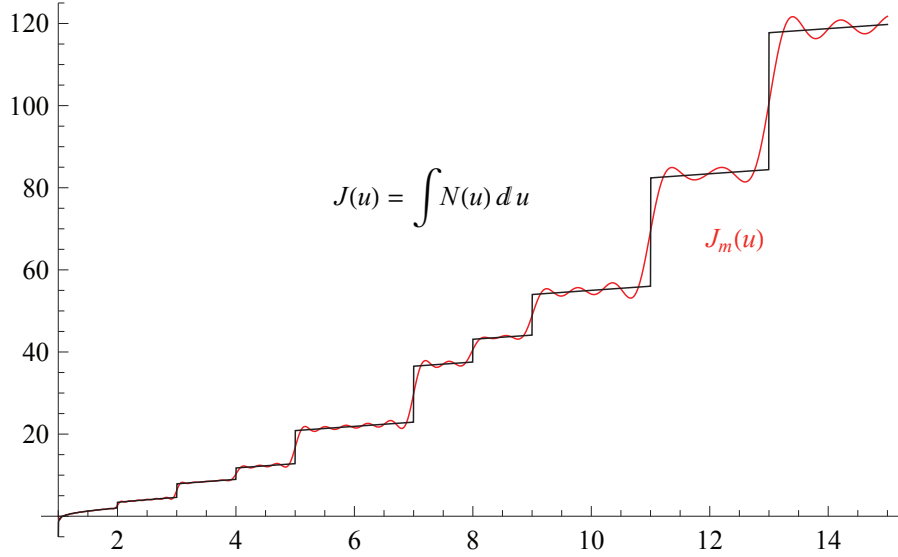


FIGURE 5. Primitive $J(u)$ of $N(u)$ and approximation using the symmetric set Z_m of first $2m$ zeros, by

$$J_m(u) = \frac{u^2}{2} - \sum_{Z_m} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} + u$$

Note that $J(u) \rightarrow -\infty$ when $u \rightarrow 1+$.

Theorem 6.7. The distribution $N(u)$ such that (108) holds, i.e.

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^{\infty} N(u) u^{-s} d^*u,$$

is positive on $]1, \infty)$ and is given on $[1, \infty)$ by

$$N(u) = u - \frac{d}{du} \left(\sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1 \quad (115)$$

where the derivative is taken in the sense of distributions, and the value at $u = 1$ of the term $\omega(u) = \sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}$ is given by

$$\omega(1) = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)} \quad (116)$$

Proof. The positivity of the distribution $N(u)$ on $]1, \infty)$ follows from (112). For $u > 1$ we define

$$\omega(u) = \sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}. \quad (117)$$

By (113) one has, for $u > 1$

$$\omega(u) = -\varphi(u) + \frac{u^2}{2} + a(u). \quad (118)$$

In a neighborhood of 1 one has $\varphi(u) = 0$ and $a(u) \sim -\frac{1}{2} \log(u-1)$ when $u \rightarrow 1+$. Thus $\omega(u)$ diverges when $u \rightarrow 1$ but is locally integrable and defines a distribution. Since $[1, \infty)$ has a boundary the derivative of the distribution depends on its boundary value and is defined as

$$\left\langle \frac{d}{du} \omega(u), f(u) \right\rangle = - \int_1^\infty \omega(u) \frac{d}{du} f(u) du - \omega(1) f(1) \quad (119)$$

for f smooth and of fast enough decay at ∞ . We apply this to $f(u) = u^{-s-1}$ for $\Re(s) > 1$. One has $-\frac{d}{du} f(u) = (s+1)u^{-s-2}$ and we get

$$\left\langle \frac{d}{du} \omega(u), f(u) \right\rangle = (s+1) \int_1^\infty \left(-\varphi(u) + \frac{u^2}{2} + a(u) \right) u^{-s-2} du - \omega(1).$$

One has by [11], Chapter I, (17), using $\varphi(u) = u\psi_0(u) - \psi_1(u)$, $\psi'(u) = \psi_0(u)$,

$$(s+1) \int_1^\infty \varphi(u) u^{-s-2} du = -\frac{\partial_s \zeta(s)}{\zeta(s)} \quad (120)$$

so that, using

$$\int_1^\infty (u+1)f(u)du = \frac{1}{s} + \frac{1}{s-1}, \quad -(s+1) \int_1^\infty \frac{u^2}{2} u^{-s-2} du = -\frac{1}{2} - \frac{1}{s-1}$$

one gets

$$\left\langle \left(u - \frac{d}{du} \omega(u) + 1 \right), f(u) \right\rangle = \frac{1}{s} - \frac{1}{2} - \frac{\partial_s \zeta(s)}{\zeta(s)} + \omega(1) - (s+1) \int_1^\infty a(u) u^{-s-2} du.$$

Finally, the following equation holds

$$\frac{1}{s} - (s+1) \int_1^\infty a(u) u^{-s-2} du = -\frac{\partial_s \Gamma(s/2)}{\Gamma(s/2)} + \frac{\zeta'(-1)}{\zeta(-1)} - \log 2 - \frac{\gamma}{2}.$$

Indeed, using integration by parts one has

$$-(s+1) \int_{1+\epsilon}^\infty \left(\text{ArcTanh}\left(\frac{1}{u}\right) - u \right) u^{-s-2} du = \int_{1+\epsilon}^\infty \frac{u^{-s+1}}{u^2-1} du + b(\epsilon)$$

$$b(\epsilon) = -\left(\text{ArcTanh}\left(\frac{1}{1+\epsilon}\right) - (1+\epsilon) \right) (1+\epsilon)^{-s-1} = -\int_{1+\epsilon}^\infty \frac{u^{-1}}{u^2-1} du + c + O(\epsilon)$$

with $c = 1 - \log 2$. Moreover one also knows that

$$\frac{\partial_s \Gamma(s/2)}{\Gamma(s/2)} = -\frac{\gamma}{2} + \int_1^\infty \frac{u^{-1} - u^{-s+1}}{u^2-1} du.$$

Thus one gets

$$\left\langle \left(u - \frac{d}{du} \omega(u) + 1 \right), f(u) \right\rangle = -\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)}$$

provided that

$$\omega(1) = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}. \quad (121)$$

To check this equality one cannot use the explicit formula (113) which is not valid at $u = 1$ since the term $\text{ArcTanh}(\frac{1}{u})$ is infinite displaying the discontinuity of the

function $\omega(u)$ at $u = 1$. Hence, to check (121) we consider the following formula (cf. [11] III, (25))

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \sum_{\mathcal{Z}} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) + \log(2\pi) - 1 - \frac{\gamma}{2}$$

when $s \rightarrow 1$. The left hand side tends to γ and the right hand side, using the symmetry $\rho \rightarrow 1 - \rho$ of the zeros, tends to (using a symmetric summation and the formula $\frac{\Gamma'}{\Gamma} \left(\frac{3}{2} \right) = 2 - \gamma - 2 \log 2$)

$$2 \sum_{\mathcal{Z}} \frac{1}{\rho} - 2 + \log(4\pi).$$

Thus we obtain

$$\sum_{\mathcal{Z}} \frac{1}{\rho} = \frac{\gamma}{2} + 1 - \frac{1}{2} \log(4\pi).$$

One then concludes using the equalities (cf. [11] IV, Theorem 28)

$$\sum_{\mathcal{Z}} \frac{1}{\rho(\rho+1)} = \frac{1}{2} - \log(4\pi) + \frac{\zeta'(-1)}{\zeta(-1)}$$

and (using a symmetric summation)

$$\sum_{\mathcal{Z}} \frac{1}{\rho+1} = \sum_{\mathcal{Z}} \frac{1}{\rho} - \sum_{\mathcal{Z}} \frac{1}{\rho(\rho+1)}.$$

□

Remark 6.8. According to [19] the value $N(1)$ of the counting function is the Euler characteristic of the hypothetical “curve” C over \mathbb{F}_1 . Since C has infinite genus one thus gets a priori that $N(1) = -\infty$, hence creating a tension with the expected positivity of $N(q)$ for $q > 1$. This tension is resolved in Theorem 6.7, since the distribution $N(q)$ is positive for $q > 1$ but its value at $q = 1$ is given formally by

$$N(1) = 2 - \lim_{\epsilon \rightarrow 0} \frac{\omega(1+\epsilon) - \omega(1)}{\epsilon} \sim -\frac{1}{2} E \log E, \quad E = \frac{1}{\epsilon}$$

whose behavior when $\epsilon \rightarrow 0$ even reflects the density of zeros.

Equality (113) is a typical application of the Riemann-Weil explicit formulae which become natural when lifted to the idèle class group. It seems therefore natural to expect that the hypothetical “curve” C is of adèlic nature and that it possesses an action of the idèle class group. This fits perfectly with the interpretation of the explicit formulae as a trace formula using the adèle class space (cf. [2], [3], [4]).

7. APPENDIX: ABELIAN GROUPS AND FINITE FIELDS

In this paper we have developed the analogy between monoids of the form $\mathbb{F}_1[H] = H \cup \{0\}$ and fields, when H is an abelian group. It is natural to ask in what sense the former are obtained as a degenerate case of fields, when the characteristic p becomes 1. This is clarified by the following simple result

Theorem 7.1. *Let H be an abelian group. Let s be a bijection of the set $\mathbb{F}_1[H] = H \cup \{0\}$ onto itself such that s commutes with its conjugates under the action of H by multiplication on the monoid $\mathbb{F}_1[H]$. Then, if $s(0) \neq 0$, the following*

$$x + y = \begin{cases} y & \text{if } x = 0 \\ s(0)^{-1}xs(s(0)y/x) & \text{if } x \neq 0 \end{cases} \quad (122)$$

defines a commutative group law on $\mathbb{F}_1[H]$. With this law as addition, the monoid $\mathbb{F}_1[H]$ becomes a commutative field \mathbb{K} .

Proof. Replacing s by its conjugate, one can assume that $s(0) = 1$. Let $A(x, y)$ be given by (122). One has $A(0, y) = y$ by definition. For $x \neq 0$, $A(x, 0) = xs(0) = x$ since $s(0) = 1$. Thus 0 is a neutral element. The commutation of s with its conjugate by the multiplication by $x \neq 0$ means that one has

$$s(xs(y/x)) = xs(s(y)/x), \quad \forall y \in \mathbb{F}_1[H]. \quad (123)$$

Taking $y = 0$, and using $s(0) = 1$, the above equation gives

$$s(x) = xs(x^{-1}), \quad \forall x \neq 0 \quad (124)$$

Assume now that $x \neq 0, y \neq 0$, then

$$A(x, y) = xs(y/x) = x(y/x)s(x/y) = ys(x/y) = A(y, x).$$

It follows that A is a commutative law of composition. Let us now check that it is associative. This follows from the commutation of left and right addition which is a consequence of the commutation of the conjugates of s . More concretely one has, assuming first that all elements involved are non zero

$$A(A(x, y), z) = A(xs(y/x), z) = A(z, xs(y/x)) = zs(xs(y/x)/z) = zs(x/zs(y/x)).$$

Using (123) one obtains

$$zs(x/zs(y/x)) = zx/zs(s(y/z)z/x) = xs(s(y/z)z/x)$$

$$A(x, A(y, z)) = A(x, s(y/z)z) = xs(s(y/z)z/x)$$

which yield the required equality. Now we have to look at the special cases. If x, y or z is zero, the equality follows since 0 is a neutral element. Since we never had to divide by $s(a)$, the above argument applies without restriction. Finally let $\theta = s^{-1}(0)$, then for any $x \neq 0$ one has $A(x, \theta x) = xs(\theta) = 0$ which shows that θx is the inverse of x for the law A . We have thus proven that this law defines an abelian group structure on $\mathbb{F}_1[H]$.

Let us now show that the distributive law holds, *i.e.* that for any $a \in \mathbb{F}_1[H]$ one has

$$A(ax, ay) = aA(x, y), \quad \forall x, y \in \mathbb{F}_1[H].$$

We can assume that all elements involved are $\neq 0$. One has

$$A(ax, ay) = axs(ax/ay) = axs(x/y) = aA(x, y).$$

This suffices to show that $\mathbb{F}_1[H]$ is a field. □

In fact one also obtains the following general uniqueness result

Theorem 7.2. *Let H be a finite commutative group and let s_j ($j = 1, 2$) be two bijections of $\mathbb{F}_1[H] = H \cup \{0\}$ fulfilling the conditions of Theorem 7.1. Then H is a cyclic group of order $m = p^\ell - 1$ for some prime p and there exists an automorphism $\alpha \in \text{Aut}(H)$ and an element $g \in H$ such that*

$$s_2 = T \circ s_1 \circ T^{-1}, \quad T(x) = g\alpha(x), \quad \forall x \in H, \quad T(0) = 0$$

Proof. By replacing s_j by its conjugate by $g_j = s_j(0)$, one can assume that $s_j(0) = 1$. Each s_j defines on the monoid $\mathbb{F}_1[H] = H \cup \{0\}$ an additive structure which turns this set into a field. This field is isomorphic to \mathbb{F}_q for some prime power $q = p^\ell$. Let m be the order of H , then $m = p^\ell - 1 = q - 1$ and H is the cyclic group of order m . Let \mathbb{F}_q be a field with $q = p^\ell$ elements. This field is unique up to isomorphism. Thus there exists a bijection α from $\mathbb{F}_1[H]$ to itself, which is an isomorphism for both the multiplicative and the additive structures given by s_j . This map sends 0 to 0 and 1 to 1, and thus transforms the addition of 1 for the first structure into the addition of 1 for the second. Since it respects the multiplication, it is given by an automorphism $\alpha \in \text{Aut}(H)$. \square

In the degenerate case $s(0) = 0$ of Theorem 7.1, with the bijection s given by the identity map, (122) becomes the indeterminate expression $0/0$.

8. APPENDIX: PRIMES OF BAD REDUCTION OF AN ELLIPTIC CURVE

In this appendix we give more details on the counting of points used in the proof of Lemma 6.3. In order to reduce modulo p an elliptic curve E over \mathbb{Q} , the first step is to take an equation of the curve which is in minimal Weierstrass form over the local field \mathbb{Q}_p *i.e.* of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (125)$$

with $a_j \in \mathbb{Z}_p$ such that the power of p dividing the discriminant is minimal. Such a minimal Weierstrass equation can always be found (*cf.* [18] Proposition VII 1.3) and is unique up to a change of variables of the form

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$

for $u \in \mathbb{Z}_p^*$ and $r, s, t \in \mathbb{Z}_p$. One then considers the reduced equation

$$y^2 + \bar{a}_1xy + \bar{a}_3y = x^3 + \bar{a}_2x^2 + \bar{a}_4x + \bar{a}_6 \quad (126)$$

where \bar{a}_j is the residue of a_j modulo p . This equation is unique up to the standard changes of coordinates on \mathbb{F}_p and defines a curve over \mathbb{F}_p . One then looks for points of this curve *i.e.* solutions of equation (126) (together with the point at ∞), in the extensions \mathbb{F}_q , $q = p^\ell$. When $p > 2$, one can always write the reduced equation in the form

$$y^2 = x^3 + \bar{a}_2x^2 + \bar{a}_4x + \bar{a}_6. \quad (127)$$

Thus assuming $p \neq 2$, a singular point has second coordinate $y = 0$ while x is the common root of $x^3 + \bar{a}_2x^2 + \bar{a}_4x + \bar{a}_6 = 0$, with $3x^2 + 2\bar{a}_2x + \bar{a}_4 = 0$. This proves that the elliptic curve has a unique singular point and that both its coordinates belong to \mathbb{F}_p . After a translation of x one can then write the equation of the curve in the form $y^2 = x^2(x - \beta)$ with $\beta \in \mathbb{F}_p$. There are three cases

- $\beta = 0$ is the case of additive reduction: one has a cusp at $(0, 0)$.
- $\beta \neq 0$, $-\beta \in \mathbb{F}_p^2$ means split multiplicative reduction.

- $\beta \neq 0, -\beta \notin \mathbb{F}_p^2$ is the remaining case of non-split multiplicative reduction.

Let now \mathbb{K} be a perfect field of characteristic 2. One starts with a Weierstrass equation in general form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \quad (128)$$

Then when the curve is singular it can be written, using affine transformations of (x, y) with coefficients in \mathbb{K} , in the form

$$y^2 = x^3 + a_4x + a_6 \quad (129)$$

We refer to [18] Appendix A, Propositions 1.2 and 1.1 for details. Indeed, the curve is singular if and only if its discriminant vanishes (*cf.* Proposition 1.2 in *op.cit.*) then one can use the description of the curve as in Proposition 1.1 c) of *op.cit.* to deduce that $a_3 = 0$, since $\Delta = 0$. The point at ∞ is always non-singular. A point (x, y) is singular if and only if (129) holds together with $3x^2 + a_4 = 0$. This equation is equivalent to $x^2 = a_4$ and uniquely determines x since the field is perfect. Then one gets $y^2 = a_6$ since $x^3 + a_4x = x(x^2 + a_4) = 0$. This in turns uniquely determines y since \mathbb{K} is perfect. Thus, for any finite field of characteristic two, one gets that if the curve is singular it has exactly one singular point. In fact there are only 4 cases to look at in characteristic two, for elliptic curves over \mathbb{Q} . They correspond to the values $a_4 \in \{0, 1\}$ and $a_6 \in \{0, 1\}$.

One can always assume that $a_6 = 0$ by replacing y with $y + 1$, if needed. For $a_4 = a_6 = 0$ one has a cusp at $(0, 0)$ and the points over \mathbb{F}_q are labeled by the x coordinate. Thus there are $q + 1$ points over \mathbb{F}_q when $q = 2^\ell$. For $a_4 = 1, a_6 = 0$, the singular point is $P = (1, 0)$. After shifting x by 1 the equation becomes $y^2 = x^2(x + 1)$ and thus the singular point is also a cusp. The points over \mathbb{F}_q are labeled as $(1 + t^2, t(1 + t^2))$ for $t \in \mathbb{F}_q$ (together with the point at ∞). Again, there are $q + 1$ points over \mathbb{F}_q when $q = 2^\ell$.

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A. CONNES: COLLÈGE DE FRANCE, 3, RUE D’ULM, PARIS, F-75005 FRANCE, I.H.E.S. AND VANDERBILT UNIVERSITY

E-mail address: alain@connes.org

C. CONSANI: MATHEMATICS DEPARTMENT, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218 USA

E-mail address: kc@math.jhu.edu