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ASPHERICAL GRAVITATIONAL MONOPOLES

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Abstract

We show how to construct *non-spherically-symmetric* extended bodies of uniform density behaving exactly as pointlike masses. These "gravitational monopoles" have the following equivalent properties: (i) they generate, outside them, a spherically-symmetric gravitational potential $M/|\,\mathbf{x}-\mathbf{x}_{\rm O}|$; (ii) their interaction energy with an external gravitational potential $U(\mathbf{x})$ is $-M\,U(\mathbf{x}_{\rm O})$; and (iii) all their multipole moments (of order $\ell \geq 1$) with respect to their center of mass O vanish identically. The method applies for any number of space dimensions. The free parameters entering the construction are: (1) an arbitrary surface Σ bounding a connected open subset Ω of $\mathbb{R}^3$; (2) the arbitrary choice of the center of mass O within Ω ; and (3) the total volume of the body. An extension of the method allows one to construct homogeneous bodies which are *gravitationally equivalent* (in the sense of having exactly the same multipole moments) to any given body.

Though our method generally assumes that the domain Ω is bounded (which leads to bounded monopoles with *closed internal cavities*), it can also generate unbounded monopoles, with exponentially-decreasing thicknesses at infinity, and cylindriclike internal cavities reachable from infinity. This may be useful for optimizing the shape of test masses in high-precision Equivalence Principle experiments, such as the planned Satellite Test of the Equivalence Principle (STEP). By suppressing the couplings to gravity gradients, one can design, with great flexibility in the choice of shapes, differential accelerometers made of nested bodies, which are (exactly or exponentially) insensitive to all external gravitational disturbances. Alternatively, one can construct nested bodies of arbitrary densities having identical (or proportional) sequences of multipole moments, thereby also suppressing any differential acceleration caused by external gravity gradients.

1 INTRODUCTION.

The Equivalence Principle states that test bodies of different compositions must "fall" exactly in the same way in an external gravitational field. This fundamental principle, which lies at the basis of Einstein's theory of gravitation, deserves to be tested with the utmost precision available. Testing the Equivalence Principle is also the most sensitive way to search for hypothetical non-electromagnetic long range forces. Such forces could be due to the exchanges of new spin-1 or spin-0 particles (as suggested in some supersymmetric or superstring theories [1, 2]), whose existence might be required for a consistent theory of quantum gravity. Because of their composition-dependence[4], these forces would, by superposing their effects to those of gravitation, lead to (apparent) violations of the Equivalence Principle.

Improving upon the results of the classic Eötvös and Dicke experiments, recent laboratory experiments have checked the validity of the Equivalence Principle at the $\approx 3\times10^{-12}$ level [3]. In addition, the Lunar Laser Ranging experiment has established that the Earth and the Moon "fall" towards the Sun with equal accelerations, to within $\approx 5\times10^{-13}$ [4]. A proposed satellite experiment known as STEP (Satellite Test of the Equivalence Principle) [5], presently studied by ESA, CNES and NASA [6, 7, 8, 9], aims at improving the sensitivity down to the $\approx 10^{-17}$ level.

The STEP experiment consists essentially of measuring the differential accelerations between several pairs of nested cylindrically-symmetric coaxial test masses of different compositions freely falling within a "drag-free" satellite in low orbit around the Earth. At the impressive sensitivity level of 10^{-17}, one has to worry about many possible disturbances. In particular, the coupling to external gravity gradients of the two differently-shaped test masses making up one differential accelerometer causes a parasitic differential acceleration even if the Equivalence Principle holds exactly. The origin of this disturbance is that an extended body is generally not equivalent to a pointlike mass in its interaction with an external gravitational field. Instead, the gravitational structure of an extended body can be described by the infinite set of its multipole moments, starting with the monopole moment – identical to the total mass. Choosing as origin the center of mass of the body ensures the vanishing of its dipole moment ($\ell = 1$) [5], and allows one to describe its extension-dependent interactions with an external gravitational

[4]New interactions due to spin-1 exchanges are generally expected to act on a linear combination of baryon and lepton numbers, or more specifically on their difference $B-L$, in grand-unified theories [1]. The spin-0 exchange forces suggested by string theory are expected to act effectively on a linear combination of B, L and of the nuclear electrostatic energy [2]. New spin-1 or spin-0 induced forces could also act, in addition, on particle spins.

[5]The common vanishing of the dipole moments of two nested masses – i.e. the fact that their two centers of mass can be made to coincide – is realized to a high precision in the

field in terms of its successive multipole moments Q_ℓ of order $\ell \geq 2$. The existence of a moment of order ℓ leads, in the presence of a disturbing mass at distance $\mathcal{R}$, to an extension-dependent acceleration $\propto Q_\ell / \mathcal{R}^{\ell+2}$.

These parasitic accelerations are a serious concern in the STEP experiment. The way this problem is currently addressed is multi-pronged. First, while the inner mass M^{in} is shaped as a straight cylinder with vanishing quadrupole moment $Q_2^{\text{in}} = 0$, but non-zero (even) higher moments Q_4^{in}, Q_6^{in}, ..., the outer mass M^{out} is shaped as a "belted cylinder" [10, 6, 11] having a vanishing quadrupole moment, $Q_2^{\text{out}} = 0$, and a non-zero hexadecapole moment chosen so that $Q_4^{\text{out}}/M^{\text{out}} = Q_4^{\text{in}}/M^{\text{in}}$. The latter matching condition for the reduced hexadecapole moments Q_4/M ensures the cancelling of the differential acceleration $\propto 1/\mathcal{R}^6$ induced by the two Q_4's. The remaining free parameters are optimized so as to minimize the effects of the non-matched higher multipole moments [11, 12]. Second, from the calculation of the residual sensitivity of differential accelerometers to external masses, one deduces constraints on the allowed motion of nearby masses. In particular, one has still severe constraints on the amplitude of any "helium tide" in the spacecraft's dewar surrounding the test masses. One must then design the dewar so as to meet these constraints.

In this paper, we propose a radical solution to the problem of the sensitivity to external gravity gradients. We show how to construct extended homogeneous bodies of very general shapes[6] behaving *exactly as pointlike masses* in any external gravitational field. The existence of such *aspherical gravitational monopoles* has been made plausible by Barrett [13] who found approximate numerical solutions to this problem. Here we shall prove the existence of such objects by providing an explicit method for constructing them[7]. Our construction allows us to have a clear control of the large flexi-

STEP experiment by measuring the effect of the Earth's gravity gradient and positioning the masses so as to eliminate this effect.

[6]This very large flexibility in the choice of the shapes of our monopoles could make them of practical utility in Equivalence Principle tests. The obvious solution of spherically-symmetric test masses was considered by the STEP team [6] but rejected because of the unsurmountable practical difficulties of building nested spherical test masses with independent suspension and sensing systems.

[7] After submitting this work for publication we learned of an unpublished work by Barrett [14], in which he proposed the same construction method (for monopoles with boundaries having the topology of a sphere) without, however, developing it and studying

bility in the shapes of such homogeneous[8] monopoles.

An extension of our method allows one to construct *gravitationally similar* bodies (of equal or different densities), i.e. bodies having the same infinite sequences of reduced multipole moments Q_ℓ/M . Two such bodies experience exactly the same acceleration in the presence of arbitrary external gravity gradients, as soon as their centers of mass are made to coincide.

As far as we know, the specific problem of the construction of aspherical monopoles has not been dealt with in the large literature on potential theory. (For an entry into this literature see e.g. Ref. [16].) In fact, there are theorems in inverse potential theory proving, *under certain assumptions*, the *inexistence* of such monopoles. In particular, a famous theorem of Novikov [17] states that, if we know the value $\rho_\circ$ of the supposedly uniform density, the exterior potential generated by a *starlike*[9] open set Ω (with known center O) determines uniquely that domain. A corollary of this theorem is that a homogeneous starlike monopole must be a ball. We shall in fact prove below that this inexistence of aspherical homogeneous monopoles holds under the much weaker assumption that the open set Ω be connected and have a connected boundary $\Sigma = \partial\Omega$. I.e., *aspherical solid homogeneous monopoles cannot exist* (this excludes, for instance, monopoles in the form of solid tori)[10].

This no-go theorem, however, does not exclude aspherical monopoles – either simply connected[11] or not – having a *disconnected* boundary $\Sigma = \Sigma_{\rm ext} \cup \Sigma_{\rm int}$. Indeed, we shall construct such homogeneous aspherical monopoles, which have a closed (in the sense of being unreachable from infinity)

its consequences.

[8]By contrast, the existence of *inhomogeneous* aspherical monopoles is not surprising as one can smoothly deform, within some sphere, the potential generated by a homogeneous ball, and then *define* the density "generating" this deformed potential (which remains proportional to $1/r$ outside this sphere) by Poisson's equation.

[9]An open set Ω in $\mathbb{R}^N$ is said to be starlike with respect to a center O if Ω contains the straight segment joining O to any point of Ω.

[10] After submitting this work for publication we learned from a referee of an unpublished work by Barrett [15], in which he established this result by a different, more involved method.

[11]In Ref. [17] one finds, without proof (on the ground that it is "easy to prove"!), the *incorrect* proposition that a homogeneous *simply connected* monopole (with C^2 boundary) must be a ball.

internal cavity.

Our construction allows us to choose arbitrarily, either the outer boundary, or the inner boundary (surface of the internal cavity) of the monopole. The topology of these boundaries does not need to be that of the sphere S^2. For instance, the outer boundary can be toroidal. In the simplest[12] bounded monopoles we construct, the center of mass – which is the center of symmetry of the generated external gravitational field – can be systematically located within the internal cavity. This allows for a second monopole to be placed inside the first one, with the same center of mass. But this inner monopole is then unreachable from the outside by a continuous path. It means that, in practical applications such as STEP, where one needs to control and sense the position of nested monopoles, one can only use *approximate* monopoles.

Interestingly, we can get practically useful shapes by considering a limit where the boundaries are not compact, but extend to infinity. In particular, we shall discuss in detail the case (directly relevant for STEP) where both the internal and external boundaries asymptotically approach a straight cylinder at infinity. The thickness of the corresponding monopole, around this limiting straight cylinder, then tends *exponentially* to zero at infinity. By cutting off such a monopole, one can construct a finite, cylinderlike object whose couplings with gravity gradients are suppressed with an exponentially good precision.

After formulating the problem in technical terms and proving the inexistence of "solid" aspherical homogeneous monopoles in Section 2, we present our method for constructing "hollow" aspherical monopoles in Section 3. We also indicate how, by a generalization of this method, one can construct homogeneous bodies which are "gravitationally equivalent", or "gravitationally similar" to any given body, or collection of bodies. In Section 4, we build our intuition for the resulting shapes of such monopoles by studying, to lowest order, some simple examples, in particular the thin cylinder. In Section 5, we discuss some of the mathematical properties of the construction, present a detailed analysis of the perturbations of a monopole, and outline the program of a rigourous mathematical proof of the existence of such monopoles. Physical considerations on the growth of monopoles, including a discussion

[12] As discussed below, our basic method allows generalizations and combinations which lead to monopoles of very general shapes, having, for instance, as many holes as desired.

of different practical methods for realizing them, are presented in Section 6.

2 Definitions of monopoles and inexistence of "solid" aspherical homogeneous monopoles.

2.1 Equivalent definitions of monopoles.

Let $\rho(\mathbf{x})$ denote a volumic mass density in three-dimensional space. For simplicity, we start by assuming that $\rho(\mathbf{x})$ has a compact support. The gravitational potential generated by $\rho(\mathbf{x})$ reads (with our sign convention, and setting Newton's constant G_N to unity)

$$U_\rho(\mathbf{x}) \;=\; \int d^3\mathbf{x}' \;\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \quad . \tag{1}$$

The interaction energy of this mass distribution $\rho(\mathbf{x})$ in an external gravitational potential $U_{\text{ext}}(\mathbf{x})$ is

$$E_\rho \;=\; -\int d^3\mathbf{x}\;\rho(\mathbf{x})\;U_{\text{ext}}(\mathbf{x}) \quad . \tag{2}$$

Both U_ρ and E_ρ can be skeletonized by introducing the multipole moments of the distribution ρ. They can be described by symmetric trace-free (STF) tensors of order $\ell \geq 0$,

$$Q^{i_1 i_2 \dots i_\ell}_{\rho,\,\mathbf{x}_\mathrm{O}} \;=\; \mathrm{STF}_{i_1\dots i_\ell}\int d^3\mathbf{x}\;\;\rho(\mathbf{x})\;\;(x^{i_1}-x^{i_1}_\mathrm{O})\,(x^{i_2}-x^{i_2}_\mathrm{O})\;\dots\;(x^{i_\ell}-x^{i_\ell}_\mathrm{O}) \;\;, \tag{3}$$

where $\mathrm{STF}_{i_1\dots i_\ell}$ denotes a symmetric trace-free projection over the indices $i_1\dots i_\ell$ [13], and $\mathbf{x}_\mathrm{O}$ the coordinates of some origin O in space. An equivalent description uses the spherical harmonics and reads, in the three dimensional space:

$$Q^{\ell m}_{\rho,\,\mathbf{x}_\mathrm{O}} \;=\; \int d^3\mathbf{x}\;\;\rho(\mathbf{x})\;\;r^\ell\;\;Y_{\ell m}(\,\theta,\,\varphi) \;\;, \tag{4}$$

where $x^i - x^i_\mathrm{O} = (\,r\,\sin\theta\,\cos\varphi,\;r\,\sin\theta\,\sin\varphi,\;r\,\cos\theta\,)$.

[13] E.g., $\mathrm{STF}_{ij}\;\; X^iX^j \;=\; X^iX^j \;-\; \frac{1}{N}\,\delta^{ij}\,X^sX^s$, $\mathrm{STF}_{ijk}\;X^iX^jX^k \;=\; X^iX^jX^k - \frac{1}{N+2}\,X^sX^s\;(\delta^{ij}X^k+\delta^{jk}X^i+\delta^{ki}X^j)$, etc., in N space dimensions.

The zeroth order multipole moment (or monopole moment) is simply the total mass of the distribution, $M = \int d^3\mathbf{x}\ \ \rho\,(\mathbf{x})$. We shall always fix the origin O at the center of mass of the distribution ρ, i.e. at the point such that the dipole moment $Q^i_{\rho,\mathbf{x}_\mathrm{O}} = \int d^3\mathbf{x}\ \ \rho\,(\mathbf{x})\ \ (x^i - x^i_\mathrm{O})$ vanishes identically.

A (bounded) gravitational monopole can then be equivalently defined as a distribution ρ satisfying either:

(**i**) the external potential generated by ρ is spherically symmetric:

$$U_\rho(\mathbf{x}) \ = \ \frac{M}{|\mathbf{x}-\mathbf{x}_\mathrm{O}|} \qquad \text{for } \mathbf{x} \text{ in the exterior}^{14} \text{ of the support of } \rho\,; \tag{5}$$

or (**ii**) the interaction energy of $\rho\,(\mathbf{x})$ in any external gravitational potential $U_\mathrm{ext}\,(\mathbf{x})$ is the same as for a point mass located at the center of mass $\mathbf{x}_\mathrm{O}$:

$$E_\rho \ = \ -\,M\ \ U_\mathrm{ext}(\mathbf{x}_\mathrm{O})\ \ ; \tag{6}$$

or (**iii**) all multipole moments (of order $\ell \geq 1$) vanish identically:

$$Q^{i_1 i_2 \ldots i_\ell}_{\rho,\,\mathbf{x}_\mathrm{O}} \ = 0 \qquad \text{for} \quad \ell \geq 1\ . \tag{7}$$

The equivalence between (i) and (ii) follows from considering the external distribution ρ_ext generating the gravitational potential U_ext . The corresponding interaction energy is $E_\rho[U_\mathrm{ext}] = -\int d^3\mathbf{x}\,\rho(\mathbf{x}')\,\rho_\mathrm{ext}(\mathbf{x})\,|\mathbf{x}-\mathbf{x}'|^{-1} = -\int d^3\mathbf{x}\ \ U_\rho(\mathbf{x})\ \ \rho_\mathrm{ext}(\mathbf{x})$, to be compared with $-M\ \ U_\mathrm{ext}(\mathbf{x}_\mathrm{O}) = -\int d^3\mathbf{x}\ \frac{M}{|\mathbf{x}-\mathbf{x}_\mathrm{O}|}\ \rho_\mathrm{ext}(\mathbf{x})$. Demanding the equality (6) of these two expressions for any $\rho_\mathrm{ext}(\mathbf{x})$ is equivalent to Equation (5).

The equivalence of either (i) or (ii) with (iii) follows from expanding $|\mathbf{x}-\mathbf{x}'|^{-1}$ in Equation (1) in powers of $\mathbf{x}'-\mathbf{x}_\mathrm{O}$, which ultimately yields, for $\mathbf{x}$ in the exterior of $\mathrm{supp}\,\rho$:

$$U_\rho(\mathbf{x}) \ = \ \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell\,!}\ \ Q^{i_1\ldots i_\ell}_{\rho,\mathbf{x}_\mathrm{O}}\ \ \partial_{i_1\ldots i_\ell}\,\frac{1}{|\mathbf{x}-\mathbf{x}_\mathrm{O}|}\ \ , \tag{8}$$

and from expanding $U_\mathrm{ext}(\mathbf{x})$ in Equation (2) in powers of $\mathbf{x}-\mathbf{x}_\mathrm{O}$, which yields:

$$E_\rho \ = \ -\sum_{\ell=0}^{\infty} \frac{1}{\ell\,!}\ \ Q^{i_1\ldots i_\ell}_{\rho,\mathbf{x}_\mathrm{O}}\ \ \partial_{i_1\ldots i_\ell}\,U_\mathrm{ext}(\mathbf{x}_\mathrm{O})\ \ . \tag{9}$$

[14] If the support of the bounded distribution ρ divides space in several regions, we mean by "exterior" the outside region, connected to infinity.

The properties (**i**) – (**iii**) are also equivalent to:

(**iv**): the spatial Fourier transform $\hat{\rho}\,(\mathbf{k}) = \int d^3\mathbf{x}\; e^{-i\,\mathbf{k}\cdot\mathbf{x}}\; \rho\,(\mathbf{x})$ (choosing here O as the origin) may be expanded (around $\mathbf{k} = \mathbf{0}$) as

$$\hat{\rho}\,(\mathbf{k}) \;=\; \hat{\rho}\,(\mathbf{0}) \;+\; \mathbf{k}^2\; f(\mathbf{k})\;, \tag{10}$$

as we shall discuss later (cf. subsection 4.2).

Essentially all our results straightforwardly extend to any number of space dimensions $N \geq 2$. It suffices to replace the basic Newtonian potential r^{-1} by $(N-2)^{-1}\; r^{-(N-2)}$ when $N \neq 2$ (and by $\ln 1/r$ for $N = 2$), and the coefficient $4\,\pi$ in the Poisson equation

$$\Delta\; U_{\rho}(\mathbf{x}) \;=\; -\;4\,\pi\;\rho\,(\mathbf{x}) \tag{11}$$

by the surface of the unit sphere in $\mathbb{R}^N$, $\omega_N = 2\;\pi^{N/2}/\,\Gamma(N/2)$. To simplify the discussion we shall phrase our results by assuming $N = 3$.

An *homogeneous* monopole is defined by a density ρ uniform within some domain[15] Ω of $\mathbb{R}^3$. For simplicity, we assume that Ω is bounded and admits a sufficiently regular boundary $\Sigma = \partial\,\Omega$ (assumptions that we shall relax later).

It is useful to introduce also the concepts of *gravitationally equivalent* and *gravitationally similar* distributions. Two density distributions $\rho_1\,(\mathbf{x})$ and $\rho_2\,(\mathbf{x})$ will be said "gravitationally equivalent" if they generate the same external gravitational potential, i.e. if all their multipole moments coincide:

$$Q^{i_1 i_2 \ldots i_\ell}_{\rho_1,\,\mathbf{x}_{\mathrm{O}}} \;=\; Q^{i_1 i_2 \ldots i_\ell}_{\rho_2,\,\mathbf{x}_{\mathrm{O}}} \qquad \text{for} \quad \ell \geq 0\;\;. \tag{12}$$

(the choice of origin O being arbitrary). The distributions will be said "gravitationally similar" if the external gravity potentials they generate are proportional. This is equivalent to requiring that their "reduced" multipole moments coincide:

$$Q^{i_1 i_2 \ldots i_\ell}_{\rho_1,\,\mathbf{x}_{\mathrm{O}}}\;/\;M_1 \;=\; Q^{i_1 i_2 \ldots i_\ell}_{\rho_2,\,\mathbf{x}_{\mathrm{O}}}\;/\;M_2\;\;. \tag{13}$$

This latter concept is invariant under a (constant) rescaling of the densities ρ_1 and ρ_2. In this nomenclature, a monopole can be defined as a distribution which is gravitationally equivalent, or similar, to a point mass.

[15]By "domain" we mean a *connected* open subset of $\mathbb{R}^N$.

2.2 Inexistence of aspherical solid homogeneous monopoles.

Let us prove that if Ω is a (connected) homogeneous monopole, and if the boundary $\Sigma = \partial\,\Omega$ of the domain Ω is *connected*, then Ω must be a solid sphere. To do so we study the action, on the potential $U_\rho(\mathbf{x})$ generated by the monopole Ω, of the generators of infinitesimal rotations[16] around O,

$$\boldsymbol{L} \;=\; (\mathbf{x} - \mathbf{x}_{\rm O}) \;\times\; \frac{\partial}{\partial\,\mathbf{x}} \quad . \tag{14}$$

The rotational symmetry of the Laplacian implies that the (Lie) derivative of $U_\rho(\mathbf{x})$, say $\boldsymbol{u} = \;\boldsymbol{L}\; U_\rho$, satisfies everywhere, in the sense of distributions,

$$\Delta\; \boldsymbol{u} \;=\; -\;4\,\pi\; \boldsymbol{L}\;\rho \;, \tag{15}$$

in which the source $\boldsymbol{L}\,\rho$ is a “single layer” surface density distribution[17].

Let us assume that Ω defines a homogeneous monopole with a *connected* boundary $\Sigma = \partial\,\Omega$. This implies that Σ divides space into two regions: the domain Ω (“interior of Σ”) and the complement of Ω (“exterior of Σ”). Each component $u^i \;=\; L^i\, U_\rho$ of the infinitesimal variation of the potential appears itself as a potential, generated by the single layer surface density $L^i\,\rho = \sigma^i\,\delta_\Sigma$, distributed on the boundary Σ. We shall now prove that this surface density σ_i must in fact vanish identically. Indeed the potential u^i it generates vanishes everywhere in the exterior of Σ (since $U_\rho(\mathbf{x})$ is rotation-invariant), and is harmonic everywhere inside it. On the surface Σ itself u^i must be continuous, while its normal derivative might a priori have a discontinuity, equal to $-\,4\,\pi$ times the surface density σ^i.

[16]If the number of space dimensions is not $N = 3$, we use

$$L^{j\,k} \;=\; (x^j - x^j_{\rm O})\,\frac{\partial}{\partial\,x^k} \;-\; (x^k - x^k_{\rm O})\,\frac{\partial}{\partial\,x^j}\;,$$

the subsequent argument remaining the same.

[17]To be explicit, we have

$$\boldsymbol{L}\;\rho \;=\; \boldsymbol{\sigma}\;\delta_\Sigma \;=\; -\;(\mathbf{x} - \mathbf{x}_{\rm O}) \;\times\; \mathbf{n}\;\;\rho_\circ\,\delta_\Sigma$$

We have used the fact that the gradient of the characteristic function of the domain Ω is $-\,\mathbf{n}\,\delta_\Sigma$, $\mathbf{n}$ being the outgoing normal to the boundary $\Sigma = \partial\,\Omega$. δ_Σ denotes the standard uniform δ-function surface distribution on Σ (such that $\int d^3\,\mathbf{x}\;f(\mathbf{x})\,\delta_\Sigma \;=\; \int_\Sigma d\,\Sigma\;[f(\mathbf{x})]_\Sigma$, where $d\,\Sigma$ is the area element on Σ).

The vanishing of u^i in the exterior and its continuity across Σ implies that it vanishes on Σ. It must then vanish everywhere inside Σ (by the uniqueness of the solution of the Dirichlet problem). Since it also vanishes outside, its derivative cannot have a discontinuity on the surface Σ, which implies that the density σ^i distributed on Σ has to vanish. The resulting equation,

$$\boldsymbol{L}\,\rho \;=\; \mathbf{0}\;, \tag{16}$$

expresses that the monopole Ω should be spherically symmetric[18]. It must be the solid sphere limited by the sphere Σ of center O.

We have not restricted in advance Ω to be simply connected. Our result therefore excludes, for example, "solid" tori as possible homogeneous monopoles.

The crucial assumption is that the connected domain Ω have a connected boundary Σ (i.e. that we are dealing with a "solid" monopole without any internal cavity). By contrast, in the case where the boundary of Ω is made of two disconnected surfaces, one inside the other (i.e. $\partial\,\Omega = \Sigma_{\rm ext} \cup \Sigma_{\rm int}$, as for a "thickened" topological sphere or topological torus, i.e. the volumes between two nested surfaces having the topology of S^2 or T^2), or more than two[19], we have no way to conclude that the monopole must be spherically symmetric. Indeed we shall see that, in such cases, one can construct infinitely many aspherical homogeneous monopoles.

Our analysis can easily be further extended to aspherical monopoles $\Omega = \cup\ \Omega_l$ that would be constituted of several *disconnected* parts (with boundaries $\Sigma_l = \partial\,\Omega_l$). If the exterior boundary $\Sigma_{{\rm ext}\ l}$ of one of them does not enclose the center O inside it, and is reachable from infinity, it follows from Gauss' theorem that the total mass included within it must vanish. As a result, *the exterior boundary of a gravitational monopole must always be connected*. For example a collection of solid objects (e.g. solid tori, even intertwined) cannot constitute a gravitational monopole. If a monopole is constituted of several components all but one should be located within the interior of one of them, as in the case of Russian dolls, for example.

[18] Technically, the formula of the previous footnote shows that $\boldsymbol{L}\,\rho = 0$ implies that every normal $\mathbf{n}$ to Σ should pass through O, i.e. that the connected surface Σ is a sphere of center O, and the monopole Ω the corresponding solid sphere.

[19] This is the case of a "Swiss-cheese-like" domain Ω with a disconnected boundary $\Sigma = \partial\,\Omega = \Sigma_{\rm ext} \cup \{\Sigma_{{\rm int}\,i}\}$.

3 Constructing aspherical (hollow) monopoles.

3.1 Thin aspherical monopoles.

The starting point of our method is a simple consequence of a well-known property of electrostatics (see e.g. Ref. [18]). Let Ω be a domain with a connected boundary Σ. Let us consider Σ as a grounded conducting surface, and introduce a unit (positive) electric charge at some given point O (with coordinates $x^i_{\rm O}$) within Ω. The charge $+1$ at O induces on the surrounding grounded conducting surface Σ a (negative) charge density, say $-\sigma$, which has the well-known property of generating a potential which exactly *screens* the $1/|\mathbf{x}-\mathbf{x}_{\rm O}|$ potential created by the charge $+1$, *outside* Σ. (This is mathematically evident from the uniqueness of the solution of the Dirichlet problem $\Delta U = 0$, in the complement of Ω.) By reversing the sign of the surface charge density, we conclude that the positive surface density $+\sigma$ on Σ generates, outside this surface, the exactly spherical potential $+1/|\mathbf{x}-\mathbf{x}_{\rm O}|$.

In mathematical terms, the potential generated by the charge $+1$ introduced at O is the Dirichlet Green function of the domain Ω, $G(\mathbf{x},\mathbf{x}_{\rm O})$, solution of

$$\Delta_x\, G(\mathbf{x},\mathbf{x}_{\rm O}) \;=\; -\,4\,\pi\,\delta(\mathbf{x}-\mathbf{x}_{\rm O})\;, \tag{17}$$

which vanishes when $\mathbf{x}\in\Sigma=\partial\,\Omega$. Let us define the following surface density on Σ

$$\sigma^{\rm O}_\Sigma(\mathbf{x}) \;=\; -\,\frac{1}{4\pi}\,\partial_n\,G(\mathbf{x},\mathbf{x}_{\rm O})\;, \tag{18}$$

where ∂_n denotes the outgoing normal derivative at $\mathbf{x}\in\Sigma$. The density $\sigma^{\rm O}_\Sigma$ is everywhere positive[20] and integrates to $+1$:

$$\int_\Sigma d\,\Sigma\;\sigma^{\rm O}_\Sigma \;=\; -\,\frac{1}{4\pi}\int_\Sigma d\,\Sigma\;\partial_n\,G \;=\; -\,\frac{1}{4\pi}\int_\Omega d^3\mathbf{x}\;\;\Delta_x\,G \;=\; +\,1\;. \tag{19}$$

[20] $\sigma^{\rm O}_\Sigma$ is positive even if Ω has a non trivial topology, e.g. that of a multi-handled solid torus. Indeed, $G(\mathbf{x})$ being positive (and $\simeq 1/|\mathbf{x}-\mathbf{x}_{\rm O}|$) near $\mathbf{x}_{\rm O}$, zero on the boundary of Ω, and harmonic within $\Omega-\{{\rm O}\}$ cannot (by the maximum-minimum principle) reach negative values within Ω. Therefore $G(\mathbf{x})$ decreases from positive to zero values when crossing Σ.

Applying Green's identity

$$\int_\Omega d^3\mathbf{x}\ \left(u\,\Delta\,v - v\,\Delta\,u\right) \ = \ \int_\Sigma d\,\Sigma\ \ \left(u\ \partial_n\,v - v\ \partial_n\,u\right) \tag{20}$$

with u equal to any external potential $U_{\rm ext}$ (satisfying $\Delta\,U_{\rm ext} = 0$ within $\Omega \cup \Sigma$), and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_{\rm O})$, yields

$$U_{\rm ext}(\mathbf{x}_{\rm O}) \ = \ \int_\Sigma d\,\Sigma\ \ \sigma_\Sigma^{\rm O}(\mathbf{x})\ \ U_{\rm ext}(\mathbf{x})\ . \tag{21}$$

Equation (21) expresses precisely the property (6) of a monopole: the surface layer with density $\sigma_\Sigma^{\rm O}$ interacts with any external potential as if it were a unit point mass located at O. (The more "active" definition (5) is obtained by taking $U_{\rm ext}(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_{\rm P}|^{-1}$ for some external point P.) In the mathematical literature, the surface distribution $\sigma_\Sigma^{\rm O}\ d\,\Sigma$ is called the "harmonic measure" of Σ with respect to the origin O. Equation (21) says that the value, at a given point $\rm O \in \Omega$, of any harmonic function can be written as a weighted average of the values taken by this harmonic function on the boundary of Ω. In other words, $\sigma_\Sigma^{\rm O}(\mathbf{x})$, if it is known for all points $\rm O \in \Omega$, gives, via Eq. (21), the explicit (unique) solution of the Dirichlet problem: determining a harmonic function in Ω from its values on $\Sigma = \partial\,\Omega$.

In words, the consideration of the surface density $\sigma_\Sigma^{\rm O}$ solves the problem of constructing infinitely thin aspherical monopoles. Note that Σ does not need to have the topology of the sphere. It can have the topology of a torus, or of a more complicated surface with many handles. As long as Σ is the connected boundary of a (sufficiently regular[21]) connected open set Ω, one can construct a thin monopole having the shape of Σ. It is surprising, but true, that one can choose arbitrarily a complicated multi-handled shape Σ, and an arbitrary origin O enclosed within Σ, and construct, by depositing a positive surface layer $\propto \sigma_\Sigma^{\rm O}$ on Σ, an object generating a potential which is exactly proportional to $|\mathbf{x} - \mathbf{x}_{\rm O}|^{-1}$ outside Σ. At this stage, we assume that Ω is bounded, but we shall later consider the limiting case where Ω becomes unbounded in a way which ensures that the mass distribution of the monopole falls off exponentially fast at infinity.

[21]We refer to the literature on potential theory, e.g. Ref. [19], for the characterization of the domains admitting a Green function and a harmonic measure. We shall briefly discuss later what happens when Σ is not smooth.

3.2 Thick aspherical monopoles.

To construct "thick" aspherical monopoles, it suffices to build them by growing successive layers having a thickness proportional to $\sigma_{\Sigma}^{\mathrm{O}}$. More precisely, let us start with some initial closed surface Σ which is the (connected) boundary of a domain Ω and let us choose, once and for all, some origin O within Ω. It is convenient to use as parameter t measuring the continuous growing of a thick monopole around Σ the algebraic volume of the constructed monopole, counted positive for monopoles obtained by adding layers to the outside of Σ, and negative when growing Σ towards the inside. At each "time" t the monopole consists of the volume Ω_t delimited by two boundaries: a fixed boundary Σ and a moving one, say Σ_t.

We can (locally) represent the moving boundary Σ_t by equations of the form

$$\mathbf{x} \in \Sigma_t \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{X}\,(t, \boldsymbol{\xi})\ , \tag{22}$$

where $\boldsymbol{\xi} = (\xi^1, \xi^2)$ denote two (curvilinear) coordinates on the fixed boundary Σ. In mathematical terms, $\mathbf{X}(t,\cdot)$ is an embedding of Σ into $\mathbb{R}^3$. We define this embedding (as can always be done) so that points with fixed coordinates $\boldsymbol{\xi}$ propagate orthogonally to the instantaneous Σ_t. The condition of adding successive infinitesimal layers of constant density and thickness proportional to the harmonic measure $\sigma_{\Sigma_t}^{\mathrm{O}}$ reads

$$\frac{\partial}{\partial t}\,\mathbf{X}\,(t, \boldsymbol{\xi}) \;=\; \sigma_{\Sigma_t}^{\mathrm{O}}\,(\mathbf{X}(t, \boldsymbol{\xi}))\;\; \mathbf{n}_{\Sigma_t}\,(\mathbf{X}(t, \boldsymbol{\xi}))\ , \tag{23}$$

where $\mathbf{n}_{\Sigma_t}$ denotes the outward normal to Σ_t. The differential version of Equation (23),

$$d\,\mathbf{X} \;=\; \sigma_{\Sigma_t}^{\mathrm{O}}\;\mathbf{n}_{\Sigma_t}\;dt\ , \tag{24}$$

together with the fact that the surface integral $\int d\,\Sigma_t\;\sigma_{\Sigma_t}^{\mathrm{O}}$ is $+1$ (Equation (19)), makes it clear that the infinitesimal volume between Σ_t and Σ_{t+dt} is indeed dt, and that each layer $d\,\mathbf{X}$ adds an infinitesimal *homogeneous* monopole $d\,\Omega_t$ with fixed origin O. (The constant density $\rho_\circ$ of the monopole is set to unity for simplicity.)

This can also be verified by a simple calculation. Namely, Equations (23) or (24) are equivalent to requiring that the t-derivative of the integral of any function $u(\mathbf{x})$ over the volume Ω_t contained between Σ and Σ_t be equal to the surface integral $\int d\,\Sigma_t\ \sigma^{\rm O}_{\Sigma_t}\ u\,(\mathbf{X}(t,\boldsymbol{\xi}))$. If $u(\mathbf{x})$ is a harmonic function, say $u(\mathbf{x}) = U_{\rm ext}(\mathbf{x})$, the latter surface integral is, according to (21), equal to the value of $U_{\rm ext}$ at the origin O. Integrating this result over t we get

$$|\,t\,|\ U_{\rm ext}(\mathbf{x}_{\rm O})\ =\ \int_{\Omega_t} d^3\,\mathbf{x}\ U_{\rm ext}(\mathbf{x})\ , \tag{25}$$

which expresses the fact that the volume Ω_t contained between Σ and Σ_t is a homogeneous monopole, of volume $|\,t\,|$.

This construction exhausts the possible homogeneous monopoles, only in the simple case where: (i) they belong to a continuous[22] family of solutions with variable mass in which only one of the boundaries of the monopole is allowed to move, and (ii) the moving boundary $\Sigma_{\boldsymbol{\alpha}}$ (where $\boldsymbol{\alpha}$ denotes a set of parameters) coincides with the fixed one Σ for some value of $\boldsymbol{\alpha}$, for which the total mass (i.e. volume $|\,t\,(\boldsymbol{\alpha})\,|$) vanishes. (We assume here, as above, that the two boundaries Σ and $\Sigma_{\boldsymbol{\alpha}}$ are connected.)

Indeed, let us consider a (*a priori* multi-parameter) family of monopoles filling the volume $\Omega_{\boldsymbol{\alpha}}$ contained between the fixed boundary Σ and the moving one $\Sigma_{\boldsymbol{\alpha}}$. Let us try to perturb this monopole, at fixed volume t (and therefore fixed mass), keeping the boundary Σ fixed. The perturbation of the other boundary $\Sigma_{\boldsymbol{\alpha}}$ is defined by its orthogonal displacement $\delta h = (\mathbf{X}(\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\boldsymbol{\xi}) - \mathbf{X}(\boldsymbol{\alpha},\boldsymbol{\xi}))\ .\,\mathbf{n}_{\Sigma_{\boldsymbol{\alpha}}}$. This δh defines a single layer surface density distributed on $\Sigma_{\boldsymbol{\alpha}}$. Since it generates a vanishing potential outside this (connected) surface, it must vanish identically. Therefore, the only possible variations of $\Sigma_{\boldsymbol{\alpha}}$ are those in which the volume (i.e. the mass) changes. For these, the uniqueness of the solution of Eq. (21), considered as an equation for the harmonic measure σ, implies that δh is necessarily equal to $\sigma^{\rm O}_{\Sigma_{\boldsymbol{\alpha}}}\,\delta t(\boldsymbol{\alpha})$. This shows that, when one boundary Σ is fixed, the other one $\Sigma_{\boldsymbol{\alpha}}$ must depend only on the single volume parameter $t = t(\boldsymbol{\alpha})$, and evolve according to Eq. (23). Under the further assumption that $\Sigma_{\boldsymbol{\alpha}}$ coincides with Σ when $t\,(\boldsymbol{\alpha}) = 0$, we recover the "thickening" process of Σ defined above.

[22]This basic construction method may easily be extended to take into account discontinuous changes, such as, for example, the inclusion of additional cavities during the growth of the monopole. There could exist, furthermore, isolated solutions which cannot be continuously deformed.

This proof of the uniqueness of the contruction of monopoles breaks down if we consider continuous families of solutions in which both the outermost boundary $\Sigma^{\rm ext}_{\boldsymbol{\alpha}}$, and one or several inner boundaries $\Sigma^{\rm int}_{\boldsymbol{\alpha}}$ are allowed to move. We shall see, in subsection 3.3, that the possibility to compensate arbitrary variations of $\Sigma^{\rm int}_{\boldsymbol{\alpha}}$ by appropriate variations of $\Sigma^{\rm ext}_{\boldsymbol{\alpha}}$ allows us to construct more general families of monopoles than the basic ones described by the thickening process of Eq. (23) – even if this process is generalized to allow for discontinuous changes leading, for instance, to the inclusion of additional cavities during the growth process.

Let us mention in passing that one can trivially extend our method to constructing many sorts of *inhomogeneous* monopoles. For instance, if we want to prescribe the volumic density $\rho(\mathbf{x})$ within the monopole it suffices to divide the right-hand sides of Equations (23) or (24) by $\rho\,(\mathbf{x})$ when propagating Σ_t ($|t|$ now denoting the total mass of the monopole Ω_t). Alternatively, one can prescribe the shape of the monopole, and define, from any slicing of the monopole by interpolating Σ_t 's a (positive) density $\rho\,(\mathbf{x})$ so that the orthogonal distance between Σ_t and Σ_{t+dt} is $\sigma^{\rm O}_{\Sigma_t}/\rho$.

Instead of writing the fundamental propagation law for homogeneous monopoles in the differential form (23), one could as well write it as an "eikonal" type equation. Namely, by eliminating the two surface coordinates $(\xi^1,\, \xi^2)$ from the three embedding equations (22), we can write an equation determining the position of the surface Σ_t in the form

$$\mathbf{x} \in \Sigma_t \quad \Leftrightarrow \quad \varphi\,(\mathbf{x}) = t\ . \tag{26}$$

In this form, the outward normal is $\mathbf{n} = \boldsymbol{\nabla}\varphi / |\boldsymbol{\nabla}\varphi|$, and the infinitesimal vectors $d\mathbf{x}$ connecting Σ_t and Σ_{t+dt} satisfy $dt = \boldsymbol{\nabla}\varphi \cdot d\mathbf{x} = |\boldsymbol{\nabla}\varphi|\ \mathbf{n}\cdot d\mathbf{x}$. The propagation law $d\mathbf{x} = \sigma\,\mathbf{n}\,dt +$ tangent vector becomes $|\boldsymbol{\nabla}\varphi|\,\sigma = 1$, i.e.

$$(\,\boldsymbol{\nabla}\varphi(\mathbf{x})\,)^2 \;=\; (\,\sigma^{\rm O}_{\Sigma_{\mathbf{x}}}(\mathbf{x})\,)^{-2}\,. \tag{27}$$

Contrary to the usual eikonal ($(\,\boldsymbol{\nabla}\varphi(\mathbf{x})\,)^2 \;=\; (\,n(\mathbf{x})\,)^2$) or Hamilton-Jacobi equations, Equation (27) is not an ordinary local partial differential equation for $\varphi(\mathbf{x})$ because its right-hand side is a non-local functional of the function $\varphi(\cdot)$ obtained by solving an elliptic problem[23]. Correspondingly,

[23]The meaning of $\sigma^{\rm O}_{\Sigma_{\mathbf{x}}}(\mathbf{x})$ appearing on the right-hand side of Equation (27) is the

the propagation equation (23) is local in "time" but non-local in the two spatial variables ξ^1, ξ^2. We shall briefly discuss below, in Section 5, in what functional spaces we expect the differential equation (23) to admit unique solutions (at least for t varying in some interval).

Finally, let us mention that, when working in any number of space dimensions[24] $N \geq 2$, the propagation law (23) (or (27)) takes the same form. The only difference is that Σ_t denotes a moving hypersurface (with $N-1$ internal coordinates $\boldsymbol{\xi}$). The harmonic measure $\sigma_\Sigma^{\rm O}$ is still uniquely defined by (21), or explicitly by $\sigma_\Sigma^{\rm O} = -\,\omega_N^{-1}\ \partial_n\, G(\mathbf{x}, \mathbf{x}_{\rm O})$ where, as mentioned in Section 2, ω_N denotes the surface of the unit sphere in $\mathbb{R}^N$ and where the Green function has a pole $\simeq (N-2)^{-1}\, r^{-(N-2)}$ when $N \neq 2$ (or $\ln 1/r$ in $N = 2$) when $r = |\mathbf{x} - \mathbf{x}_{\rm O}| \to 0$.

We shall now extend the basic construction process of Equations (23) or (27) to more general situations.

3.3 Gravitationally-similar bodies and more general monopoles.

We have defined, at the end of subsection 2.1, the concepts of gravitationally equivalent and gravitationally similar bodies, which have identical (or proportional) sequences of multipole moments. Let us show how, by an extension of our method, we can construct such bodies. A by-product of this extension will lead to more general monopoles than the ones obtained earlier in subsection 3.2.

Our electrostatics considerations of subsection 3.1 admit the following wide generalization. Let Ω denote as above a domain with a connected

following: Given the point $\mathbf{x}$ and the function $\varphi(\cdot)$, one determines the surface $\Sigma_{\mathbf{x}}$ passing through $\mathbf{x}$ as the set of $\mathbf{y} \in \mathbb{R}^N$ satisfying $\varphi(\mathbf{y}) = \varphi(\mathbf{x})$. Then one determines the value, at the point $\mathbf{x}$, of the density of the harmonic measure on $\Sigma_{\mathbf{x}}$ with respect to the fixed origin O.

[24]It is amusing to note that, in dimension $N = 1$, any bounded object, say any (finite) reunion of disjoint finite intervals, constitutes a homogeneous monopole. Indeed, a general harmonic function is simply $u(x) = a + bx$ so that the condition (6) holds simply by linearity. Still, if we consider the analog of our general construction, i.e. the growing of two intervals located around their fixed center of gravity, one finds that the dynamics of their growth keeps (in a simplified form) some of the features of (23).

boundary Σ. Let us consider Σ as a grounded conducting surface, and introduce within it an arbitrary (fixed) volumic charge distribution $-\rho_1(\mathrm{Q})$, where Q denotes a generic point of Ω. This charge distribution induces on the grounded boundary Σ a surface charge density obtained by superposing all the elementary charge densities induced by point charges at the running point Q. If $\sigma_\Sigma^Q(\mathbf{x})$ denotes the surface density (18) taken for a pole O located at Q (see also Eq. (77) below), the resulting surface density induced on Σ by $-\rho_1(\mathrm{Q})$ reads:

$$\sigma_\Sigma^{\rho_1}(\mathbf{x}) \;=\; \int d\,V_{\mathrm{Q}}\;\; \rho_1(\mathrm{Q})\;\; \sigma_\Sigma^Q(\mathbf{x}) \quad . \tag{28}$$

By construction, this surface density generates, in the exterior of Σ, the same potential as the *arbitrary* volumic distribution ρ_1 contained within Σ. In other words, the surface distribution $\sigma_\Sigma^{\rho_1}(\mathbf{x})$ is *gravitationally equivalent* to the volumic distribution $+\rho_1(\mathrm{Q})$.

To construct an homogeneous[25] volumic distribution ρ_2 equivalent to ρ_1, we can now turn on continuously the distribution ρ_1 (by considering $t\,\rho_1$, with t increasing from 0 to 1), while growing, at the same time, successive surfaces Σ_t from the initial surface Σ, in such a way that the elementary volume $d\,\Omega_t$ of density ρ_2 enclosed between Σ_t and Σ_{t+dt} is gravitationally equivalent to the infinitesimal mass distribution $\rho_1\; dt$. This new growing process is defined by the following differential equation (which generalizes Eqs. (23) or (24)):

$$d\,\mathbf{X} \;=\; \frac{1}{\rho_2}\;\sigma_{\Sigma_t}^{\rho_1}\;\; \mathbf{n}_{\Sigma_t}\; dt \;\; , \tag{29}$$

where $\mathbf{n}_{\Sigma_t}$ denotes the outward[26] normal to Σ_t, and $\sigma_{\Sigma_t}^{\rho_1}$ the surface distribution (28) taken for the moving surface Σ_t. The total mass of the corresponding layer $d\,\Omega_t$ – which by construction generates at the exterior of Σ_t the same potential as the infinitesimal distribution $\rho_1\; dt$ – is easily verified to be (as it should)

$$d\,M_2 \;=\; \rho_2 \int_{\Sigma_t} d\,\Sigma_t\;\; \mathbf{n}_{\Sigma_t}\,.\,d\,\mathbf{X}$$

[25]The construction given below trivially extends to inhomogeneous distributions.

[26]To fix ideas we consider here an outward growth process, which has the advantage of being stable (as discussed later); we can define as well an inward growth process, by changing the sign of the right-hand side of Eq. (29).

$$= \int_{\Sigma_t} d\,\Sigma_t \;\; \sigma_{\Sigma_t}^{\rho_1} \;\; dt \;\; = \;\; \int d\,V_{\rm Q} \;\; \rho_1\,({\rm Q}) \;\; dt \;\; = \;\; M_1\, dt \;\; . \qquad (30)$$

By ending the growth process at $t_f = 1$ we obtain a thickened version of Σ which, when filled with density ρ_2, is gravitationally equivalent to the volumic distribution ρ_1. Distributions gravitationally similar to ρ_1 can be obtained by a rescaling of ρ_2, or by ending the growth process at $t_f \neq 1$ (thereby generating a distribution gravitationally equivalent to $t_f \rho_1$).

This construction is interesting in several respects. From a theoretical point of view, it solves a generalization of the monopole construction problem by providing, for any (connected) surface Σ and any mass distribution ρ_1 located within Σ, an homogeneous thickened version of Σ which is gravitationally equivalent (or similar) to ρ_1. (Note that ρ_1, being totally arbitrary, can correspond to a collection of disconnected bodies with arbitrary densities.)

It also leads to the construction of more general monopoles than the ones obtained by the basic growth process (23, 24), even when it is generalized by introducing discontinuities, with the inclusion of additional cavities during the growth. (This generalization can be viewed as the superposition of several "basic" monopoles, all of them having the same center of mass O.)

To show this we start from an homogeneous monopole $\mathcal{M}$ of center O (e.g. a solid sphere, or one of the monopoles constructed above). Let us dig arbitrarily various holes within this monopole. The mass distribution $\rho_1\,({\rm Q})$ that we take off corresponds to a subset $\mathcal{H}$ of $\mathcal{M}$. This density $\rho_1\,({\rm Q})$ taken away from the initial monopole is gravitationally equivalent, as we have seen, to a new homogeneous body $\Delta\mathcal{M}$ (filled with the same density as $\mathcal{M}$), obtained by thickening, towards the outside, the outer boundary Σ of our initial monopole $\mathcal{M}$. Altogether the resulting object $((\mathcal{M} - \mathcal{H}) \cup \Delta\mathcal{M})$ is a new homogeneous monopole, of the same mass M and center of mass O as the initial one. But it now presents a new set of holes $\mathcal{H}$, corresponding to one or several disconnected volumes of arbitrary shapes.

The construction (29) is also quite interesting from a practical point of view. One motivation of this work is to define *differential* accelerometers which are – ideally – totally insensitive to arbitrary external gravity gradients. This may be realized by using two gravitationally similar bodies centered at the same point, the accelerations induced by the external gravity gradients, proportional to $Q^{i_1 i_2 \ldots i_\ell}_{\rho_i, \mathbf{x}_\mathrm{O}} \; / \; M_{\rho_i}$, being identical.

Given some inner body with mass distribution ρ_1 (Q) (e.g. a solid straight cylinder) and an arbitrary surface Σ enclosing it, we can now define, from the growth process (29) considered with any value of the density ρ_2 and any final "time" t_f, a one-parameter family of homogeneous outer bodies gravitationally similar to the inner one. These bodies have the form of homogeneous thick shells built on the boundary Σ. We shall see later how this construction can be extended to the case where Σ has, for example, a cylindric or cylindriclike shape.

Before discussing in Section 5 some of the mathematical properties of the evolution equations (23) or (29), we shall build up our intuition on the growing of monopoles by solving explicitly, in Section 4, the first step of the construction in some simple geometries.

4 Simple examples of thin monopoles.

4.1 The plane and the sphere.

The most trivial example would be the limiting case where the domain Ω is a half of $\mathbb{R}^3$, $\Sigma = \partial\,\Omega$ being an infinite plane. The Green function with pole at $\mathrm{O} \in \Omega$ is simply $G(\mathbf{x}, \mathbf{x}_\mathrm{O}) = |\mathbf{x} - \mathbf{x}_\mathrm{O}|^{-1} - |\mathbf{x} - \mathbf{x}_{\bar{\mathrm{O}}}|^{-1}$, where $\bar{\mathrm{O}}$ is the image of O through the "mirror" Σ. Equation (18) yields the surface distribution on Σ,

$$\sigma^\mathrm{O}_\Sigma \;=\; \frac{a}{2\,\pi\; r^3} \;, \tag{31}$$

where $r = |\mathbf{x} - \mathbf{x}_\mathrm{O}|$ and a is the orthogonal distance between O and the plane Σ. As seen from the other half-space that is the complement of Ω, the surface distribution (31) generates exactly the potential $|\mathbf{x} - \mathbf{x}_\mathrm{O}|^{-1}$. However, as the asymptotic fall off of σ is rather slow it does not make sense to compute the multipole moments of the corresponding surface distribution, nor to say that we have generated an interesting thin monopole.

The second most trivial example consists of the case where Ω is a ball of radius R in $\mathbb{R}^3$, $\Sigma = \partial\,\Omega$ being a sphere. Let C be the center of the ball and O be any point within Ω. The Green function is again given by the method of images:

$$G(\mathbf{x},\mathbf{x}_{\rm O}) \;=\; |\mathbf{x}-\mathbf{x}_{\rm O}|^{-1} - \lambda\,|\mathbf{x}-\mathbf{x}_{\bar{\rm O}}|^{-1} \quad \text{where} \quad \lambda = R/{\rm CO}\;, \tag{32}$$

and where $\bar{\rm O}$ is the inverse of O through the sphere Σ (i.e. $\mathbf{C\bar{O}} = R^2\,\mathbf{CO}/|\mathbf{CO}|^2$). The corresponding surface distribution is given by the well-known Poisson formula (see e.g. Ref. [19])

$$\sigma_\Sigma^{\rm O} \;=\; \frac{R^2\;-\;\mathbf{CO}^2}{4\pi\,R}\;\frac{1}{r^3}\;, \tag{33}$$

which exhibits the same r^{-3} dependence (where $r\equiv|\mathbf{x}-\mathbf{x}_{\rm O}|$ with $\mathbf{x}\in\Sigma$) as the planar distribution (31). (In N space dimensions the Poisson formula is simply obtained from (33) by replacing $4\,\pi\to\omega_N$ and $r^3\to r^N$.)

We can now consider the surface distribution (33) on this sphere (of geometrical center C) as defining a thin monopole (of center of mass O). By construction, it generates a Newtonian potential which is exactly $|\mathbf{x}-\mathbf{x}_{\rm O}|^{-1}$ outside Σ, its interaction energy with an external potential is exactly that of a unit mass point located at O, and all its multipole moments (of order $\ell\geq1$) around O vanish identically.

The lesson we learn from (31) and (33) (at least when one starts with a surface Σ with a simple shape varying only on length scales larger than the distance between the pole O and Σ) is that the growing of a monopole will consist of successive layers on Σ forming bumps roughly concentrated around the points of Σ which minimize the distance $|\mathbf{x}-\mathbf{x}_{\rm O}|$. When growing the monopole towards the outside, one expects the bumps to become smeared (because their centers move away from O), and the surface Σ_t to asymptotically approach a very large sphere of center O. On the contrary, when growing Σ_t towards the inside, one expects the bumps to start developing into fingers rapidly advancing towards O. We shall further discuss later the instabilities associated with the inward evolution and the smoothing character of the outward propagation.

4.2 The thin cylinder.

A less trivial – and much more interesting – example of a thin monopole is obtained by choosing for surface $\Sigma=\partial\,\Omega$ a straight cylinder (of radius R) in

$\mathbb{R}^3$. We choose, for simplicity, the pole O on its symmetry axis. In cylindrical coordinates $(\rho,\, \varphi,\, z)$ the pole is at $\rho = z = 0$ and the equation of Σ is $\rho = R$. There are (at least) two equivalent ways of finding the corresponding Green function. Let us start with one before explaining the other.

Writing the Green function $G(\mathbf{x}) = \frac{1}{r} + U\,(\rho, \varphi, z)$, where $r = \sqrt{\rho^2 + z^2}$, the function U (manifestly φ-independent) must be harmonic within Ω,

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} \;=\; 0\;, \tag{34}$$

and equal, when $\rho = R$, to $-1/r = -1/\sqrt{R^2+z^2}$. Taking a Fourier transform with respect to z, $U\,(\rho, z) = \int_{-\infty}^{+\infty} dk\; U_k\,(\rho)\; e^{ikz}$, we find from (34) that $U_k(\rho)$ is proportional to a *modified* Bessel function of order zero: $U_k(\rho) = c_k\; I_0(k\rho)$. The coefficient c_k is obtained by writing that $U_k(R)$ is the inverse Fourier transform of $U(R,z) = -1/\sqrt{R^2+z^2}$. The latter transform is given by a modified Bessel function of the second kind: $\int_0^\infty dz\, \cos kz/\sqrt{z^2+R^2} = K_0(|k|R)$. Finally, we get

$$G(\rho, z) = \frac{1}{\sqrt{\rho^2+z^2}} + U(\rho,z) = \frac{1}{\sqrt{\rho^2+z^2}} - \frac{1}{\pi}\int_{-\infty}^{+\infty} dk\, \frac{K_0(|k|R)}{I_0(kR)}\, I_0(k\rho)\, e^{ikz}\;. \tag{35}$$

The corresponding surface density (18) can be written (when writing also the $(\rho^2+z^2)^{-1/2}$ term as a Fourier integral with respect to z, and using the Wronskian $\kappa\,[\,I_0'(\kappa)\, K_0(\kappa) - I_0(\kappa)\, K_0'(\kappa)\,] = 1$) as

$$\sigma(z) \;=\; \frac{1}{2\pi\, R^2}\; F\left(\frac{z}{R}\right)\;, \tag{36}$$

where (using the dimensionless variables $\zeta = z/R$ and $\kappa = kR$)

$$F(\zeta) \;=\; \int_{-\infty}^{+\infty} \frac{d\kappa}{2\pi}\, \frac{e^{i\kappa\zeta}}{I_0(\kappa)}\;. \tag{37}$$

A different method of getting G and σ is to take advantage of the vanishing of $G\,(\rho,\, z)$ at $\rho = R$ (for any z) to expand the Green function $G\,(\rho,\, z)$ as a Fourier-Bessel series

$$G\,(\rho, z) \;=\; \frac{2}{R}\,\sum_{\ell=1}^{\infty}\, a_\ell\,(z)\; J_0\,(k_\ell\, \rho)\;, \tag{38}$$

where $k_\ell R \equiv j_\ell \simeq 2.405, 5.520, 8.654, \ldots$ are the successive zeroes of the Bessel function J_0. Inserting this expansion into Poisson's equation $\Delta G = -4\pi\delta$ (i.e. Eq. (34) completed with the source term $-4\pi\,\delta(z)\,\delta_\perp(\boldsymbol{\rho})$), one gets: (i) the information that $a_\ell(z) = A_\ell\, e^{-k_\ell|z|}$, and (ii), from the strength of the delta function, the values of the coefficients A_ℓ. (More details on this, and on the properties of the resulting distribution function F, will be given elsewhere.) Finally, we get

$$G(\rho, z) = \frac{2}{R} \sum_{\ell=1}^{\infty} \frac{1}{j_\ell \; J_1(j_\ell)^2} \; e^{-k_\ell|z|} \; J_0(k_\ell \rho) \,, \tag{39}$$

in which enters the Bessel function of order one, $J_1(x) = -J_0'(x)$, evaluated at the zeroes of $J_0(x)$. From (39) one deduces that the surface density (18) can be written as in Equation (36), with

$$F(\zeta) = \sum_{\ell=1}^{\infty} \frac{1}{J_1(j_\ell)} \; e^{-j_\ell|\zeta|} \,. \tag{40}$$

The identity of the different-looking results (37) and (40) is easily verified by folding (when, say, $\zeta > 0$) the κ contour of integration in (37) (after closing it by an infinite half-circle in the upper complex κ plane) around the upper imaginary axis $\kappa = ix$, thereby picking up an infinite series of contributions due to residues at the simple poles of $1/I_0(ix) = 1/J_0(x)$. The Fourier-integral representation (37) is valid for all values of ζ and makes it easy to compute numerically the shape of the function $F(\zeta)$. The series (40) is absolutely convergent only when $\zeta \neq 0$ [the slow decrease of $J_1(j_\ell) \sim (-)^{\ell+1}\,\ell^{-1/2}$ as $\ell \to \infty$ causes the series (40) to diverge when $\zeta = 0$], but gives a useful representation of $F(\zeta)$ when $|\zeta| \gtrsim 1$ because it captures well its asymptotic fall off. In particular, the leading behaviour for $|\zeta| > 1$ is given by the first term in (40), coming from the first zero of $J_0(x)$: $j_1 \simeq 2.40483$ (and $J_1(j_1) \simeq 0.51915$). Therefore, for $|z| \gtrsim R$ the surface density (36) is approximately given by

$$\sigma(z) \simeq \frac{1.926}{2\pi R^2} \; e^{-2.405\frac{|z|}{R}} \,. \tag{41}$$

The function $F(\zeta)$ (which integrates to unity) is represented graphically in Figure 1.

By construction the surface density (36) laid on an infinite straight cylinder of radius R generates a $1/|\mathbf{x}-\mathbf{x}_{\rm O}|$ potential everywhere outside the cylinder. This monopole is of infinite extent, but, thanks to the exponential fall off (41) of the density it makes sense to consider its multipole moments of arbitrarily high order, all the integrals $\int d\Sigma\, \sigma\, {\rm STF}(x^{i_1}\ldots x^{i_\ell})$ being convergent. It is clear (by considering, for instance, the infinite cylinder as the limit of a finite cylinder closed by two caps) that our construction guarantees that all the multipole moments of the distribution (36) vanish identically (for $\ell \geq 1$).

This can be verified by explicit computations. A very simple way to proceed (which extends to all distributions which are bounded, or have an exponential fall off) is to consider the spatial Fourier transform of the density distribution: $\hat{\rho}\,(\mathbf{k}) = \int d^3\,\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\,\rho\,(\mathbf{x})$. This is an analytic function of $\mathbf{k}$ which can be analytically continued to (sufficiently small) complex values of $\mathbf{k}$. We define the "inertia moment" of order ℓ of $\rho\,(\mathbf{x})$, say $I^{i_1\cdots i_\ell}$, as the integral on the right-hand side of Eq. (3) without the STF projection. These inertia moments are the Taylor coefficients of the (convergent) expansion of $\hat{\rho}\,(\mathbf{k})$ in powers of $\mathbf{k}$. Each inertia moment $I^{i_1\cdots i_\ell}$ is equal to its trace-free projection $Q^{i_1\cdots i_\ell}$ plus some "trace terms" containing at least one Kronecker $\delta^{i_a i_b}$. It is easily seen that a monopole ($Q^{i_1\cdots i_\ell}=0$) can be characterized by requiring that all terms of the Taylor expansion of $\hat{\rho}\,(\mathbf{k})$ contain a factor $\mathbf{k}^2=\delta_{ij}\,k^i k^j$. In other words, a monopole is characterized by requiring that the Taylor expansion of $\hat{\rho}\,(\mathbf{k})$ reduce to $\hat{\rho}\,(\mathbf{0})$ when the complexified $\mathbf{k}$ is only restricted to satisfy $\mathbf{k}^2=0$. This general characteristic property of a monopole can also be proven by taking the Fourier transform of the Poisson equation satisfied by the difference potential $V(\mathbf{x})=U(\mathbf{x})-\frac{M}{r}$, where $U(\mathbf{x})$ denotes the potential generated by the monopole (centered at the origin) and M its mass. Indeed, $\triangle_x\,(U-\frac{M}{r}) = -\,4\,\pi\;(\rho\,(\mathbf{x})-M\,\delta(\mathbf{x}))$, which implies $\hat{\rho}\,(\mathbf{k})=\hat{\rho}\,(\mathbf{0})+\frac{1}{4\,\pi}\,\mathbf{k}^2\;\hat{V}(\mathbf{k})$, where $\hat{V}(\mathbf{k})$ denotes the Fourier transform of the *compact-support* function $V(\mathbf{x})$.

For the case at hand, the explicit calculation of $\hat{\rho}(\mathbf{k}) = \int d\,\Sigma\;\sigma(\mathbf{x})\;e^{-i\mathbf{k}\cdot\mathbf{x}}$, with $\mathbf{k}=(\,h\cos\phi,\,h\sin\phi,\,k)$, is very easy using (37) and yields $\hat{\rho}\,(\mathbf{k})=J_0\,(h)/I_0\,(k)$. The fact that it reduces to $\hat{\rho}\,(\mathbf{0})=1$ when $h=ik$ (i.e. $\mathbf{k}^2=h^2+k^2=0$) establishes directly the vanishing of all the multipole moments. The same verification can be done from inserting (40) in (3) at the price of more Bessology.

The important lesson we learn from this last example, by truncating σ at a finite $|z| > R$, is the possibility to construct *exponentially-accurate* approximations to exact monopoles having the shape of *finite* cylinders, opened at both ends, and thickened in the appropriate way in the middle. From (41) we see that the accuracy with which the multipole moments of such a truncated thin cylinder vanish is of order $e^{-j_1 |z|/R}$. More generally, one can get such exponential decreases of σ at infinity, whenever, when some longitudinal dimension (playing the role of z) tends to infinity, the transverse dimensions remain bounded[27].

Such bodies could be very good candidates for being the "external cylinder" of a differential accelerometer of the STEP type. They are exponentially insensitive[28] to external gravity gradients, and allow easily for an "internal cylinder" (together with its suspension and sensing system) to be placed inside them.

Alternatively we can use the generalized construction of subsection 3.3 to define, for a given internal *straight* cylinder of finite length – with non-vanishing multipole moments – a corresponding "external cylinder" having exactly the same reduced multipole moments $Q^{i_1 i_2 \ldots i_\ell} / M$. In the thin cylinder approximation, the appropriate surface distribution σ is obtained by generalizing the solution (35-40) to the case where Eq. (34) is replaced by $\triangle G^\rho (\mathbf{x}) = -4\pi\ \rho(\mathbf{x})$, where $\rho(\mathbf{x})$ denotes the mass distribution of the internal cylinder. The exponential fall-off of σ will be given by the same exponential factor as in Eq. (40) (with, in general, a different prefactor). We shall come back, at the end, to the possible use of this construction for Equivalence Principle experiments.

[27] The asymptotic behavior of σ is then primarily determined by studying the eigenvalues of the Schrödingerlike equation $-\triangle_{\rm transverse}\,\psi = E\,\psi$ within the interior of Σ, with Dirichlet boundary conditions. The crucial criterion for an exponential decrease of σ is the existence of a positive lower bound on E (typically proportional to the inverse of the square of the transverse dimensions).

[28] The suppression factor is roughly proportional to $\exp(-\,j_1\,L/2R)$, where L is the total length of the cylinder.

5 Mathematical aspects of the construction of aspherical monopoles.

5.1 The mathematical setting.

We have shown in Section 3 that the problem of constructing aspherical monopoles with a given center of mass[29] amounts to solving a differential equation in the functional space describing closed[30] surfaces Σ. It seems intuitively obvious that, if one starts from a sufficiently smooth surface Σ_0 at "time" $t = 0$, the differential propagation law (24), $d\,\mathbf{X} = \sigma_{\Sigma_t}\,\mathbf{n}_{\Sigma_t}\,dt$, will generate, at least for a finite interval of values of the parameter t around zero, a continuously evolving family of neighbouring surfaces Σ_t. The problem of characterizing the class of good initial data Σ_0 which admit such an evolution is, actually, highly non trivial. From the considerations we shall now present, it is clear that (if we demand evolution for both signs of t around zero) we need to work with *analytic* (C^ω) surfaces embedded in $\mathbb{R}^N$. We have, however, not precisely determined what class of norms one must use to guarantee that Σ_0 is "sufficiently analytic". We shall see also that the problem of *outward* propagation ($t > 0$) is very different from the *inward* propagation one, and allows one to start with much less regular (say, only continuous) initial data[31].

A first clear hint that one needs to work with analytic data is the fact that, in two dimensions, the propagation law (23) is reminiscent of the Cauchy-Riemann equations for the mapping $(t, \xi) \to \mathbf{X}(t, \xi) = (X, Y)$, $\partial X/\partial t = \partial Y/\partial \xi$, $\partial Y/\partial t = -\,\partial X/\partial \xi$, which can be written as $\partial\,\mathbf{X}(t, \xi)/\partial t = |\,\partial\,\mathbf{X}/\partial \xi|\;\mathbf{n}(\mathbf{X})$. Clearly the local functional $|\,\partial\,\mathbf{X}/\partial \xi\,|$ is a much simplified version of the non-local functional $\sigma_\Sigma(\mathbf{X})$ appearing in (23) which has, as we shall see, a worse sensitivity on the regularity of the function $\mathbf{X}(\xi)$ than $|\,\partial\,\mathbf{X}/\partial \xi\,|$.

[29] In this section, the origin O used in the construction of Section 3 is fixed from the beginning. We henceforth drop the superscript O on $\sigma^{\mathrm{O}}_{\Sigma_t}$.

[30] By "closed", we mean more precisely that Σ is the connected boundary of a connected, open subset of $\mathbb{R}^N$.

[31] The fact that the evolution is stable outwards and unstable inwards has been known for some time in a very similar two-dimensional problem (injection of fluid in a Hele-Shaw cell; see e.g. Ref.[20]).

Having recognized the need to work within a (real) analytic setting, we can abstractly formulate our problem as a functional equation (for u belonging to some functional space) of the form

$$\frac{d\,u(t)}{dt} \;=\; \mathcal{F}\,[\,u(t)\,]\;, \tag{42}$$

which is the ultimate generalization of the Cauchy-Kovaleskaya problem. Nirenberg [21] and Nishida [22] have proven a general theorem about the existence and uniqueness of solutions to the general abstract non-linear Cauchy-Kovalevskaya problem (42). The crucial point of their work was to formulate the abstract analog of the condition (in the usual Cauchy-Kovalevskaya problem) that the functional $\mathcal{F}\,[\,u\,]$ on the right-hand side of Equation (42) involve at most first-order (spatial) partial derivatives of u and be linear in them. Their formulation assumes that u varies in a "scale of Banach spaces" $S = \{\,B_s;\, 0 < s \leq 1\,\}$ such that, $B_s \subset B_{s'}$, when $s' \leq s$, with the norms satisfying $\|\cdot\|_{s'} \leq \|\cdot\|_s$. Then the crucial "quasi-linearity" condition (expressing the analog of the linearity in first-order derivatives) is that $\mathcal{F}\,[\,u\,]$ must satisfy a Lipschitz-type inequality:

$$\|\,\mathcal{F}\,[\,u\,] - \mathcal{F}\,[\,v\,]\,\|_{s'} \;\leq\; \frac{\mathrm{C}}{s-s'}\;\|\,u-v\,\|_s\;, \qquad \text{for } s' < s\;. \tag{43}$$

Under (43), and some other mild assumptions, the solution of (42) exists and is unique (for sufficiently small t). The norms $\|\;\|_s$ appropriate to our problem are the ones standard when dealing with analytic functions (e.g. $\left\|\sum\limits_{n=0}^{\infty} a_n\, x^n\right\|_s = \sum\limits_{n=0}^{\infty} |a_n|\, e^{ns}$) or slight adaptations of these to deal with the embedding functions $\boldsymbol{\xi} \in \Sigma \;\rightarrow\; \mathbf{X} \in \mathbb{R}^N$.

Actually, the functional of $\mathbf{X}(\boldsymbol{\xi})$ appearing on the right-hand side of the propagation equation (23) cannot satisfy the condition (43) because it contains the *product* of two "dangerous" functionals: the normal $\mathbf{n}_\Sigma$ (which depends on the first derivatives $\partial\,\mathbf{X}/\partial\boldsymbol{\xi}$) and the harmonic measure σ_Σ. In order to apply the abstract theorem of Refs. [21, 22], one should transform the original evolution equation (23) into an equivalent "quasilinear" system that we shall introduce later, in subsection 5.6. But, first, to understand the subtleties of Equation (23) let us discuss the linearization of our evolution problem, by studying the small perturbations of a monopole.

5.2 A linear propagation equation for the perturbations of a monopole.

In this subsection we discuss how a monopole, built from an "initial" surface Σ_0, reacts to a small deformation of this surface Σ_0. The deformation of Σ_0 is parametrized, at first order, by a "height function" $h\,(t=0,\,\boldsymbol{\xi})$ defined on Σ_0. This deformation, which extends to the whole monopole, i.e. to the whole family of surfaces Σ_t, is described – still at first order – by a function $h\,(t,\,\boldsymbol{\xi})$ which gives the local orthogonal displacement of the intermediate surface Σ_t. The quantity $h(t,\boldsymbol{\xi})$ obeys a *linear* propagation equation, in terms of the "time" evolution variable t (i.e. the algebraic volume of the monopole Ω_t included between the two surfaces Σ_0 and Σ_t). We shall establish below, by different methods, this evolution equation of h, and discuss, in subsection 5.3, some general features of its solutions. In subsections 5.4 and 5.5 we shall explicitly solve it in the case of the outward propagation, and discuss, as an illustrative example, the simple case for which the initial (unperturbed) monopole has a spherical symmetry of center O.

The function $h\,(t,\,\boldsymbol{\xi})$ will be shown to satisfy a linear (real "Schrödinger-type" or "heat-type") propagation equation,

$$\frac{\partial}{\partial t}\,h(t) \;=\; -\,\mathcal{H}(t)\;h(t)\;, \tag{44}$$

in which the (self-adjoint) operator $\mathcal{H}(t)$ is the one defining the variation of the harmonic measure σ_{Σ_t} under the perturbation $h(t)$, according to the equation

$$\delta\,\sigma_{\Sigma_t} \;=\; -\,\mathcal{H}(t)\;h(t)\;. \tag{45}$$

This "Hamiltonian" $\mathcal{H}$ is explicitly given by

$$\mathcal{H}:\quad h \;\rightarrow\; \mathcal{H}\,h \;\equiv\; \mathcal{N}_{\Sigma_t}\,(\sigma_{\Sigma_t}\,h) \,+\, k\;\sigma_{\Sigma_t}\;h\;, \tag{46}$$

in which k denotes the mean curvature of the surface Σ_t (i.e. $\frac{1}{R_1}+\frac{1}{R_2}$, in three dimensions), and $\mathcal{N}_{\Sigma_t}$ is the "Neumann operator" which associates, to every continuous function (u) on Σ_t, the value $(\,\partial_n\,\tilde{u}\,)$ of the normal derivative of its harmonic extension (within the inside of Σ_t). (This propagation equation of the deformation h, incidentally, must admit as a solution σ_{Σ_t} itself, since $h(t)=\sigma_{\Sigma_t}\,\delta t$ corresponds to an obvious and rather trivial deformation of the monopole associated with an infinitesimal shift δt in the variable t.) The essential positivity of this Hamiltonian (meaning that it can be made positive by adding a constant) will imply that the *outward* ($t > 0$) propagation of the perturbation h will be well-defined and smoothing for all t, as we shall discuss in subsections 5.3 and 5.4. We shall also solve explicitly this propagation equation (44), in the case of the *outward* propagation.

Let us now assume, after this general presentation, that we know some exact solution of the evolution problem (23), defining a monopole. (At this stage, the only exact solution we can write down easily corresponds to the trivial case where Σ_0 is a sphere *and* the origin O is chosen at its center, the Σ_t's constituting concentric spheres parametrized by the algebraic volume between Σ_0 and Σ_t.) We consider a small perturbation of this solution. If $\mathbf{X}(t, \boldsymbol{\xi})$ denotes the embedding associated with the unperturbed solution Σ_t, let $\mathbf{X}'(t, \boldsymbol{\xi}) = \mathbf{X}(t, \boldsymbol{\xi}) + \varepsilon \, \mathbf{X}_1(t, \boldsymbol{\xi})$, with ε some infinitesimal parameter[32], denote the embedding associated with some neighbouring solution Σ'_t. Note that we consider here variations of end surfaces, $(\Sigma_0, \Sigma_t) \to (\Sigma'_0, \Sigma'_t)$, keeping fixed the volume t spanned between the surfaces.

Modulo a (t-dependent) reparametrization of Σ'_t, the perturbation of Σ_t into Σ'_t can be described by a simple scalar quantity: the orthogonal distance between Σ_t and Σ'_t (i.e. the local "height" of Σ'_t with respect to Σ_t), say (after division by ε)

$$h(t, \boldsymbol{\xi}) \;=\; \mathbf{n}(t, \boldsymbol{\xi}) \,\cdot\, \mathbf{X}_1(t, \boldsymbol{\xi}) \,. \qquad (47)$$

This scalar quantity also appears naturally if we perturb the representation (26) instead of the embedding representation (22). More precisely, if $\varphi(\mathbf{x}) = t$ is the equation of the unperturbed family of surfaces Σ_t, and $\varphi'(\mathbf{x}') \equiv \varphi(\mathbf{x}') + \varepsilon \, \varphi_1(\mathbf{x}') = t$ that of the perturbed family Σ'_t, one finds $h = -\,\varphi_1/|\boldsymbol{\nabla}\varphi| = -\,\sigma(\mathbf{x}) \; \varphi_1(\mathbf{x})$.

An evolution equation for h can be obtained by perturbing (and projecting) the vectorial equation (23), or, equivalently, by perturbing the (scalar) eikonal equation (27). To describe the result we must first discuss some mathematical objects entering the change of the harmonic measure σ_Σ under an infinitesimal variation of the surface Σ.

[32] Henceforth, we work to first order in ε, disregarding all terms of order ε^2, ...

Let us first introduce the "Neumann operator" $\mathcal{N}_\Sigma$ associated to $\Sigma = \partial\,\Omega$, where Ω is some domain in $\mathbb{R}^N$. The functions which are harmonic within Ω (and continuous on its closure $\bar{\Omega} = \Omega \cup \Sigma$) are uniquely determined by their values on the boundary $\Sigma = \partial\,\Omega$ ("Dirichlet problem"). If $\boldsymbol{\xi}$ denote some coordinates on Σ, any sufficiently regular (say continuous) function $u\,(\boldsymbol{\xi})$ on Σ defines a unique harmonic function $\tilde{u}\,(\mathbf{x})$ within Ω ($\Delta\tilde{u} = 0$), called the "harmonic extension" of $u(\boldsymbol{\xi})$. Under some mild regularity conditions for $u\,(\boldsymbol{\xi})$ and Σ, the normal derivative of $\tilde{u}\,(\mathbf{x})$ on Σ exists. Evidently, $[\,\partial_n\,\tilde{u}\,]\,(\boldsymbol{\xi})$ is a linear functional of $u\,(\boldsymbol{\xi})$. This linear functional is called the Neumann operator:

$$[\,\partial_n\,\tilde{u}\,]_\Sigma \;=\; \mathcal{N}_\Sigma\;u\;. \tag{48}$$

This operator $\mathcal{N}_\Sigma$ acts within the space of (sufficiently regular) functions on Σ. It is a positive, elliptic pseudodifferential operator of order 1, which means essentially that its kernel $\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\xi}')$ is singular as $\boldsymbol{\eta} = \boldsymbol{\xi} - \boldsymbol{\xi}' \to 0$ in a manner such that its Fourier transform with respect to $\boldsymbol{\eta}$ is of order $|\mathbf{k}|$ as $|\mathbf{k}| \to \infty$ ($\mathbf{k}$ denoting the momentum associated with $\boldsymbol{\eta}$). It is also known that the leading short-distance singularity of $\mathcal{N}$ is given by $\sqrt{-\,\Delta_\Sigma}$ where Δ_Σ is the intrinsic Laplacian on Σ, as defined by the inner metric induced from the ambient euclidean metric on $\mathbb{R}^N$ (see e.g. Ref. [23]).

By using the second Green's identity

$$\int_\Sigma d\,\Sigma\;\;(\tilde{u}\;\partial_n\,\tilde{v}\;-\;\tilde{v}\;\partial_n\,\tilde{u})\;\;=\;\;\int_\Omega d^N\,\mathbf{x}\;\;(\tilde{u}\;\Delta\,\tilde{v}\;-\;\tilde{v}\;\Delta\,\tilde{u})\;\;=\;\;0\;, \tag{49}$$

one sees that the Neumann operator is *self-adjoint* with respect to the scalar product $(u,v) \equiv \int_\Sigma d\,\Sigma\;u\,v$ between functions on Σ: $(u,\mathcal{N}v) = (\mathcal{N}u,\,v)$. Moreover, $\mathcal{N}$ is a *positive* operator, with respect to this scalar product, because of the first Green's identity

$$\int_\Sigma d\,\Sigma\;\;\tilde{u}\;\;\partial_n\,\tilde{u}\;\;=\;\;\int_\Omega d^N\,\mathbf{x}\;\;\boldsymbol{\nabla}\,(\tilde{u}\,\boldsymbol{\nabla}\,\tilde{u})\;\;=\;\;\int_\Omega d^N\,\mathbf{x}\;\;[\,(\boldsymbol{\nabla}\,\tilde{u})^2 + \tilde{u}\;\Delta\,\tilde{u}\,]\;, \tag{50}$$

applied to harmonic functions:

$$(\,u,\;\mathcal{N}u\,)\;\;=\;\;\int_\Omega d^N\,\mathbf{x}\;\;(\boldsymbol{\nabla}\,\tilde{u})^2\;\;>\;\;0\;. \tag{51}$$

Note that the positivity of $\mathcal{N}$ has resulted from using the *outward* normal in its definition (48).

The operator $\mathcal{N}_\Sigma$ determines the modification of the harmonic measure $\sigma^{\rm O}_\Sigma$ under an infinitesimal variation of the surface Σ. Let us describe, as in Equation (47), an infinitesimal variation of Σ by the orthogonal distance between Σ and Σ', i.e. as the map

$$(\mathbf{X}(\boldsymbol{\xi}) \in \Sigma) \quad \to \quad (\mathbf{X}'(\boldsymbol{\xi}) \; = \; \mathbf{X}(\boldsymbol{\xi}) + \varepsilon \, h(\boldsymbol{\xi}) \, \mathbf{n} \; \in \Sigma') \; . \tag{52}$$

We can compute the change $\delta \, \sigma^{\rm O}_\Sigma(\boldsymbol{\xi}) = \sigma^{\rm O}_{\Sigma'}(\boldsymbol{\xi}) - \sigma^{\rm O}_\Sigma(\boldsymbol{\xi})$ (evaluated, following (52), at two orthogonally corresponding points on Σ and Σ') by varying the defining relation

$$\tilde{u}({\rm O}) \; = \; \int_\Sigma d\,\Sigma \;\; \sigma^{\rm O}_\Sigma(\boldsymbol{\xi}) \;\; \tilde{u} \, (\, \mathbf{X}(\boldsymbol{\xi}) \,) \; , \tag{53}$$

in which $\tilde{u}(\mathbf{x})$ is an arbitrary harmonic function. The variation of the left-hand side of (53) vanishes, since one keeps both the origin O and the function $\tilde{u}\,(\mathbf{x})$ fixed. The variation of the right-hand side of (53) yields three contributions.

The first one is due to the change of the area element $d\,\Sigma$ (corresponding to a given coordinate span $d^{N-1}\boldsymbol{\xi}$) under the orthogonal deformation (52). This is described by the trace of the second fundamental form of Σ. We recall that, using some Gaussian coordinates, the metric of $\mathbb{R}^N$ around Σ can be written as $(\alpha, \beta, \gamma = 1, \ldots, N-1)$

$$ds^2 \; = \; dn^2 \; + \; g_{\alpha\beta}(n, \xi^\gamma) \; d\xi^\alpha \, d\xi^\beta \; , \tag{54}$$

where $n = 0$ is the equation of Σ (so that $g_{\alpha\beta}$ is the first fundamental form of Σ) and where the curves $\xi^\alpha = {\rm const.}$ are ambient geodesics (i.e. straight lines in $\mathbb{R}^N$) *orthogonal* to Σ. The second fundamental form (or extrinsic curvature tensor) of Σ is defined as one-half the normal (Lie) derivative of $g_{\alpha\beta}$:

$$k_{\alpha\beta}(\xi^\gamma) \;\; \equiv \;\; \frac{1}{2} \left[\frac{\partial \, g_{\alpha\beta}(n, \xi^\gamma)}{\partial \, n} \right]_{n=0} \; . \tag{55}$$

For a surface in $\mathbb{R}^3$, the diagonalization of the mixed tensor[33] $k^\alpha{}_\beta = g^{\alpha\gamma} \, k_{\gamma\beta}$ defines the two principal curvatures $k_1 = 1/R_1$ and $k_2 = 1/R_2$. From

[33]The mixed form of k can also be defined as the "Weingarten" map, $\nabla_{\mathbf{V}} \, \mathbf{n} = k(\mathbf{V})$, where $\mathbf{V}$ is a vector tangent to Σ, and $[k(\mathbf{V})]^\alpha = k^\alpha{}_\beta \, V^\beta$.

$d\,\Sigma = (\det g_{\alpha\beta})^{1/2}\, d^{N-1}\,\boldsymbol{\xi}$ it is easy to see that the variation of $d\,\Sigma$ under the orthogonal deformation (52) depends only, at first order in ε, on the local value of $h(\boldsymbol{\xi})$ (but not on its derivatives). It is given by

$$d\,\Sigma' \;=\; d\,\Sigma\;\left[1 \,+\, \varepsilon\; h(\boldsymbol{\xi})\;k(\boldsymbol{\xi})\right]\,, \tag{56}$$

where

$$k(\boldsymbol{\xi}) \;\equiv\; g^{\alpha\beta}\; k_{\alpha\beta} \;\equiv\; k^{\alpha}_{\;\;\alpha} \tag{57}$$

is the trace of the second fundamental form, also called the "mean curvature". In the most relevant case of a surface in $\mathbb{R}^3$, $k = \frac{1}{R_1} + \frac{1}{R_2}\,.$

Coming back to the variation of the right-hand side of Equation (53), we find the following three contributions

$$0 \;=\; \int_{\Sigma} d\,\Sigma\;\left[\,(\varepsilon\,h\,k)\,\tilde{u} \;+\; \delta\,\sigma^{\mathrm{O}}_{\Sigma}\;\,\tilde{u} \;+\; \sigma^{\mathrm{O}}_{\Sigma}\;(\varepsilon\,h\;\partial_n\,\tilde{u})\,\right]\,. \tag{58}$$

The third contribution (which comes from the variation of $\tilde{u}\,(\mathbf{X}(\boldsymbol{\xi})\,)$ caused by the change (52) of its argument) can be rewritten in terms of $\mathcal{N}_{\Sigma}\,u\,.$ Finally, using the self-adjointness of the Neumann operator, we can rewrite (58) as a linear form in $u\,(\boldsymbol{\xi})\,.$ The arbitrariness of $u\,(\boldsymbol{\xi})$ then yields the looked for formula for the "normal" variation of $\sigma^{\mathrm{O}}_{\Sigma}$

$$\delta\,\sigma^{\mathrm{O}}_{\Sigma}(\boldsymbol{\xi}) \;=\; \sigma^{\mathrm{O}}_{\Sigma'}(\boldsymbol{\xi}) - \sigma^{\mathrm{O}}_{\Sigma}(\boldsymbol{\xi}) \;=\; -\,\varepsilon\;[\mathcal{N}_{\Sigma} + k\,]\,(\sigma^{\mathrm{O}}_{\Sigma}\,h)\,. \tag{59}$$

Here, $\mathcal{N}(\sigma h) = \int d\,\Sigma(\boldsymbol{\xi}')\;\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\xi}')\;\sigma(\boldsymbol{\xi}')\;h(\boldsymbol{\xi}')$ is a non-local functional of $h\,,$ while $k\,\sigma\,h = k(\boldsymbol{\xi})\,\sigma(\boldsymbol{\xi})\,h(\boldsymbol{\xi})$ depends only on the local value of h at $\boldsymbol{\xi}\,.$ (The operator $\mathcal{N}_{\Sigma} + k$ is independent from the choice of the pole O; therefore, by linear superposition, the equation (59) gives also the infinitesimal variation of the weighted harmonic measure $\sigma^{\rho_1}_{\Sigma}$ defined by Eq. (28) and entering our generalized construction (29).)

We are now (nearly) in measure of deriving an evolution equation for the perturbed "height" $h(t,\boldsymbol{\xi})$, defined by Equation (47). A direct perturbation of the propagation law (23) yields

$$\varepsilon\;\;\frac{\partial\,\mathbf{X}_1(t,\boldsymbol{\xi})}{\partial\,t} \;\;=\;\; \sigma^{\mathrm{O}}_{\Sigma'_t}(\mathbf{X}')\;\mathbf{n}'(\mathbf{X}') \;-\; \sigma^{\mathrm{O}}_{\Sigma_t}(\mathbf{X})\;\mathbf{n}(\mathbf{X})\;. \tag{60}$$

Projecting this vectorial equation along the normal $\mathbf{n}$ yields (to first order in ε) the following expression for the evolution of the height $h = \mathbf{n} \cdot \mathbf{X}_1$

$$\frac{\partial}{\partial t} \, h(t, \boldsymbol{\xi}) \; = \; \frac{\partial}{\partial t} \, (\mathbf{n} \cdot \mathbf{X}_1) \; = \; \sigma^{\mathrm{O}}_{\Sigma'_t}(\mathbf{X} + \varepsilon \, \mathbf{X}_1) - \sigma^{\mathrm{O}}_{\Sigma_t}(\mathbf{X}) + \frac{\partial \, \mathbf{n}}{\partial \, t} \cdot \mathbf{X}_1 \, . \tag{61}$$

Because the displacement $\mathbf{X}_1$ has both a normal and a tangential component, $\mathbf{X}_1 \equiv \mathbf{n} \, h + \mathbf{X}_T$, the variation of $\sigma^{\mathrm{O}}_{\Sigma}$ appearing on the right-hand side is the sum of two contributions: (1) the term (59) due to the orthogonal displacement $h = \mathbf{n} \cdot \mathbf{X}_1$, and (2) the supplementary contribution $+ \mathbf{X}_T \cdot \boldsymbol{\nabla} \, \sigma^{\mathrm{O}}_{\Sigma}$ due to the tangential displacement. However, the latter is precisely cancelled by the last term on the right-hand side of Equation (61). Indeed, the t-derivative of the normal field $\mathbf{n}(t, \boldsymbol{\xi})$ is given by

$$\frac{\partial}{\partial t} \, \mathbf{n}(t, \boldsymbol{\xi}) \; = \; - \boldsymbol{\nabla}_T \, \sigma^{\mathrm{O}}_{\Sigma_t} \; , \tag{62}$$

where $\boldsymbol{\nabla}_T$ denotes the tangential gradient (along Σ_t).

One way to prove (62) is to notice that, by definition of the t and $\boldsymbol{\xi}$ parametrizations, the metric element of $\mathbb{R}^N$, $ds^2 = d \, \mathbf{X}^2$, has the form, in $(t, \boldsymbol{\xi})$ coordinates,

$$ds^2 \;\; = \;\; \sigma^2(t, \xi^\gamma) \, dt^2 \; + \; g_{\alpha\beta}(t, \xi^\gamma) \;\; d\xi^\alpha \; d\xi^\beta \; . \tag{63}$$

Writing the curvilinear components of the absolute acceleration of the lines $\xi^\gamma = \text{const.}$ is very easy in the coordinate system (63) (only one Christoffel symbol survives) and yields $(d^2 \, \mathbf{X}/ds^2)_\alpha = (d \, \mathbf{n}/ds)_\alpha = - \, \partial_\alpha \, \sigma / \sigma$, which is equivalent to (62).

Altogether, using (59) we get from (61) the simple evolution law

$$\frac{\partial}{\partial t} \, h(t, \boldsymbol{\xi}) \;\; = \;\; - \, [\, \mathcal{N} + k \,] \, (\sigma \, h) \; . \tag{64}$$

Apart from h, all quantities in (64) refer to the unperturbed embedding $\mathbf{x} = \mathbf{X}(t, \boldsymbol{\xi})$ and its associated family of surfaces Σ_t. The explicit expression of the right-hand side of (64) may be written as follows:

$$- \int d^{N-1} \boldsymbol{\xi}' \; \sqrt{g(\boldsymbol{\xi}')} \; \mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\xi}') \; \sigma(\boldsymbol{\xi}') \; h(t, \boldsymbol{\xi}') \;\; - \;\; k(\boldsymbol{\xi}) \; \sigma(\boldsymbol{\xi}) \; h(t, \boldsymbol{\xi}) \; , \tag{65}$$

where $g = \det g_{\alpha\beta}$. The evolution law (64) applies also to the perturbation of the generalized growth process (29) leading to the construction of gravitationally-equivalent bodies, if we interpret σ as the weighted harmonic measure $\sigma_\Sigma^{\rho_1}$ of Eq. (28).

An alternative way of deriving (64) consists of perturbing the (scalar) eikonal equation (27). Inserting

$$\varphi'(\mathbf{x}) \;=\; \varphi(\mathbf{x}) + \varepsilon\,\varphi_1(\mathbf{x}) \quad \text{and} \quad \sigma^{\mathrm{O}}_{\Sigma'_{\mathbf{x}}}(\mathbf{x}) \;=\; \sigma(\mathbf{x}) + \varepsilon\,\sigma_1(\mathbf{x}) \tag{66}$$

into (27) yields $\boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\varphi_1 = -\,\sigma^{-3}\,\sigma_1$. The perturbed harmonic measure is obtained from (59) with the subtlety that the height h is not the expected $-\,\varphi_1/|\boldsymbol{\nabla}\varphi| = -\,\sigma\,\varphi_1$, but instead $h(t,\boldsymbol{\xi}') = -\,\sigma(\boldsymbol{\xi}')\,[\,\varphi_1(\boldsymbol{\xi}') - \varphi_1(\boldsymbol{\xi})\,]$, the last contribution coming from the fact that, in this formulation, the surface $\Sigma_{\mathbf{x}}$ is constrained to pass through the point $\mathbf{x}$ when $\varphi \to \varphi'$ (so that the volume parameter $t' = \varphi'(\mathbf{x}) \neq t = \varphi(\mathbf{x})$). The resulting equation yields back (64) upon using the following expression for the t-derivative of the harmonic measure:

$$\frac{\partial}{\partial t}\,\sigma(t,\boldsymbol{\xi}) \;=\; -\,[\,\mathcal{N} + k\,]\,(\sigma^2)\;. \tag{67}$$

The latter result follows directly by applying the general result (59) to the case where $\Sigma = \Sigma_t$ and $\Sigma' = \Sigma_{t+\varepsilon}$ are two neighbouring surfaces within the unperturbed family of surfaces satisfying (23). Note that Equation (67) says that $h = \sigma$ is a particular solution of the general perturbation equation (64). [This is indeed, as we have already mentioned, a “trivial” solution corresponding to considering the t-shifted embedding $\mathbf{x} = \mathbf{X}(t+\varepsilon,\boldsymbol{\xi})$ as a “perturbation” of $\mathbf{x} = \mathbf{X}(t,\boldsymbol{\xi})$.]

A third, instructive, way of deriving the perturbed evolution equation (64) consists of directly varying the fundamental property defining a monopole, i.e.

$$|\,t\,|\;\tilde u\,(\mathrm{O}) \;=\; \int_{\Omega_t} d^N\mathbf{x}\;\;\tilde u(\mathbf{x})\;, \tag{68}$$

where $\tilde u(\mathbf{x})$ is an arbitrary harmonic function, and Ω_t the domain (of volume $|\,t\,|$) extending between the initial surface Σ_0 and Σ_t. Keeping fixed the origin O, the volume t and the harmonic function $\tilde u(\mathbf{x})$, the change of (68) under variations of the two end surfaces Σ_0 and Σ_t (both described by a normal displacement as in (52)) is easily seen to lead to a "conservation"[34] equation $0 = \varepsilon\,(I_t - I_0)$ where

$$I_t \;=\; \int_{\Sigma_t} d\,\Sigma_t \;\; h(t,\mathbf{x}) \;\; \tilde{u}(\mathbf{x}) \tag{69}$$

is a surface contribution associated with the variation of the "upper cap". Passing as above to the (unperturbed) variables $(t, \boldsymbol{\xi})$, one finds easily (using the tools given above) that the "time" derivative of I_t can be rewritten as

$$\frac{d\,I_t}{dt} \;=\; \int_{\Sigma_t} d\,\Sigma_t \;\; \tilde{u} \;\left\{ \frac{\partial\, h}{\partial\, t} \,+\, [\,\mathcal{N} + k\,]\,(\sigma\, h) \right\} \quad (\,=\,0\,)\;, \tag{70}$$

thereby recovering (64).

5.3 Behavior of the solutions of the propagation equation for the perturbation of a monopole.

One can consider Equation (64) as a Schrödinger equation for the "wave function" $h(t, \boldsymbol{\xi})$ with respect to the imaginary time t (or, directly, as a type of "heat equation"). To do this, it is convenient to endow the space of functions on a surface Σ with a new scalar product differing from the canonical "metric" one $[\,(u,v) \equiv \int_\Sigma d\,\Sigma \;\; u\,v]$ used above. Let us define the "harmonic" scalar product (for some given origin O)

$$\langle\, u, v\,\rangle \;\;\equiv\;\; \int_\Sigma d\,\Sigma \;\; \sigma_\Sigma^{\mathrm{O}} \;\; u\,v\;. \tag{71}$$

We also define the following operators acting on the space of functions on some surface Σ,

$$\begin{cases} \mathcal{H}_0 : \quad h \;\;\rightarrow\; \mathcal{H}_0\, h \;\;\equiv\;\; \mathcal{N}(\sigma\, h)\,, \\[2ex] \mathcal{H}_k : \quad h \;\;\rightarrow\; \mathcal{H}_k\, h \;\;\equiv\;\; (\mathcal{H}_0 + k\,\sigma)\, h = [\,\mathcal{N} + k\,]\,(\sigma\, h)\;. \end{cases} \tag{72}$$

The important point is that both $\mathcal{H}_0$ and $\mathcal{H}_k$ are *self-adjoint* with respect to the new scalar product (71). Indeed, $\langle\, u, \mathcal{H}_0\, v\,\rangle \;=\; (\,\sigma\, u, \mathcal{N}(\sigma\, v)\,) \;=$

[34] By "conservation" we mean t-independence, as in classical mechanics when t denotes the time.

$(\mathcal{N}(\sigma\, u\,),\, \sigma\, v) = \langle\, \mathcal{H}_0\, u,\, v\,\rangle$. Another important point is that the positivity of $\mathcal{N}$ with respect to the "metric" scalar product, implies the *positivity* of $\mathcal{H}_0$ with respect to the "harmonic" scalar product (71). Indeed, $\langle\, u,\, \mathcal{H}_0\, u\,\rangle = (\,\sigma\, u,\, \mathcal{N}(\sigma\, u)\,)$.

The evolution equation (64),

$$\frac{\partial}{\partial\, t}\, h(t) \;=\; -\, \mathcal{H}_k(t)\; h(t)\;, \tag{73}$$

appears then as the imaginary-time version of the Schrödinger equation with the self-adjoint, time-dependent, Hamiltonian $\mathcal{H}_k$. (The time-dependence of $\mathcal{H}_k$ comes from the Σ-dependence of $\mathcal{N}$, k and σ.) The formal "Dyson" solution of (73) reads

$$h(t) \;=\; T\, \exp\left(-\int_0^t dt'\;\, \mathcal{H}_k(t')\right)\, h(0)\,. \tag{74}$$

If the mean curvature k is positive, the Hamiltonian $\mathcal{H}_k$ will be positive. If k is not positive, one can reduce the problem to a positive Hamiltonian $\mathcal{H}'_k = \mathcal{H}_k + c$ (where c is a sufficiently large constant) by considering the modified variable $h'(t) = e^{-ct}\, h(t)$. This positivity of the Hamiltonian makes it clear that the solution (74) will exist for *positive* values of t under very mild conditions on the regularity of the initial data $h(0\,)$ as a function of the surface coordinates $\boldsymbol{\xi}$. Like in the case of the heat equation, the evolution (74) for *positive* t will be well defined, and *smoothing* for all t. (The explicit solution of Eq. (73) given in the next subsection will show that $h(t, \boldsymbol{\xi})$ is analytic in $\boldsymbol{\xi}$.)

In other words, we expect that our evolution, when considered in the *outward* direction ($t > 0$) will be well defined for all t, starting from a very general class of initial data (i.e. initial surfaces Σ_0) with weak regularity properties. The surfaces Σ_t generated by the outward evolution will be very smooth (probably analytic) even if the initial surface is barely regular. By contrast, the situation is completely different in the case of the evolution in the *inward* direction ($\,t < 0\,$). In this case, we expect that the initial data need in general to be an analytic function of $\boldsymbol{\xi}$, for the solution of (73) to exist at all. The domain of existence of the solution (how "long" it exists for) will depend on how analytic the initial data is. This is precisely this type of feature that the family of norms entering the Nirenberg-Nishida theorem [21, 22] are supposed to capture.

5.4 Explicit solution of the propagation equation for the perturbation of a monopole.

We can actually write down *directly* (and explicitly) the solution $h(t)$ of our propagation equation for the perturbation of a monopole, without even having to write this equation, in the case of the *outward* propagation ($t > 0$). To do so we consider, for some given value of t, the variation of the homogeneous monopole Ω_t as described by the variation of both its inner (Σ_0) and outer (Σ_t) boundaries, expressed by the two height functions $h(0)$ and $h(t)$, respectively.

We recall that, in this variation, the total volume t spanned between the end surfaces is kept fixed. Therefore, the total mass $M = |t|$ of the homogeneous monopole (of unit density) does not vary, which implies that the variation of the external gravitational potential $M/|\mathbf{x}-\mathbf{x}_{\mathrm{O}}|$ must vanish. On the other hand, the variation of the external potential is equal to the potential generated by the variation of the volumic density of the monopole. The latter quantity is clearly made of two single layers: a layer of algebraic thickness $+\varepsilon\, h(t,\boldsymbol{\xi})$ on the outer boundary Σ_t and a layer of algebraic thickness $-\varepsilon\, h(0,\boldsymbol{\xi})$ on the internal boundary Σ_0. The problem is therefore, given the internal layer $-\varepsilon\, h(0,\boldsymbol{\xi})$, to determine the outer layer $+\varepsilon\, h(t,\boldsymbol{\xi})$ which is such that the external potential generated by these two layers vanish identically. This problem is solved by simple electrostatics considerations.

Let us consider Σ_t as a grounded conductor, and deposit on the surface Σ_0, enclosed within Σ_t, a charge layer of surface density $-\varepsilon\, h(0,\boldsymbol{\xi})$. By definition of a grounded conductor, the surface density induced on Σ_t by the presence of the given layer distribution $-\varepsilon\, h(0,\boldsymbol{\xi})$ on Σ_0 will be the unique solution $+\varepsilon\, h(t,\boldsymbol{\xi})$ of our problem. In mathematical terms, if $G_{\Sigma_t}(\mathrm{P},\mathrm{Q})$ denotes the Dirichlet Green function of the domain interior to Σ_t, the potential induced at any point P within Σ_t by the layer $-\varepsilon\, h(0)$ on Σ_0 reads

$$v\,(\mathrm{P}) \;=\; \int_{\mathrm{Q}\in\,\Sigma_0} G_{\Sigma_t}\,(\mathrm{P},\,\mathrm{Q})\;\left(-\,\varepsilon\;h\,(0,\,\mathrm{Q})\right)\;d\,\Sigma_0\quad . \qquad (75)$$

To simplify the notation we write $h(0,\mathrm{Q})$ instead of a more explicit $h(0,\,\boldsymbol{\xi}_{\mathrm{Q}})$.

By definition of the Dirichlet Green function this potential vanishes when $\mathrm{P}\in\Sigma_t$. The corresponding density induced on Σ_t is given by the normal derivative $\frac{1}{4\pi}\partial_n v$ on Σ_t, so that, after division by ε,

$$h\,(t,\,\mathrm{P}) \;=\; \int_{\mathrm{Q}\,\in\,\Sigma_0} d\,\Sigma_0\;\; \sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P})\;\; h\,(0,\,\mathrm{Q}) \;\; . \tag{76}$$

The quantity $\sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P})$ entering Eq. (76) is nothing but the previously defined surface density (18) ("harmonic measure") considered for $\Sigma = \Sigma_t$ and, as in subsection 3.3, with pole taken at an arbitrary point Q within the interior of Σ_t (instead of the fixed origin O):

$$\sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P}) \;=\; -\,\frac{1}{4\,\pi}\;\partial_n\; G_{\Sigma_t}(\mathrm{P},\,\mathrm{Q}) \;, \tag{77}$$

where ∂_n denotes the outgoing normal derivative at $\mathrm{P} \in \Sigma_t$.

Eq. (76) solves explicitly the problem of determining $h(t)$ in terms of $h(0)$. The kernel $K(t,\mathrm{P};0,\mathrm{Q})$ defining the propagation of $h(t)$, i.e. $h(t,\mathrm{P}) = \int K(t,\mathrm{P};\,0,\mathrm{Q})\;\; h(0,\mathrm{Q})\; d\,\Sigma_0$, written earlier in Eq. (74) as a formal Dyson operator exponential, is in fact identical to $\sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P})$. An alternative way of proving this result is to combine the "conservation" equation (69), which expresses the invariance of the gravitational energy of the monopole in an external potential $U\,(\mathbf{x})$,

$$\int_{\mathrm{P}\,\in\,\Sigma_t} d\,\Sigma_t\;\; h\,(t,\mathrm{P})\;\; U(\mathrm{P}) \;\;=\;\; \int_{\mathrm{Q}\,\in\,\Sigma_0} d\,\Sigma_0\;\; h\,(0,\mathrm{Q})\;\; U(\mathrm{Q}) \;, \tag{78}$$

with the property (21) of the harmonic measure $\sigma^{\mathrm{Q}}_{\Sigma}(\mathrm{P})$ of determining the unique solution of the Dirichlet problem – i.e. the harmonic extension of data given on a surface Σ. More precisely, Eq. (21), written by changing $\mathrm{O} \to \mathrm{Q}$, allows us to express

$$U\,(\mathrm{Q}) \;=\; \int_{\mathrm{P}\,\in\,\Sigma_t} d\,\Sigma_t\;\; \sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P})\;\; U\,(\mathrm{P}) \;, \tag{79}$$

if Q is in the interior of Σ_t and U is harmonic within Σ_t. Combining these two equations, and using the arbitrariness of the boundary data $U(\mathrm{P})$ on Σ_t, precisely yields the propagation equation (76) of $h(t)$. This argument based on the duality (78) between the perturbation h and the external potential U makes it clear why the kernel solving the (outward) propagation of $h(t)$ (Eq. 76) is the *adjoint* of the kernel solving the (inward) harmonic extension problem (Eq. 79).

If the unperturbed family of surfaces Σ_t is analytic, the Green function $G_{\Sigma_t}(\mathrm{P},\mathrm{Q})$ (which solves an elliptic problem) and its normal derivative, proportional to $\sigma^{Q}_{\Sigma_t}(\mathrm{P})$, will be analytic in all their arguments. As a consequence, the outward propagated height $h(t,\mathrm{P})$, given by the integral (76) (where the points P and Q stay always separate when $t \neq 0$) will be, when $t > 0$, analytic in its arguments under very weak regularity assumptions for the initial height $h(0,\mathrm{Q})$ [35]. The kernel $\sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P})$ being positive, an initial disturbance which is everywhere positive on Σ_0 is propagated to an everywhere positive $h(t,\mathrm{P})$ on Σ_t. Furthermore, even if the change $h(0,\mathrm{Q})$ has compact support on Σ_0, the propagated change $h(t,\mathrm{P})$ will, in general, be spread all over Σ_t. Finally, the property (19) of the kernel σ ensures that the integral of h over Σ is preserved by the "time-evolution", which expresses that the volume of the monopole is kept constant (and also follows from taking $U =$ const. in Eq. (78)).

The kernel representation (76) of the solution of the outward propagation problem does not help for solving the inward propagation one. By reversing the arguments above, it is clear that the inward propagation of a perturbation $h(0)$ makes sense only if this perturbation is analytic in the spatial variables. In cases where, given an analytic $h(0)$, the integral (76) can be explicitly performed to give some well-defined analytic function $h(t,\boldsymbol{\xi})$ for $t > 0$, its analytic continuation to negative values of t would, when it is possible, define the (unique) inward-propagated perturbation.

Most of the results of this section hold, *mutatis mutandis*, when considering the perturbation of a body which is gravitationally-equivalent to some *fixed* distribution $\rho_1(\mathrm{Q})$. The perturbation may be caused, for example, by the variation of some inner boundary, and compensated by the appropriate variation of the outermost boundary. There is no difficulty in adapting the above reasoning to this case, as well as to the perturbation of the more general (e.g. "Swiss-cheese-like") monopoles constructed at the end of subsection

[35]Note however that if $h(0)$ has sharp variations in a given region of Σ_0 (e.g. if it is angulous or even discontinuous) $h(t)$ is expected to vary rapidly in the regions of Σ_t which are close to this region of Σ_0. Furthermore, $h(t)$ may even present irregularities, reflecting those of $h(0)$, at points $\boldsymbol{\xi}$ which are common to Σ_0 and Σ_t. This situation is expected to occur, as discussed later in Section 6, whenever the surface Σ_0 presents sharp angles ($\beta < \pi/2$) protruding towards its exterior. These points are expected to remain, at least for some time, fixed points of the successive surfaces Σ_t built from Σ_0, both in the cases of outward and inward growths.

3.3.

5.5 The perturbation of a spherical monopole.

An illustrative example of the propagation of perturbation is the case of a spherically-symmetric monopole Ω_t, i.e. the volume contained between two concentric spheres of radii $R(0) \equiv R_0$ and $R(t)$. The center of gravity O must be the center of spherical symmetry. The parameter t is related to the radii by $t = \frac{4\pi}{3}(R^3(t) - R_0^3)$. If of the point P of Σ_t at which we evaluate $h(t)$, and $\mathbf{Q}$ those of the point Q at which the initial perturbation of Σ_0 is located, the kernel solving the outward propagation problem ($t > 0$) is the Poisson kernel (33) with pole at Q, i.e.

$$\sigma^{\mathrm{Q}}_{\Sigma_t}(\mathrm{P}) \;=\; \frac{R^2(t) - R_0^2}{4\pi\, R(t)}\;\frac{1}{|\mathbf{P} - \mathbf{Q}|^3}\quad , \tag{80}$$

which tends asymptotically, for large t's, towards a uniform distribution $\sigma \approx \frac{1}{4\pi\ R^2(t)}$, smoothing out all initial disturbances of Σ_0.

An instructive way to see the effect of the propagation (especially in the inward direction for which it would be incorrect to use the kernel (80)) is to solve directly the "Schrödinger" equation (73). For the case at hand, the harmonic measure is $\sigma_{\Sigma_t} = (4\pi\, R^2)^{-1}$, the curvature $k = 2/R$ and the Neumann operator $\mathcal{N} = \hat{\ell}/R$ where $\hat{\ell}$ is the operator acting on functions on the sphere as does ℓ on the spherical harmonics $Y_{\ell m}(\theta, \varphi)$: $\hat{\ell}\, Y_{\ell m} = \ell\, Y_{\ell m}$. The explicit form of the evolution equation (73) reads

$$\frac{d}{dR}\, h\,(\,R,\,\theta,\,\varphi) \;=\; -\;\left[\;\frac{\hat{\ell} + 2}{R}\;h\;\right](\,R,\,\theta,\,\varphi)\quad . \tag{81}$$

Its solution, given the "initial" perturbation of Σ_0 described by the orthogonal displacement $h(R_0, \theta, \varphi)$, is as follows. If we decompose h in spherical harmonics, $h(R, \theta, \varphi) = \sum_{\ell,m} h_{\ell m}(R)\ Y_{\ell m}(\theta, \varphi)$, we have

$$h_{\ell m}\,(R) \;=\; \left(\frac{R_0}{R}\right)^{\ell+2} h_{\ell m}\,(R_0)\quad . \tag{82}$$

Equation (82) shows very explicitly the difference between the outward and the inward evolutions. In the outward case ($R > R_0$), the high-harmonic fluctuations of the initial deformation are *exponentially* damped as $\exp[-(\ell+2)\ln(R/R_0)]$. This is a typical analyticizing transformation. In the inward case ($R < R_0$), the high-harmonics are enlarged by an exponential factor $\exp[+(\ell+2)\ln(R_0/R)]$. Therefore the solution will exist (i.e. the spherical harmonics series will converge) only if the initial data were such that $h_{\ell m}(R_0)$ was an exponentially decreasing function of ℓ, which means that the initial shape must be analytic. This illustrates the general properties of the solutions $h(t)$, that we have discussed in subsections 5.3 and 5.4.

Evidently, one does not need the full apparatus of the Neumann operator, etc., to get Equation (81) and its solution (82). Let us briefly indicate how one can directly derive these results (generalized to any number of spatial dimensions N). Consider a system of polar coordinates $(r,\boldsymbol{\theta})$ [where $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_{N-1})$ denote standard angular coordinates on the unit sphere in $\mathbb{R}^N$] centered on the origin O which is to be the center of mass (and center of gravity). A would-be monopole having the topology of the domain contained between two nested spheres can be described by the polar equations of the inner (Σ_0) and outer (Σ_1) boundaries, namely $r=\rho_0(\boldsymbol{\theta})$ and $r=\rho_1(\boldsymbol{\theta})$, respectively.

Let $Y_{\ell\mathbf{m}}(\boldsymbol{\theta})$ denote a basis of spherical harmonics in N dimensions (with $N-2$ "magnetic" quantum numbers $\mathbf{m}$; in $N=2$, $\mathbf{m}$ reduces to a ± 1 discrete index to distinguish $e^{i\ell\theta}$ from $e^{-i\ell\theta}$, or $\cos\ell\theta$ from $\sin\ell\theta$). The multipole moments of the would-be monopole (with unit density) read

$$Q_{\ell\mathbf{m}} \;=\; \int_{\Sigma_0}^{\Sigma_1} d^N\mathbf{x}\;\; r^\ell\;\, Y_{\ell\mathbf{m}}(\boldsymbol{\theta})\;. \tag{83}$$

Replacing $d^N\mathbf{x} = r^{N-1}\,dr\,d\Omega_\theta$ where $d\Omega_\theta$ is the area element on the unit sphere in $\mathbb{R}^N$, and using the polar equations of the boundaries Σ_0 and Σ_1, we get

$$Q_{\ell\mathbf{m}} \;=\; \int d\Omega_\theta\;\, Y_{\ell\mathbf{m}}(\boldsymbol{\theta})\;\, \frac{1}{\ell+N}\;\,[\;\rho_1(\boldsymbol{\theta})^{\ell+N} - \rho_0(\boldsymbol{\theta})^{\ell+N}\;]\;. \tag{84}$$

In the case where Σ_0 and Σ_1 are perturbed spheres of polar equations $\rho_A(\boldsymbol{\theta}) = R_A + \varepsilon\, h(R_A,\boldsymbol{\theta})$ (with $A=0,1$, and $\varepsilon \ll 1$), one finds, to first order in ε, and for $\ell \geq 1$,

$$Q_{\ell\mathbf{m}} \;=\; \varepsilon\;[\;R_1^{\ell+N-1}\; h^*_{\ell\mathbf{m}}(R_1) \;-\; R_0^{\ell+N-1}\; h^*_{\ell\mathbf{m}}(R_0)\;]\;, \tag{85}$$

where $h_{\ell\mathbf{m}}(R_A) = \int d\Omega_\theta \; Y^*_{\ell\mathbf{m}}(\boldsymbol{\theta}) \; h(R_A, \boldsymbol{\theta})$ is the $\ell\mathbf{m}$ spherical harmonics component of $h(R_A, \boldsymbol{\theta})$. Equating $Q_{\ell\mathbf{m}}$ (for $\ell \geq 1$) to zero gives

$$h_{\ell\mathbf{m}}(R_1) \;=\; \left(\frac{R_0}{R_1}\right)^{\ell+N-1} \; h_{\ell\mathbf{m}}(R_0) \;, \tag{86}$$

which is the N-dimensional generalization of (82).

5.6 Towards a rigorous mathematical treatment of the growth of a monopole.

To complete this Section dealing with the mathematical aspects of our construction, let us sketch the rewriting of the evolution equation (23) describing the growth of a monopole, as an evolution system that we anticipate to be amenable to a rigourous treatment along the assumptions of the abstract theorem of Nirenberg and Nishida. The basic idea is, like in many proofs of the Cauchy-Kovaleskaya theorem, to transform (23) (whose right-hand side is nonlinear in the "dangerous" first-order terms, i.e. $\partial_{\boldsymbol{\xi}} \mathbf{X}$ and the pseudodifferential dependence hidden in σ_Σ) into an equivalent "quasilinear" system. Essentially, such a system is obtained by introducing new independent variables, with corresponding evolution equations, until one obtains a closed, "quasilinear" system. We think that the system obtained by adding evolution equations for σ, as well as for the $\boldsymbol{\xi}$-derivatives of the embedding map, $p^i_\alpha = \partial X^i(t, \boldsymbol{\xi}) / \partial \xi^\alpha$, defines a "good quasilinear" system which should provide a basis for proving rigourously the existence of solutions of (23) in suitable functional spaces (of the type mentioned at the beginning of this Section). This system reads (suppressing the $\boldsymbol{\xi}$-dependence):

$$\left\{ \begin{array}{lcl} \dfrac{\partial X^i}{\partial t} & = & \sigma \, n^i(p^j_\alpha) \;, \\[2ex] \dfrac{\partial \sigma}{\partial t} & = & - \left[\mathcal{N}_{\mathbf{X}} + k\left(\mathbf{p}, \dfrac{\partial \mathbf{p}}{\partial \xi^\alpha}\right) \right] (\sigma^2) \;, \\[2ex] \dfrac{\partial p^i_\alpha}{\partial t} & = & \dfrac{\partial \sigma}{\partial \xi^\alpha} \, n^i(\mathbf{p}) + \sigma \, \dfrac{\partial n^i}{\partial p^j_\beta} \, \dfrac{\partial p^j_\beta}{\partial \xi^\alpha} \;. \end{array} \right. \tag{87}$$

The important point is that, here, $\mathbf{n}$ and k no longer denote the functionals of the embedding $(t, \xi^\alpha) \to X^i(t, \xi^\alpha)$ defined above, but some explicit (rational) functions of p^j_β and $\partial p^j_\beta / \partial \xi^\alpha$ obtained by replacing $\partial X^i / \partial \xi^\alpha$ by p^i_α in the original expressions of the normal vector $\mathbf{n}\,[\partial X]$ and of the mean curvature $k\,[\partial X, \partial^2 X]$. Similarly, σ no longer denotes the harmonic measure, solution of an elliptic problem, but only some numerical function of $\boldsymbol{\xi}$. The non-locality (in $\boldsymbol{\xi}$) of the system (87) is all contained in the Neumann operator $\mathcal{N}_{\mathbf{X}}$ associated with the surface $\mathbf{x} = \mathbf{X}(t, \xi^\alpha)$. The information that σ, p and $\partial p / \partial \xi$ are to be related to $\sigma^{\mathrm{O}}_\Sigma$, $\partial X / \partial \xi$ and $\partial^2 X / \partial \xi^2$ is fed in as suitable initial conditions taken at $t = 0$. The evolution system (87) will then preserve this property.

The reasons why we are confident that one could establish the quasi-linearity of the system (87), i.e. the validity of inequalities of the type (43), are the following. First, the facts that the normal $\mathbf{n}\,(\mathbf{p})$ does not depend on the derivatives of $\mathbf{p}$ with respect to $\boldsymbol{\xi}$, and that the mean curvature $k\,(\mathbf{p}, \partial\,\mathbf{p}/\partial \xi^\alpha)$ is *linear* in the derivatives $\partial\,\mathbf{p}/\partial \xi^\alpha$, ensure that all the algebraic terms on the right-hand sides of the system (87) are quasi-linear. The only difficulty resides in the Neumann operator term. Here, the work of Verchota [24] should be sufficient to prove the quasilinearity of the term involving $\mathcal{N}_{\mathbf{X}}$. Indeed, if (using the notation of Ref. [24]) K denotes the in-surface double-layer potential operator and K^* its adjoint, Verchota has shown that one could control the operators $\frac{1}{2} + K$ and $\frac{1}{2} + K^*$ even when the domain Ω_t bounded by Σ_t is Lipshitzian. Knowing that the inequalities he derived depend only of the Lipshitzian constant of Ω_t, and that $\mathcal{N}$ can be expressed in terms of $\left(\frac{1}{2} + K\right)^{-1}$ or $\left(\frac{1}{2} + K^*\right)^{-1}$ and some *domain-independent* pseudo-differential operators, it seems clear to us that, with some work, one can establish the quasi-linearity of the term involving $\mathcal{N}$.

The system (87) can also be applied to the evolution (29) of gravitationally-equivalent bodies. The information that σ is to be related with the weighted harmonic measure (28) is to be fed in as an initial condition taken at $t = 0$.

We wish to mention that the simple representation (76) of the linearized perturbation problem suggests an alternative route towards giving a rigourous proof of the existence of exact monopoles. If one rewrites the exact problem as the linearized problem (around the exact solution) plus nonlinear terms, it may be possible to use the form (76) to set up a convergent iteration scheme using at each step the exact "propagator" of the previous iteration.

Finally, let us mention that Elliott and Janovský have proven, for the case of two dimensions, the existence of a unique solution to an analogous problem – the injection of fluid in a Hele-Shaw cell – with, however, a different boundary condition near the origin [25]. It would be interesting to see whether their method, based on an elliptic variational inequality reformulation of the moving boundary problem, can be adapted to our situation.

5.7 The two-dimensional case.

In the two-dimensional case one can prove the existence of particular classes of exact solutions of our nonlinear evolution problem (23) by means of conformal mapping techniques. In $N=2$ space dimensions, our problem is equivalent to the zero-surface-tension limit of the Saffman-Taylor problem [26] (see also the existence theorem of Ref. [25]). As shown, e.g. in Refs. [27, 26], some multi-parameter classes of exact solutions of the latter problem can be constructed by considering suitable t-dependent conformal mappings of the plane. Let us briefly discuss the application of this idea to our problem (which has different boundary conditions than the usual Saffman-Taylor problem).

One introduces the complex variables $z = x + iy$ (where (x, y) are coordinates in the two-dimensional space) and $\zeta = e^{-(G+iH)}$, where $G(x, y)$ is the Green function, with pole at O, of the simply connected domain Ω interior to some contour Σ, and where $H(x, y)$ is its harmonic conjugate $(\partial G/\partial x = \partial H/\partial y,\ \partial G/\partial y = -\partial H/\partial x)$. ζ is a holomorphic function of z. The (Riemann) mapping $z \to \zeta$ maps in a one-to-one conformal manner the domain Ω onto the unit disc in the ζ plane, the pole O being mapped on the center $\zeta = 0$ of the disc (the logarithmic singularity of the Green function $G \simeq -\ln|z - z_O|$ in 2 dimensions ensures that $\zeta \sim z - z_O$ near the pole O). Actually, it is more convenient to consider the inverse mapping, $z = f(\zeta)$, from the unit disc onto the considered domain Ω. The t-evolution of a family of contours $\Sigma_t = \partial\,\Omega_t$ can be described as the "dynamics" of t-dependent mappings $z = f_t(\zeta)$, Ω_t denoting the image of the unit disc under f_t.

The equation (23) governing the evolution of the boundary Σ_t is equivalent to enforcing the constraint

$$\mathrm{Re}\,[\,\zeta\;\partial_\zeta\, f_t\;\;\partial_t\, f_t^*\,] \;=\; \mathrm{Im}\,[\,\partial_\theta\, f_t\;\;\partial_t\, f_t^*\,] \;=\; \frac{1}{2\,\pi} \tag{88}$$

(independently of θ and t) on the boundary of the unit disc, $\zeta = e^{\,i\theta}$. The three ingredients needed to establish this constraint are:

i) along the unit circle $\zeta = e^{\,i\theta}$, H is identical to $-\,\theta$;

ii) the harmonic measure $\sigma^{\rm O}_{\Sigma_t}$ is $-\frac{1}{2\,\pi}$ times the normal derivative of G, i.e. the tangential derivative of H; therefore the differential form $\frac{d\theta}{2\,\pi}$ is mapped by f_t into the harmonic measure $\sigma^{\rm O}_{\Sigma_t}\, dl$, where dl is the line element along Σ_t ;

iii) the elementary area element swept dy dl when t is increased by dt is $d\,\mathcal{A} = dl\;dh = dl\;\sigma^{\rm O}_{\Sigma_t}\;dt = dt\;\frac{d\theta}{2\,\pi}$ – also equal to $\frac{i}{2}\,df_t \wedge df_t^*$. This leads to Eq. (88).

Remarkably, one can find multi-parameter classes of mappings $f(\zeta; p_1, p_2, \ldots, p_n)$ such that $f_t(\zeta) \equiv f(\zeta; p_1(t), \ldots, p_n(t))$ satisfies Eq. (88) when the $p_i(t)$ satisfy some (complex) ordinary differential equations $dp_i/dt = F_i(p_1, \ldots, p_n)$ [27, 26]. The p_i's parametrize singularities (zeroes and poles) of the derivative $\partial f_t/\partial\zeta$. In other words, one can reduce the dynamics of the contours Σ_t to a particular dynamics of the critical points p_i's.

Let us illustrate this method by a very simple example. We consider a mapping of the form $z = f_t(\zeta) = a(t)\,\zeta + \frac{1}{2}\,b(t)\,\zeta^2$. The derivative mapping $\partial f_t/\partial\zeta = a(t) + b(t)\,\zeta$ has a zero (and therefore a singularity) on the unit disc when $|b(t)| = |a(t)|$. Let us consider the simple case where $a(t)$ and $b(t)$ are real. The constraint (88) is equivalent to the two differential equations

$$\left\{ \begin{array}{rcl} a\,\dot a \,+\, \dfrac{1}{2}\,b\,\dot b & = & \dfrac{1}{2\pi}\ , \\[2ex] b\,\dot a \,+\, \dfrac{1}{2}\,a\,\dot b & = & 0\ . \end{array} \right. \tag{89}$$

The second equation shows that $a^2\,b \equiv C$ is a constant. The solution of the t-evolution is then implicitly given by

$$\frac{1}{2}\,a^2 \,+\, \frac{1}{4}\,b^2 \;=\; \frac{1}{2}\,a^2 \,+\, \frac{1}{4}\,\frac{C^2}{a^4} \;=\; \frac{t}{2\pi} \,+\, \mathrm{const}\ . \tag{90}$$

The outward evolution of some initial "epicycloid" $z = a_0\, e^{i\theta} + \frac{1}{2}\, b_0\, e^{2i\theta}$ (with $|b_0| < |a_0|$) exhibits a growth of the "radius" $a(t)$ and a fast decrease of the "ellipticity" b/a. The inward evolution develops a finger which becomes a cusp when $|a| = |b|$, i.e. in a finite time. This cusp is *located away from the origin* O, and locally described by $y \sim (x - x_0)^{3/2}$. We leave to future work a more thorough discussion of the usefulness of conformal mapping techniques in our context.

6 Physical considerations.

6.1 On the growth of aspherical monopoles.

The examples of Section 4 and the results of Section 5 give a clear picture of the growth of a monopole[36] obtained by thickening, towards the *outside*, some initial surface Σ. It does not matter whether this one has a somewhat irregular shape (say a continuous shape with edges, corners, conical points, etc.). Generally speaking, the thickening of Σ will, at first, be most important in the regions of Σ located closest to the chosen origin O. The evolution stabilizes itself in that regions of fast growth recede fast away from O and thereby slow down their evolution. The shape of Σ becomes smoother and smoother and, in the long term, rounder and rounder. Note, however, that in cases where the initial surface Σ_0 contains narrow gaps between sub-parts of Σ_0 (e.g. when Σ_0 is obtained by bending a "sausage-like" surface until the extremities get close to each other) – or if such a situation develops during the evolution – the subsequent outward evolution can develop a singularity ("collision" of surfaces) in a finite time.

The fate of edges is interesting to consider. Consider on Σ an edge with angle β (opening towards the inside domain Ω which includes O). The harmonic measure σ is, initially, locally proportional to $\rho^{\frac{\pi}{\beta}-1}$ [18], where ρ denotes the distance (on Σ) away from the edge. This suggests that very acute edges, with $\beta < \frac{\pi}{2}$, remain edgelike (with the same value of β) for a while, while the surface around curves out until, eventually, the edge disappears. Less acute edges, with $\beta > \frac{\pi}{2}$ (and even $\beta = \frac{\pi}{2}$), are expected to disappear instantly (i.e. for any $t > 0$) because of the overlap (when

[36] Most of the following considerations also apply, *mutatis mutandis*, to the generalized case of the growth (29) of a "gravitationally-equivalent" body.

$\frac{\pi}{2} < \beta < \pi$) of the growing layers or (when $\beta > \pi$) because of their formally infinite growth rate.

If we consider an infinite cylinder as initial Σ, with O on the axis, we expect from Section 4 the development of an axially-symmetric bulge on the outside, with an exponentially-decreasing tail for large $|z|$, proportionally to $e^{-\frac{j_1|z|}{R}}$ (the exact exponent j_1 in (41) being preserved during the evolution). During the thickening process the bulge should become rounder and rounder (and therefore less and less well approximated by the zeroth-order result of Section 4, represented earlier in Fig.1), tending asymptotically toward the shape of a sphere.

On the other hand the growth of a monopole obtained by thickening, towards the *inside*, some initial surface Σ corresponds to a completely different picture. Consider first the case where Σ is analytic (e.g. an ellipsoid or an infinite cylinder). Then the solution of Equation (23) should exist and stay analytic for some "time" t. However, there will be a tendency for bumps, and then "fingers", to grow towards the origin O. These fingers will become sharper and sharper. The process probably ends (assuming a "sufficiently analytic" initial shape Σ) by the development of an infinitely sharp spine at some distance of O, or, at the latest, when the fastest growing finger reaches the origin. In any case, there will be a finite "time" (i.e. volume) $|t|$ before the regular growth process breaks down, i.e. a maximum filling factor for generating a monopole with a given external surface. Even in simple cases (such as a sphere with O not being at the center, or an ellipsoid or a cylinder) it seems very difficult to estimate analytically this maximum filling factor.

If the initial shape Σ is non-analytic (and, say, contains corners, edges, conical points, ...) it is not yet clear whether the inward growth process admits any rigourous solution. The formula given above for corners ($\sigma \propto \rho^{\frac{\pi}{\beta}-1}$) suggests that very acute corners ($\beta \leq \frac{\pi}{2}$) will not fill, but will become more and more deeply incrusted within the solid body. On the other hand, a less acute corner ($\beta > \frac{\pi}{2}$) might admit no uniquely defined inward evolution (when $\frac{\pi}{2} < \beta < \pi$ one has the collision of two growing layers, while when $\beta > \pi$ the tip is formally expected to grow at an infinite rate).

In the case of small deviations from a spherical shape (or more generally from a surface Σ), the results of subsections 5.4 and 5.5 show that the problem of inward propagation is closely akin to the problem of defining the *outward* analytic extension of the inward analytic extension of a function given on the sphere (or on the surface Σ). As explained in subsection 5.4, the two problems are actually *dual* to each other. This duality leads to a simple explicit relation between the two problems in the case of deviations from a sphere. Indeed, if one starts from a function on the unit sphere, given, say, by its spherical harmonic expansion $f(\boldsymbol{\theta}) = \sum_\ell f_\ell(\boldsymbol{\theta})$, it admits a unique inward harmonic extension within the sphere, $\tilde{f}(r, \boldsymbol{\theta}) = \sum_\ell r^\ell f_\ell(\boldsymbol{\theta})$ with $r < 1$. This well-defined inward harmonic extension is closely related[37] to the result (82) (with $R_0/R < 1$) giving the well-defined outward growth of a nearly spherical initial Σ. The problem of continuing the extension of $\tilde{f}(r, \boldsymbol{\theta})$ for $r > 1$ is thereby related to that of inwardly propagating a nearly spherical Σ (Equation (82) with $R_0/R > 1$).

Because of this analogy we might expect the inward growth problem to be mathematically ill-defined when the initial Σ is non-analytic. Physically, the inward growth problem can make sense if one introduces new (regularizing) physical phenomena taking place at small distances. In fact, there is a large literature on such inward "Laplacian growth" problems (with cut off), and they lead to a complicated zoology of fractallike structure (for an entry into the literature see Ref. [28]). It does not seem that the resulting objects could be of practical interest for defining useful aspherical monopoles (or test masses).

Let us finally remark that some interesting phenomena can take place in presence of symmetries (as for a parallepipedic or cubic box with the origin O at the center of symmetry). In such a case two or more bumps or fingers may grow at the same rate. In the case of inward growth, the symmetry between the fingers is unstable to small perturbations, which should lead to interesting phenomena of spontaneous symmetry breaking.

[37] More precisely the two problems can be identified if we consider the function $\tilde{f}(R, \theta, \varphi) = (R/R_0)^2 \; h(R, \theta, \varphi) = \Sigma \; (R_0/R)^\ell \; h_{\ell m}(R_0) \; Y_{lm}(\theta, \varphi)$, and replace R_0/R by r.

6.2 On the practical realization of aspherical monopoles.

The construction we have presented could be used for actually fabricating aspherical (homogeneous) monopoles (or gravitationally similar test masses), in several different ways.

A first type of approach[38] consists in numerically computing beforehand an ideal shape, then machining the object according to this calculated shape, using some computer-driven machine. There are several practical methods one can use for computing an ideal shape. One is to follow strictly the differential construction given in Section 3, for instance by iterating an electrostatic code (solving the concerned Laplacian problem) in small successive "time" steps. Another method (a "spectral" method) is to parametrize the looked for final shape Σ_t by a finite but large number of free parameters[39], and to determine them by solving a correspondingly large number of constraints, say the vanishing of as many multipole moments as possible. For instance, if we define the boundaries of the monopole by polar equations (as in subsection 5.5) the multipole moments are given by the simple formula (84) (or more complicated ones if it is not enough to introduce two polar contours $\rho_0(\theta, \varphi)$ and $\rho_1(\theta, \varphi)$). If, for instance $\rho_0(\theta, \varphi)$ is given, we can approximately solve for $\rho_1(\theta, \varphi)$ by expanding it as a linear combination of K basis functions (which may but do not need to be spherical harmonics) and numerically solve the first $K - 1$ equations $Q_{\ell m} = 0$ ($\ell \geq 1$). The optimal choice of basis functions depends on the geometry of the problem and the expected qualitative shape of the solution. For instance the work of Section 4 on the cylinder suggests that one should use basis functions that capture the basic features of the lowest-order solution, namely a height function above (or below) the surface $\rho = R$ written as a function of $\zeta = |z|/R$ which exhibits a smooth hump for $|\zeta| \lesssim 1$ followed by an exponential decay for $|\zeta| \gtrsim 1$.

[38]This approach applies as well to the construction of gravitationally-equivalent bodies. By contrast the electrodeposition and brownian motion methods described below require more work to be applied to the case of Eq. (29).

[39]The "belted cylinder" approach introduced by the European STEP team [10, 6, 11] is a simple version of this approach. The better understanding we now have for the theoretically-optimal shapes can help to improve upon this approach (if nothing else, by considering suitable multi-belted cylinders with adequate tapering!).

A second type of approach for realizing a monopole may consist of using electrodeposition. Under certain conditions this may be a very direct way of realizing the inward thickening of an initial surface Σ. Let us consider the space Ω inside some given surface Σ to be (uniformly) filled with an electrolyte solution (of conductivity η) and let us apply a difference of potential between the cathode Σ and an almost pointlike anode located at the origin O. The electric field $\mathbf{E}$ induces, in the quasi-stationary regime, a current density $\mathbf{j} = \eta \mathbf{E}$, satisfying $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ except at the origin O. This current density obeys the same differential equation as $\mathbf{E}$. Provided that we can consider the surface of the growing cathode as an equipotential, the magnitude of $\mathbf{j}$ on Σ is proportional to the harmonic measure σ. If we take for electrolyte a metallic solution, the thickness of the metal sheet deposited on the cathode Σ per unit time, fixed by $|\mathbf{j}|$, is proportional to σ. Assuming the surface of the growing cathode to remain an equipotential under conditions allowing for the stable[40] growth of a dense pattern, this method of electrodeposition can provide us with a practical mean to thicken Σ precisely according to the law (23).

An alternative way of getting this growth law is to have a stationary Brownian motion with a source of particles at O and to kill it when the particles first hit Σ. The statistics of the hittings on Σ is precisely given by the harmonic measure σ (see e.g. Ref. [19]), so that, if we conceive the particles as sticking on Σ when they hit it, we have a Monte Carlo version of electrodeposition. This could be useful either as a computational method, or, possibly (if one can think of a suitable implementation), as a practical realization method.

7 CONCLUSIONS.

We have shown how to construct aspherical homogeneous bodies behaving exactly as point masses, both from the point of view of the gravitational field

[40]Many physical instabilities, in addition to the mathematical unstabilities mentioned above if the initial shape Σ is not smooth enough, might alas show up. In particular, in order to avoid such instabilities we may have to ensure that the resistivity of the electrolyte is small enough, compared to that of the growing aggregate [29], a requirement which tends to be in conflict with the other requirement that the surface of the growing aggregate remains an equipotential.

they generate, and from the point of view of their interaction with external gravitational fields. All their multipole moments (of order $\ell \geq 1$) vanish exactly. We have proven that these gravitational monopoles cannot occupy a "solid" region of space (in the sense of a solid sphere) but must possess (at least) one internal cavity. There is a very large flexibility in the construction of such objects (in any number of space dimensions). The free parameters entering this construction of monopoles in N spatial dimensions (besides the value of the uniform density) are (i) an arbitrary "initial" hypersurface Σ bounding a connected open subset Ω of $\mathbb{R}^N$; (ii) the arbitrary choice of the final center of mass (and "center of gravity") O within Ω; and (within some limits) the total volume of the constructed body.

An extension of this method allows one to construct homogeneous bodies which are gravitationally equivalent (in the sense of having exactly the same infinite sequence of multipole moments) or gravitationally similar (with proportional sequences of multipoles) to any given body, or collection of bodies. There is a very large flexibility in the construction of these objects, given in particular the arbitrariness in the choice of the initial surface upon which the construction process is based. By combining and superposing the two constructions, we can obtain swiss-cheese-like monopoles having an arbitrary number of cavities, with arbitrarily chosen shapes.

The basic construction consists of thickening the arbitrary initial hypersurface Σ by depositing homogeneous layers with infinitesimal successive heights proportional to the harmonic measure, with respect to the fixed pole[41] O, of the moving surface Σ_t. The natural evolution parameter for this growing process is the (N-dimensional) volume contained between Σ and Σ_t. The topology of Σ is not restricted to be that of a sphere in $\mathbb{R}^N$; e.g. in 3 dimensions Σ can be a multi-handled "torus". The growth process has very different properties when performed outwardly, or inwardly[42], with respect to Σ. Our study of the linearized evolution equation (a kind of imaginary-time Schrödinger equation, i.e. similar to the heat equation), and of its solutions, shows that the outward growth process is stable and

[41]The construction of gravitationally-equivalent bodies is similar, provided one replaces the harmonic measure $\sigma^{\mathrm{O}}_{\Sigma_t}$ by the weighted measure $\sigma^{\rho_1}_{\Sigma_t}$ of Eq. (28).

[42]Though we phrased our construction in the text as being either outward or inward, it is evident that one may want, in some practical applications, to thicken some "initial" Σ on both sides at once.

smoothing. It is probably well defined even if the initial hypersurface has a very limited regularity. By contrast, the inward growth process is, probably, mathematically well defined only for analytic initial data. We presented the program of a rigourous mathematical proof of the existence of aspherical monopoles without, however, completing such a proof. We also exhibited explicitly a simple example of a two-dimensional aspherical monopole.

We indicated several practical ways of realizing gravitational monopoles (including computer-aided machining and electrodeposition). In particular, once one knows, from the general construction method, that such objects exist with many possible shapes, one can try to find numerical approximations to some of them (as Barrett [13] did some years ago) without necessarily following into the footsteps of the harmonic measure construction (see Section 6.2). Though our method generally assumes that the domain Ω is bounded, it can also generate unbounded monopoles, with cylindriclike internal cavities, and exponentially-decreasing thicknesses at infinity.

Apart from its conceptual interest, our construction might be of practical interest for optimizing the shape of test masses in high-precision Equivalence Principle experiments, such as STEP. By suppressing the coupling to gravity gradients, it allows one to design, with great flexibility in the choice of shapes, differential accelerometers, made of nested bodies, which are insensitive to external gravitational disturbances. Even very close disturbing masses would have no effect[43] on the differential accelerometer, which is then *"gravitationally screened"* from all external gravity gradients. Truncated versions of the cylindriclike monopoles with exponentially falling off thicknesses may define particularly useful shapes for the STEP experiment. If their length L is large enough compared to their radius R such cylinders would be exponentially insensitive to disturbing masses[44], even if these are very close.

Using the extended construction method described in subsection 3.3 we can choose any (non-monopole) shape for some inner body, e.g a straight cylinder, and construct an enclosing (or cylinderlike) outer body which has exactly (or almost exactly) the same set of reduced multipole moments

[43] When the monopoles are concentric.

[44] Our lowest-order solution – and its successive iterations – indicate that the quality factor of the decoupling goes like $e^{2.405\,L/2R}$, increasing by about an order of magnitude each time the total length L is increased by $2R$.

Q_ℓ/M as the inner body. This is enough to ensure that the differential accelerometer they define is insensitive to all external gravity gradients.

Finally our techniques may also allow one to devise *multi-nested* differential accelerometers, made of "Russian doll" type masses, (almost) insensitive to external gravitational disturbances. This could be important for conceiving and implementing more compact (and cheaper) versions of the STEP concept. Altogether, the optimization of the test masses used in differential accelerometers should allow for a significant increase of their performances, as required for very precise tests of the Equivalence Principle.

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Figure caption.

Fig. 1: Surface density for a thin cylindrical gravitational monopole of radius R, as a function of the longitudinal coordinate z. The function $F\,(z/R)$, defined by Eqs. (37) or (40), determines the mass density $\sigma\,(z) = \frac{(M)}{2\,\pi\,R^2}\;F\,(z/R)$. All $(l \geq 1)$ multipole moments vanish identically. 86 % of the mass correspond to $|\,z\,| < R$, 98.7 % to $|\,z\,| < 2\,R$.