L^2-HOMOLOGY FOR VON NEUMANN ALGEBRAS

Alain CONNES [1] and Dimitri SHLYAKHTENKO [2]

1. Introduction.

The aim of this paper is to introduce a notion of L^2-homology in the context of von Neumann algebras. Finding a suitable (co)homology theory for von Neumann algebras has been a dream for several generations (see [KR71a, KR71b, JKR72, SS95] and references therein). One's hope is to have a powerful invariant to distinguish von Neumann algebras. Unfortunately, little positive is known about the Kadison-Ringrose cohomology $H_b^*(M,M)$, except that it vanishes in many cases. Furthermore, there does not seem to be a good connection between the bounded cohomology theory of a group and of the bounded cohomology of its von Neumann algebra.

Our interest in developing an L^2-cohomology theory was revived by the introduction of L^2-cohomology invariants in the field of ergodic equivalence relations in the paper of Gaboriau [Gab02]. His results in particular imply that L^2-Betti numbers $\beta_i^{(2)}(\Gamma)$ of a discrete group are the same for measure-equivalent groups (i.e., for groups that can generate isomorphic ergodic measure-preserving equivalence relations). Parallels between the "worlds" of von Neumann algebras and measurable equivalence relations have been noted for a long time (starting with the parallel between the work of Murray and von Neumann [MvN] and that of H. Dye [Dy]). Thus there is hope that an invariant of a group that "survives" measure equivalence will survive also "von Neumann algebra equivalence", i.e., will be an invariant of the von Neumann algebra of the group.

The original motivation for our construction comes from the well understood analogy between the theory of II_1-factors and that of discrete groups, based on the theory of correspondences [Con, Con94]. This analogy has been remarkably efficient to transpose analytic properties such as "amenability" or "property T" from the group context to the factor context [Con80] [CJ] and more recently in the breakthrough work of Popa [Popa] [Con03]. We use the theory of correspondences together with the algebraic description of L^2-Betti numbers given by Luck. His definition involves the computation of the algebraic group homology with coefficients in the group von Neumann algebra. Following the guiding idea that the category of bimodules over a von Neumann algebra is the analogue of the category of modules over a group, we are led to the following algebraic definition of L^2-homology of a von Neumann algebra M:

$$H_k^{(2)}(M) = H_k(M; M\bar{\otimes} M^o).$$

[1] Collège de France, 3 rue d'Ulm, 75 005 Paris,
I.H.E.S. and Vanderbilt University, connes@ihes.fr
[2] Department of Mathematics University of California, Los Angeles, CA 90095, USA
shlyakht@math.ucla.edu. Reserach supported by the NSF and the Sloan Foundation

Here H_k stands for the algebraic Hochschild homology of M. One is thus led to consider the L^2-Betti numbers,

$$\beta_k^{(2)}(M) = \dim_{M\bar{\otimes}M^o} H_k^{(2)}(M)$$

(see Section 2.2 for more motivation behind this definition).
The L^2 Betti numbers that we associate to a II_1 factor M enjoy the following scaling property. If $p \in M$ is a projection of trace λ, then

$$\beta_k^{(2)}(pMp) = \frac{1}{\lambda^2}\beta_k^{(2)}(M).$$

This behavior is a consequence of considering the "double" $M \otimes M^o$ of M. It is quite different from the behavior of L^2-Betti numbers associated to equivalence relations [Gab02], where the factor λ^2 is replaced by λ. Note that the behavior of our invariants is consistent with the formula for the "number of generators" of compressions of free group factors, where the same factor λ^2 appears. Indeed, for $p \in L(\mathbb{F}_t) = L(\mathbb{F}_{(t-1)+1})$ a projection of trace λ, one has by the results of Voiculescu, Dykema and Radulescu (see e.g. [VDN92]) that

$$pL(\mathbb{F}_t)p \cong L(\mathbb{F}_{((t-1)\lambda^{-2})+1}.$$

In this respect, our theory seems to be quite disjoint from a kind of relative theory of L^2-invariants for factors with HT-Cartan subalgebras discovered by Popa [Popa]. Indeed, the two theories have different scaling properties (λ^{-2} versus λ^{-1}) under compression.

We have completely unexpectedly found a connection between our definition, involving algebraic homology, and the theory of free entropy dimension developed by Voiculescu and his followers ([Voi93, Voi94, Voi96, Voi98], see also [Voi02] and references therein). Such a connection between L^2-Betti numbers and free entropy dimension parallels nicely the connections between free entropy dimension and cost for equivalence relations [Gab00, Shl01, Shl03a].
Our definition seems to be robust with respect to modifications; for example, the connections with free probability theory do not seem to be accessible for certain variations of our definition.
This unexpected connection allows us to rely on several deep results in free probability theory, including the inequality between the microstates and microstates-free free entropy proved by recent fundamental work of Biane, Capitaine and Guionnet [BCG03]. Drawing on these results, we obtain strong evidence that the first L^2-Betti number associated to the von Neumann algebra of a free group $\mathbb{F}_n$ on n generators is bounded below by $n-1$ ($= \beta_1^{(2)}(\mathbb{F}_n)$). Moreover, our inability to prove a similar inequality between free entropy dimension and L^2-Betti numbers obtained by varying our definition can be taken as evidence that the present definition is the correct one. Even more encouraging are the facts that the L^2-Betti numbers can be controlled from above at least in some cases (such as abelian von Neumann algebras, or von Neumann algebras with a diffuse center).
Our results also have several implications for the questions involving computation of free entropy dimension. In particular, we show that the modified free entropy dimension $\delta_0(\Gamma)$ of any discrete group Γ is bounded from above by $\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$; a similar estimate holds for the various versions of the non-microstates free entropy dimension. If Γ has property T and if the von Neumann algebra $L(\Gamma)$ is diffuse, then $\delta_0(\Gamma) \leq 1$ (and $= 1$ if $L(\Gamma)$ can be

embedded into the ultrapower of the free group factor, as is the case when Γ is residually finite). This generalizes a result of Voiculescu [Voi99b] for the case that $\Gamma = SL(n,\mathbb{Z})$, $n \geq 3$.

1.1. **Notation.** Whenever possible, we shall use the following notation: the letter A will stand for a tracial $*$-algebra with a fixed positive faithful trace τ. The letter M will stand for the von Neumann algebra generated by A in the GNS representation associated to τ. We write $L^2(M)$ for the associated representation space.
For a group Γ, we denote by $\mathbb{C}\Gamma$ its (algebraic) group algebra, and by $L(\Gamma)$ its group von Neumann algebra.
The tensor sign $\otimes$ will always refer to the algebraic tensor product. Tensor products that involve completions (such as the von Neumann algebra tensor product or the Hilbert space tensor product) will always be denoted by $\bar{\otimes}$.
We denote by $HS(L^2(M))$ the ideal of Hilbert-Schmidt operators on $L^2(M)$, and by $FR(L^2(M))$ the ideal of finite-rank operators on $L^2(M)$.
It will be useful to identify $HS(L^2(M))$ with $L^2(M)\bar{\otimes}L^2(M^o)$ in the following way. Let $J : L^2(M) \to L^2(M)$ be the Tomita conjugation. Let $P_1 \in L^2(M)$ be the projection onto the cyclic vector. Then consider the map $\Psi : M \otimes M^o \to FR(L^2(M))$ given by

$$\Psi(x\otimes y^o)(\xi) = x\ \tau(y\,\xi) \qquad x,y\in M,\ \xi \in L^2(M). \tag{1.1}$$

The map Ψ extends to an isometric isomorphism of $L^2(M\bar{\otimes}M^o)$ with the Hilbert space $HS(L^2(M))$ of Hilbert-Schmidt operators in $L^2(M)$.
Note that $B(L^2(M))$ admits four commuting actions of M:

$$\begin{aligned} T &\mapsto aT \\ T &\mapsto Ta \\ T &\mapsto Ja^*JT \\ T &\mapsto TJa^*J. \end{aligned}$$

Here $a \in M$ and $T \in B(L^2(M))$. These four actions are intertwined by the map Ψ with the four natural actions of M on $L^2(M\bar{\otimes}M^o)$, listed in the corresponding order:

$$\begin{aligned} x\otimes y^o &\mapsto a\,x\otimes y^o \\ x\otimes y^o &\mapsto x\otimes (y\,a)^o \\ x\otimes y^o &\mapsto x\,a\otimes y^o \\ x\otimes y^o &\mapsto x\otimes (a\,y)^o, \end{aligned} \tag{1.2}$$

where $a\in M$ and $x\otimes y^o \in M\otimes M^o$.

2. Definition of L^2-Betti numbers and L^2-homology.

2.1. Review of L^2-homology for groups.

2.1.1. *Co-compact actions.* The study of L^2-cohomology has been initiated by Atiyah in [Ati76], who considered L^2 deRham cohomology of a connected manifold X endowed with a cocompact proper free action of a discrete group Γ. The k-th cohomology group is defined as the quotient of the space of closed L^2-integrable k-forms by the closure of the space of exact L^2 k-forms. In other words, one considers *reduced* L^2-cohomology (reduced means that one takes a quotient by the closure of the image of the boundary operator).
The cohomology groups in question can also be identified with the Hilbert spaces of L^2 harmonic forms, and are in a natural way modules over the group Γ. Furthermore, because of the properness and cocompactness assumptions, these spaces are actually modules over the group von Neumann algebra $L(\Gamma)$. Atiyah considered the Murray-von Neumann dimensions of these cohomology groups, and called them L^2-Betti numbers of the action. Remarkably, it turns out that these numbers are homotopy invariants; in particular, if the manifold X is contractible, then the numbers depend only on the group Γ, and are called the L^2-Betti numbers of the group, $\beta_i^{(2)}(\Gamma)$. This definition extends to cover cocompact proper free actions of Γ on a contractible simplicial complex, and gives an equivalent definition of the Betti numbers using L^2 singular homology. For example, if $\Gamma = \mathbb{F}_n$, the free group on n generators, then $\beta_i^{(2)}(\Gamma) = 0$ for $i \neq 1$, and $\beta_1^{(2)}(\Gamma) = n - 1$.

2.1.2. *Cheeger-Gromov's approach to non-cocompact actions.* Not every group can act on a contractible space in a proper free and co-compact way. For this reason, if one wants to obtain an L^2-homology theory for groups, one is forced to consider non-cocompact actions. This leads to difficulties of a technical nature. The problem is that because the spaces of L^2-(co)chains are now infinitely-dimensional modules over $L(\Gamma)$, one needs to be much more careful in taking the closure of the image of the boundary operator, and one may end up with having to consider dimensions of actions of $L(\Gamma)$ on quotients of Hilbert spaces by not necessarily closed subspaces.
This can be overcome in one of two ways. The first, which is the original approach of Cheeger and Gromov [CG86], is to "approximate" the L^2-homology of a non-cocompact action on a contractible manifold X by realizing the simplicial complex (Δ_*, ∂_*) underlying X as an inductive limit of sub-complexes $(\Delta_*^{(k)}, \partial_*^{(k)})$, which correspond to co-compact (free) actions. Let $C_*^{(k)}(X)$ the the space of L^2-chains for $\Delta_*^{(k)}$ (i.e., the Hilbert space with orthonormal basis given by simplices in $\Delta_*^{(k)}$). Denote by $i_{k,l}$ the map from $C_*^{(k)}(X)$ to $C_*^{(l)}(X)$. Then the n-th L^2-Betti number is defined as

$$\beta_n^{(2)} = \sup_{k\to\infty} \inf_{l>k} \dim_{L(\Gamma)} \frac{i_{k,l}(\ker \partial_n^{(k)})}{i_{k,l}(\ker \partial_n^{(k)}) \cap \operatorname{im} \partial_{n+1}^{(l)}}. \tag{2.1}$$

2.1.3. *Luck's approach to non-cocompact actions.* The second way, developed by Luck [Lüc98], is to extend the notion of Murray-von Neumann dimension to algebraic modules over type II_1 von Neumann algebras. This way one can, for example, assign a dimension to the quotient of an $L(\Gamma)$ module by a non-closed submodule. Luck shows that such an extension indeed exists. In the case that V is a finitely-generated module over $L(\Gamma)$, its dimension is just the supremum of the dimensions of *normal* $L(\Gamma)$ modules (i.e. finite projective modules) that can be embedded into V.

Returning briefly to the co-compact case, one can now consider *non-reduced* simplicial L^2 homology of a space X, defined as the quotient of the kernel of the boundary operator ∂_n by the (non-closed) image of ∂_{n+1}. This results in $L(\Gamma)$ modules, which are not normal. However, due to the behavior of Luck's extension of the Murray-von Neumann dimension, it turns out that the dimensions of these modules are the same as the Murray-von Neumann dimensions of the reduced homology groups (i.e., the L^2-Betti numbers).

In the case that X is connected and contractible, its ordinary homology vanishes. This means that if we denote by $C_k^{(f)}$ the vector space with basis given by the k-chains on X, then the sequence $(C_k^{(f)}, \partial_k)$ is exact. Each $C_k^{(f)}$ is a flat module over Γ, and $C_0^{(f)} \cong \mathbb{C}$, since we assume that X is connected. Thus $(C_k^{(f)}, \partial_k)$ forms a resolution of the trivial Γ-module $C_0^{(f)}$. Furthermore, the space of L^2 chains on X is very roughly $L(\Gamma) \otimes_\Gamma C_k^{(f)}$.[3] Thus the non-reduced L^2-homology of X is nothing but the (algebraic) homology group

$$H_*(\Gamma; L(\Gamma))$$

of the group Γ with coefficients in $L(\Gamma)$ viewed as a right Γ-module. Since $L(\Gamma)$ is both a right Γ-module, and a left $L(\Gamma)$-module, the $H_*(\Gamma; L(\Gamma))$ are left $L(\Gamma)$-modules. Thus one can consider $\dim_{L(\Gamma)} H_*(\Gamma; L(\Gamma))$. However, as we pointed out, the dimensions of these non-reduced homology groups are precisely the L^2-Betti numbers of the group.

Thus even in the case that Γ admits no proper free co-compact action on a contractible space, one can consider the numbers

$$\beta_*^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_*(\Gamma; L(\Gamma)), \tag{2.2}$$

where H_* stands for the algebraic group homology. Luck shows that a certain continuity property of his dimension leads to the formula

$$\dim_{L(\Gamma)} H_*(\Gamma; L(\Gamma)) = \sup_k \inf_{l>k} \dim_{L(\Gamma)} i_{k,l}(H_*(C_*^{(k)}))$$

in the notation of the previous section and of equation (2.1). Combined with (2.1), this shows that this definition produces the same numbers as the one in equation (2.1).

2.2. **From groups to von Neumann algebras.**

2.2.1. *Correspondences.* Let us recall the basis of the analogy between discrete groups and II_1-factors. A correspondence from II_1-factors is simply a Hilbert space endowed with a structure of (normal) bimodule. Correspondences on M (i.e. M-M Hilbert bimodules) play

[3] More precisely, the space of L^2-chains is $\ell^2(\Gamma) \otimes_\Gamma C_k^{(f)} = \ell^2(\Gamma) \otimes_{L(\Gamma)} L(\Gamma) \otimes_\Gamma C_k^{(f)}$. However, since by [Lüc98], the functor $\ell^2 \otimes_{L(\Gamma)} \cdot$ in the category of finitely-generated $L(\Gamma)$-modules is flat, the nuance between $\ell^2(\Gamma) \otimes_\Gamma C_k^{(f)}$ and $L(\Gamma) \otimes_\Gamma C_k^{(f)}$ is irrelevant in the foregoing.

the role of unitary representations, and the dictionary begins as follows,

Discrete Group Γ	II_1-Factor M
Unitary Representation	M-M Hilbert bimodule
Trivial Representation	$L^2(M)$
Regular Representation	Coarse Correspondence
Amenability	$L^2(M) \subset_{\text{weakly}} L^2(M)\bar{\otimes}L^2(M^o)$
Property T	$L^2(M)$ isolated

where the "coarse correspondence" is given by the bimodule $L^2(M)\bar{\otimes}L^2(M^o)$ with bimodule structure given by the first two lines of (1.2).

2.2.2. *Some homological algebra.* In order to find a suitable notion of L^2 Betti numbers of a tracial algebra (M, τ), we chose as our point of departure equation (2.2). It is perhaps best to rewrite the definition of group homology in the language of homological algebra see e.g. [CE56]:

$$H_*(\Gamma; L(\Gamma)) = \mathrm{Tor}_*^{Mod(\Gamma)}(\star; L(\Gamma)).$$

This equation involves three objects: $Mod(\Gamma)$, $\star$ and $L(\Gamma)$. The role of $Mod(\Gamma)$ is to prescribe a suitable category; in this case, it is the category of left modules over Γ. The module $\star$ stands for the trivial left module. To compute the Tor functor, one must first choose a resolution of $\star$ by flat modules

$$\star \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots . \tag{2.3}$$

This means that each C_i is flat over Γ (the definition of a flat module is not really needed here; free modules and, more generally, projective modules, are flat), and the sequence (2.3) is exact.

Now $L(\Gamma)$ plays two roles. Its first role is that of a right module over Γ. To finish the computation of Tor, one applies the functor $L(\Gamma)\otimes_\Gamma$ to the exact sequence (2.3):

$$L(\Gamma) \otimes_\Gamma \star \leftarrow L(\Gamma) \otimes_\Gamma C_0 \leftarrow L(\Gamma) \otimes_\Gamma C_1 \leftarrow \cdots . \tag{2.4}$$

This new sequence (2.4) may well fail to be exact. Note, however, that it is a differential complex: the composition of any two consecutive arrows is zero (since the original sequences (2.3) had this property). The value of Tor is precisely the homology of (2.4), i.e., Tor_k is the kernel of the k-th map, divided by the image of the $k+1$-st. It is crucial that the value of the Tor functor is well-defined and is independent of the choice of the resolution (2.3).

Finally, $L(\Gamma)$ plays its final role, which is that of not just a right Γ-module, but also of a left $L(\Gamma)$-module. It is crucial that the left action of $L(\Gamma)$ commutes with the right action of Γ. Because of this, (2.4) is is a differential complex of left $L(\Gamma)$-modules. This makes it possible to take the dimension

$$\beta_*^{(2)}(\Gamma) = \dim_{L(\Gamma)} \mathrm{Tor}_*^{Mod(\Gamma)}(\star; L(\Gamma)), \tag{2.5}$$

as the definition of the Betti numbers (2.2).

2.2.3. *Analogs for tracial $*$-algebras.* Let now (A,τ) be a tracial $*$-algebra. We would like to make sense of

$$\beta_*^{(2)}(A,\tau) = \dim_X \mathrm{Tor}_*^Y(Z,W),$$

where: X is the analog of the group von Neumann algebra $L(\Gamma)$; Y is the analog of the category of right Γ-modules; Z is the analog of the trivial module; and W is the analog of the $\Gamma, L(\Gamma)$-bimodule $L(\Gamma)$.
Let $M = W^*(A)$ in the GNS representation associated to τ.
It is fortunate that the theory of correspondences furnishes perfect analogs for all of these objects. The analog of Y, i.e., of the category of left Γ-modules, is the category Y of A,A-bimodules, or, better, of left modules over the algebraic tensor product $A\otimes A^o$ of A by its opposite algebra A^o.
In particular the trivial Γ-module Z is A viewed as a bimodule over A and since we want to view it as a *left* module over $A\otimes A^o$ we use

$$(m\otimes n^o)\cdot a := m\,a\,n\,, \quad \forall m\,,n\,,a\in A \tag{2.6}$$

We next look for X, the analog of the group von Neumann algebra $L(\Gamma)$. Note that $L(\Gamma)$ is the von Neumann algebra generated by Γ in the left regular representation. The analog of the left regular representation of Γ on $\ell^2(\Gamma)$ is the coarse correspondence, i.e. the representation of $A\otimes A^o$ on $L^2(M,\tau)\bar{\otimes}L^2(M^o,\tau)$ given by

$$(m\otimes n^o)\cdot(\sum a_i\otimes b_i^o) = \sum m\,a_i\,\otimes\,(b_i\,n)^o, \qquad m\,,n\in A\,.$$

i.e. the first two lines of (1.2). Hence $X = W^*(M\otimes M^o)$ in this representation, i.e., $X = M\bar{\otimes}M^o$ (von Neumann tensor product). Finally, $W = M\bar{\otimes}M^o$, but having two structures. One is that of a right $A\otimes A^o$-module, with the action

$$(\sum a_i\otimes b_i^o)\cdot(m\otimes n^o) = \sum a_i\,m\otimes(n\,b_i)^o, \qquad m\,,n\in A\,. \tag{2.7}$$

The other is that of an $M\bar{\otimes}M^o$-left module, given by left multiplication in $M\bar{\otimes}M^o$, i.e.

$$(m\otimes n^o)\cdot(\sum a_i\otimes b_i^o) = \sum m\,a_i\otimes(b_i\,n)^o\,, \qquad m,n\in M.$$

Note that the actions of $M\bar{\otimes}M^o$ and $A\otimes A^o$ on $W = M\bar{\otimes}M^o$ commute.
Thus we are led by our analogy to consider

$$\begin{aligned} H_k^{(2)}(A,\tau) &= \mathrm{Tor}_k^{A\otimes A^o}(A; M\bar{\otimes}M^o), \\ \beta_k^{(2)}(A,\tau) &= \dim_{M\bar{\otimes}M^o} H_k^{(2)}(A,\tau). \end{aligned}$$

It turns out that in the category of bimodules over an algebra, $\mathrm{Tor}_k^{A\otimes A^o}(A; M\bar{\otimes}M^o)$ is exactly the k-th Hochschild homology of A with coefficients in the bimodule $M\bar{\otimes}M^o$ of (2.7).

Definition 2.1. Let (A,τ) be a tracial $*$-algebra. Let $M = W^*(A)$ in the GNS representation associated to τ. Then define the k-th L^2-homology group of A to be the Hochschild homology group

$$H_k^{(2)}(A,\tau) = H_k(A; M\bar{\otimes}M^o).$$

Also define the k-th L^2-Betti number of A to be its extended Murray-von Neumann dimension (in the sense of Luck),

$$\beta_k^{(2)}(A,\tau) = \dim_{M\bar{\otimes}M^o} H_k(A; M\bar{\otimes}M^o).$$

This definition of course depends on the trace that we choose on A.

2.3. **The bar resolution.** We will now give an "explicit" description of the L^2-homology group of A, which is at the same time the only description we have at present. The situation is somewhat analogous to the case of dealing with a general group Γ, for which one does not know whether or not one can choose a "nice" topological space on which Γ can act properly and freely (which would give us a "nice" resolution with which to compute the homology). Thus one must instead resort to using the universal cover $E\Gamma$ of the universal classifying space $B\Gamma$ of Γ.

Let $C_k(A) = (A\otimes A^o)\otimes A^{\otimes k}$, $k = 0, 1, \ldots$, viewed as a left $A\otimes A^o$-module via

$$(m\otimes n^o)\cdot((a\otimes b^o)\otimes a_1\otimes\cdots\otimes a_k) = (m\,a\,\otimes(b\,n)^o)\otimes a_1\otimes\cdots\otimes a_k,$$

where $m, n, a, b, a_1, \ldots, a_k \in A$. Note that $C_k(A)$ is a free $A\otimes A^o$-module (typically on an infinite number of generators if $k > 0$). Define

$$\partial_k : C_k(A) \to C_{k-1}(A)$$

by the formula

$$\begin{aligned}(2.8)\qquad \partial_k(T\otimes a_1\otimes\cdots\otimes a_k) &= T\,(a_1\otimes 1)\,\otimes\cdots\otimes a_k + \\ (-1)^k T\,(1\otimes a_k^o)\,\otimes a_1\otimes\cdots\otimes a_{k-1} &+ \sum_{j=1}^{k-1}(-1)^j T\otimes a_1\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_k;\end{aligned}$$

where $T \in A\otimes A^o$, $a_1, \ldots, a_k \in A$ and we use the algebra structure of $A\otimes A^o$ to define $T\,(a_1\otimes 1)$ and $T\,(1\otimes a_k^o)$. Then $(C_*(A), \partial_*)$ is exact [CE56] and forms a resolution of the $A\otimes A^o$-left module A of (2.6) by $A\otimes A^o$-left modules, with the last map $C_0(A) = A\otimes A^o \mapsto A$ given by the multiplication m,

$$\sum a_i\otimes b_i^o \to \sum a_i\, b_i$$

Let $C_k^{(2)}(A) = M\bar{\otimes}M^o \otimes_{A\otimes A^o} C_k = (M\bar{\otimes}M^o)\otimes A^{\otimes k}$. Let $\partial_k^{(2)}$ be given by the same formula as in (2.8), except that we now allow $T \in M\bar{\otimes}M^o$. Then

$$(2.9)\qquad H_k^{(2)}(A,\tau) = \frac{\ker\partial_k^{(2)}}{\operatorname{im}\partial_{k+1}^{(2)}}.$$

is by construction a left $M\bar{\otimes}M^o$-module.

It is in general not clear how to compute these homology groups (or their dimensions over $M\bar{\otimes}M^o$). However, one can give a description of the L^2-Betti numbers of A as a limit of finite numbers.

Lemma 2.2. *The bar resolution of A can be written as an inductive limit of its sub-complexes $C_*^{(\ell)}$, $\ell \in I$, so that each $C_*^{(\ell)}$ is a finitely generated left module over $A \otimes A^o$. Moreover, if we denote by $i_{\ell,k} : C_*^{(\ell)} \to C_*^{(k)}$ the inclusion map, and by $H_*(\ell)$ the homology of the complex $(M\bar{\otimes}M) \otimes_{A\otimes A^o} C_*^{(\ell)}$, then we have*

$$H_n^{(2)}(A,\tau) = \varinjlim (H_n(\ell),(i_{\ell,k})_*), \tag{2.10}$$

$$\beta_n^{(2)}(A,\tau) = \sup_\ell \inf_{k>\ell} \dim_{M\bar{\otimes}M^o} (i_{\ell,k})_* H_n(\ell). \tag{2.11}$$

Proof. For each integer n, and finite subset $F \subset A$, let $V = \operatorname{span} F$, and let $V_0 = V^{\otimes n}$. Denote by m the multiplication map $m : A \otimes A \to A$, and let

$$\begin{aligned} V_1 &= \operatorname{span}((m \otimes 1^{\otimes n-2})V_0, (1 \otimes m \otimes 1^{\otimes n-3})V_0, \ldots, (1^{\otimes n-2} \otimes m)V_0) \\ V_2 &= \operatorname{span}((m \otimes 1^{\otimes n-3})V_1, (1 \otimes m \otimes 1^{\otimes n-4}V_1, \ldots, (1^{\otimes n-3} \otimes m)V_1) \end{aligned}$$

and so on, where 1 stands for the identity map $A \mapsto A$. Let

$$C_k^{(F,n)} = \begin{cases} (A \otimes A^o) \otimes V_{n-k}, & k \le n \\ 0, & k > n. \end{cases}$$

Note that $C_k^{(F,n)}$ forms a sub-complex of the bar resolution of A.
Order the pairs (F,n) by saying that $(F,n) < (F',n')$ if $F \subset F'$ and $n \le n'$. If $(F,n) < (F',n')$, let

$$i_{(F,n)} : C_k^{(F,n)} \to C_k^{(F',n')}$$

be the inclusion map. Then each $C_k^{(F,n)}$ is finitely-generated over $A \otimes A^o$ (it has at most $\dim V_k$ generators), and the bar resolution of A is the inductive limit of the sub-complexes $C_k^{(F,n)}$.
It follows that the homology group $H_n(A; M\bar{\otimes}M^o)$ is itself the inductive limit of the directed system $\{H_n(\ell), i_{\ell,k}\}$, so that (2.10) holds.
Because of the finite generation assumptions, we have that

$$\dim_{M\bar{\otimes}M^o} (i_{\ell,k})_* H_n(\ell) < \infty, \qquad \forall k > \ell.$$

Thus [Lüc98, Theorem 2.9(2)] implies (2.11). □

A reader interested in an even more explicit description, valid for the first Betti number, is urged to look ahead to section 3.1.

2.4. **Group algebras.** We have defined L^2-Betti numbers for any tracial $*$-algebra (A,τ). While the case of interest is when A is a von Neumann algebra (so that $M = A$), we want to point out that the definition works well even in the purely algebraic setting.

Proposition 2.3. *Let Γ be a discrete group, and denote by τ the von Neumann trace on the group algebra $\mathbb{C}\Gamma$. Let $\Gamma^{(2)} = \Gamma \times \Gamma^o$, with Γ^o the opposite group, and view Γ as a subgroup of $\Gamma^{(2)}$ via the diagonal inclusion map $\gamma \mapsto \Delta(\gamma) = (\gamma, (\gamma^{-1})^o)$. Then*

$$H_k^{(2)}(\mathbb{C}\Gamma,\tau) = L(\Gamma^{(2)}) \otimes_{L(\Gamma)} H_k^{(2)}(\Gamma), \tag{2.12}$$

$$\beta_k^{(2)}(\mathbb{C}\Gamma,\tau) = \beta_k^{(2)}(\Gamma). \tag{2.13}$$

Proof. Note that $L(\Gamma)\bar{\otimes}L(\Gamma^o) = L(\Gamma \times \Gamma^o) = L(\Gamma^{(2)})$. By [CE56], we have that

$$H_k(\mathbb{C}\Gamma; L(\Gamma)\bar{\otimes}L(\Gamma^o)) = H_k(\Gamma, L(\Gamma^{(2)})).$$

Since $L(\Gamma^{(2)}) = L(\Gamma^{(2)}) \otimes_{L(\Gamma)} L(\Gamma)$ as Γ-modules, and the functor $L(\Gamma^{(2)})\otimes_{L(\Gamma)}$ is flat [Lüc98, Theorem 3.3(1)], it follows that

$$H_k(\mathbb{C}\Gamma; L(\Gamma)\bar{\otimes}L(\Gamma^o)) = L(\Gamma^{(2)}) \otimes_{L(\Gamma)} H_k(\Gamma; L(\Gamma)) = L(\Gamma^{(2)}) \otimes_{L(\Gamma)} H_k^{(2)}(\Gamma).$$

Equation (2.13) now follows from [Lüc98, Theorem 3.3(2)]. □

2.5. **Compressions of von Neumann algebras.** As was shown by Gaboriau in the context of measurable measure-preserving equivalence relations [Gab02], L^2-Betti numbers behave well under restrictions of equivalence relations. More precisely, if an ergodic equivalence relation R is restricted to subset X of measure λ, then one has

$$\beta_n^{(2)}(R_\lambda) = \frac{1}{\lambda}\beta_n^{(2)}(R).$$

The analogue of this fact is given by the following theorem. It should be noted that the factor $1/\lambda$ in Gaboriau's result is replaced in our context by the factor of $1/\lambda^2$. This is explained by the fact that in constructing our L^2-homology, we have passed to the category of bimodules, so the natural object that we are working with is $M \otimes M^o$ (and not M). Compressing M to a projection of trace λ amounts to compressing $M \otimes M^o$ by a projection of trace λ^2.

Theorem 2.4. *Let M be a factor and let $p \in M$ be a projection of trace λ. Then*

$$\beta_n^{(2)}(pMp, \frac{1}{\tau(p)}\tau(p\cdot p)) = \frac{1}{\lambda^2}\beta_n^{(2)}(M, \tau). \tag{2.14}$$

Proof. Let $(C_*(M), \partial_*)$ be the bar resolution of M,

$$C_k(M) \;=\; (M \otimes M^o) \otimes M^{\otimes k}, \qquad k \geq 0$$

with ∂_* as in 2.8. Let $N = M \otimes M^o, q = p \otimes p^o$ which is an idempotent in N. The reduced algebra $N_q = q\,N\,q$ is $pMp \otimes (pMp)^o$.

Let F be the functor $V \mapsto F(V) = q\,V$ from the category of left N-modules to that of left N_q-modules. It is an exact functor since for $T : V \mapsto W$, $x \in \operatorname{im} T \cap q\,W$ one has

$$x = Ty = qTy = Tqy \in \operatorname{im}(T/qV)\,.$$

Note that in our case qN is a projective left module over N_q since pM is a projective left module over pMp and similarly for the opposite algebras. Thus F maps projective modules to projective modules. Moreover when we apply F to the "trivial" N-module M of 2.6 we get $F(M) = pMp$, the "trivial" $pMp \otimes (pMp)^o$-module.

This shows that $F(C_*(M), \partial_*)$ is a projective resolution of pMp and hence that,

$$H_*^{(2)}(pMp, \frac{1}{\tau(p)}\tau(p\cdot p)) \;=\; (p \otimes p^o)H_*^{(2)}(M, \tau).$$

Equation (2.14) now follows from the fact that for an $M\bar{\otimes}M^o$ left module V and the projection $q = p \otimes p^o \in M\bar{\otimes}M^o$, we have

$$\dim_{pMp\bar{\otimes}(pMp)^o} qV = \dim_{q(M\bar{\otimes}M^o)q} qV = \frac{1}{(\tau\otimes\tau)(q)} \dim_{M\bar{\otimes}M^o} V.$$

□

2.6. **Direct sums.** L^2-homology behaves well with respect to direct sums:

Proposition 2.5. *Let* $(A,\tau) = \bigoplus_i (A_i,\tau_i)$ *(finite direct sum), so that the trace on* A *decomposes as* $\bigoplus_i \alpha_i \tau_i$ *in terms of normalized traces* τ_i *on* A_i.

Then

$$H_k^{(2)}(A,\tau) \cong \bigoplus_i H_k^{(2)}(A_i,\tau_i)$$

and

$$\beta_k^{(2)}(A,\tau) = \sum_i \alpha_i^2 \beta_k^{(2)}(A_i,\tau_i).$$

Proof. Let $C_*^{(i)}$ be the bar resolution of A_i with its differential $d_*^{(i)}$, and let

$$C_k = \bigoplus_i C_k^{(i)}, \qquad d_k = \bigoplus_i d_k^{(i)}.$$

Then each C_k, $k \geq 1$ is a projective module over $A \otimes A^o$. This is because $\bigoplus_i A_i \otimes A_i^o$ is a direct summand of $A \otimes A^o = \bigoplus_{i,j} A_i \otimes A_j^o$. Thus (C_k, d_k) is a projective resolution of A. Using this resolution to compute the L^2-homology of A we obtain

$$H_k^{(2)}(A,\tau) = \bigoplus_i H_k^{(2)}(A_i,\tau_i).$$

Let $M_i = W^*(A_i)$, $M = W^*(A)$ (each time in the GNS representation associated to τ). The formula for Betti numbers is now immediate, once we remark that if V_i is a module over $M_i \bar{\otimes} M_i^o$, then

$$\dim_{M\bar{\otimes}M^o}(\bigoplus_i V_i) = \sum \alpha_i^2 \dim_{M_i\bar{\otimes}M_i^o} V_i,$$

the factor α_i^2 coming from the fact that

$$M\bar{\otimes}M^o = \bigoplus_{i,j} M_i \otimes M_j^o, \qquad \tau\otimes\tau = \bigoplus_{i,j} \alpha_i\alpha_j \tau_i \otimes \tau_j.$$

□

2.7. **Zeroth Betti number and zeroth L^2-homology for von Neumann algebras.** Let M be a von Neumann algebra. By definition

$$H_0^{(2)}(M,\tau) = \frac{\ker \partial_0^{(2)}}{\operatorname{im} \partial_1^{(2)}}$$

is the quotient of $M\bar{\otimes}M^o$ by the left ideal L generated by

$$V = \{1 \otimes n^o - n \otimes 1 : n \in M\}.$$

or in other words

$$H_0^{(2)}(M,\tau) = (M\bar{\otimes}M^o) \otimes_{M\otimes M^o} M.$$

Proposition 2.6. *Let* M *be a* II_1*-factor. Then* $H_0^{(2)}(M,\tau) \neq 0$ *if and only if* M *is hyperfinite.*

Proof. Since as a left $M\bar{\otimes}M^o$-module, $H_0^{(2)}(M)$ is clearly generated by the class in the quotient of the element $1\otimes 1$, $H_0^{(2)}(M)=0$ if and only if $[1\otimes 1]=0$.
By ([Con76]) M is hyperfinite if and only if the trivial correspondence is weakly contained in the coarse correspondence. That is to say, there exists a nonzero non-normal state $\theta : M\bar{\otimes}M^o\to\mathbb{C}$ with the property that

$$\theta(m\otimes n^o)=\tau(m\,n),\qquad \forall m,n\in M. \tag{2.15}$$

Assume first that M is hyperfinite. Then $\theta(\,x^*x)=0$ for $x\in V$. Thus $\theta|_J=0$, so that θ factors through to a non-zero linear functional on $H_0^{(2)}(M)$ and $H_0^{(2)}(M)\neq 0$.
Conversely, assume that $H_0^{(2)}(M)\neq 0$. Then for any n unitaries $u_i\in M$, the operator

$$T=\frac{1}{n}\sum u_i\otimes u_i^{*o}-1,$$

belongs to the left ideal L so that T and hence T^*T are not invertible, for any u_i. Thus one can find a non-normal state ϕ on $M\bar{\otimes}M^o$, for which $\phi(T^*T)=0$ for any such T. Denoting by ξ_ϕ the associated cyclic vector, we get that $T\xi_\phi=0$ and so

$$(u_i\otimes u_i^{*o})\,\xi_\phi=\xi_\phi.$$

But then we have

$$(u_i\otimes 1)\,\xi_\phi=(1\otimes u_i^o)\,\xi_\phi$$

and hence

$$(n\otimes 1)\,\xi_\phi=(1\otimes n^o)\,\xi_\phi$$

for all $n\in M$. Hence

$$(m\,n\otimes 1)\,\xi_\phi=(m\otimes 1)(n\otimes 1)\,\xi_\phi=(m\otimes n^o)\,\xi_\phi,$$

so that $\phi(m\otimes n^o)=\phi(m\,n\otimes 1)$. Lastly,

$$\begin{aligned}\phi(n\,m\otimes 1)&=\langle (n\otimes m^o)\,\xi_\phi,\xi_\phi\rangle\\ &=\langle (1\otimes m^o)\cdot(n\otimes 1)\,\xi_\phi,\xi_\phi\rangle\\ &=\langle (n\otimes 1)\,\xi_\phi,(1\otimes m^{*o})\xi_\phi\rangle\\ &=\langle (n\otimes 1)\,\xi_\phi,(m^*\otimes 1)\,\xi_\phi\rangle\\ &=\langle (m\,n\otimes 1)\,\xi_\phi,\xi_\phi\rangle=\phi(m\,n\otimes 1),\end{aligned}$$

so that $\phi|_{M\otimes 1}$ is a trace, and hence the unique trace τ on $M\cong M\otimes 1$, thus ϕ fulfills 2.15. □

We get as consequence of Luck's Theorem 1.8 [Lüc98], his definition of the projective part of a module and of Theorem 0.6 in [Lüc98], and of the fact that $H_0^{(2)}(A,\tau)=M\bar{\otimes}M^o\otimes_{A\otimes A^o}A$, $M=W^*(A)$ is finitely (in fact, singly) generated as an $M\bar{\otimes}M^o$ module, that

$$\beta_0(A,\tau)=\dim_{M\bar{\otimes}M^o}\mathrm{Hom}(M\bar{\otimes}M^o\otimes_{A\otimes A^o}A,M\bar{\otimes}M^o)$$

Proposition 2.7. *If (A,τ) contains an element x, whose distribution with respect to τ is non-atomic, then $\beta_0^{(2)}(A,\tau)=0$.*

Proof. Assume $\beta_0^{(2)}(A,\tau) \neq 0$, let $\phi \neq 0$, $\phi \in \mathrm{Hom}(M\bar{\otimes}M^o \otimes_{A\otimes A^o} A, M\bar{\otimes}M^o)$. Denote by $[1\otimes 1]$ the class of $1 \otimes 1$ in $M\bar{\otimes}M^o \otimes_{A\otimes A^o} A$. Let $\xi = \phi([1\otimes 1]) \in M\bar{\otimes}M^o$. Since $M\bar{\otimes}M^o \otimes_{A\otimes A^o} A$ is generated by $[1\otimes 1]$, $\phi \neq 0$ implies that $\xi \neq 0$.
We thus have a vector $\xi \neq 0$ in $L^2(M\bar{\otimes}M^o)$ with the property that $(m\otimes 1 - 1 \otimes m^o)\,\xi = 0$ for all $m \in A$ and hence for all $m \in A'' = M$. Identify $L^2(M\bar{\otimes}M^o)$ with the space of Hilbert-Schmidt operators on $L^2(M)$, by the map Ψ of (1.2). Then $\Psi(\xi)$ is a non-zero Hilbert-Schmidt operator, commuting with M by (1.2). But this is impossible, since M contains an element with a diffuse spectrum. □

Corollary 2.8. *If M is a type II_1 factor, then $\beta_0^{(2)}(M) = 0$.*

Proposition 2.9. *Let M be a finite-dimensional von Neumann algebra with a faithful trace τ. Decompose $M = \oplus M_i$ into factors with $M_i \cong M_{k_i\times k_i}$(the algebra of $k_i \times k_i$ matrices), and let λ_i be the trace of the minimal central projection in M corresponding to the i-th summand.*
Then

$$\beta_0^{(2)}(M) = \sum_i \frac{\lambda_i^2}{k_i^2},$$
$$\beta_k^{(2)}(M) = 0, \qquad k \geq 1.$$

Proof. Since there is no difference between $M\otimes M$ and $M\bar{\otimes}M$ in the finite-dimensional case, $\beta_k^{(2)}(M) = 0$ if $k > 0$. For $\beta_0^{(2)}$ we find easily that $\beta_0^{(2)}(\mathbb{C}) = 1$; the compression formula then gives $\beta_0^{(2)}(M_{k\times k}) = \frac{1}{k^2}$, and the direct sum formula gives us finally the desired expression. □

2.8. L^2-Betti numbers for bimodule maps.

2.8.1. *Betti numbers for group module maps.* L^2-Betti numbers can be more generally defined for maps between group modules. Let us for definiteness consider a module map f between two free left Γ-modules:

$$f : (\mathbb{C}\Gamma)^n \to (\mathbb{C}\Gamma)^m.$$

Thus f is given by an $n \times m$ matrix of right-multiplication operators in $\mathbb{C}\Gamma$. Consider now

$$f^{(p)} : (\ell^p(\Gamma))^n \to (\ell^p(\Gamma))^m,$$

given by the same matrix. The kernel of $f^{(p)}$ may be larger than the ℓ^p-closure of the kernel of f. To measure the difference, consider for $p \leq 2$

$$\beta^{(2,p)}(f) = \dim_{L(\Gamma)} \frac{\overline{\ker f^{(p)}}^{\ell^2}}{\overline{\ker f}^{\ell^2}}.$$

Set

$$\beta^{(2)}(f) = \beta^{(2,2)}(f)$$

for convenience.
Note that if Γ acts freely and cocompactly on some contractible simplicial complex X, then $\beta_j^{(2)}(\Gamma)$ is then just $\beta^{(2)}(\partial_j)$, where ∂_j is the boundary operator of X. Indeed, contractibility implies that $\ker \partial_k = \mathrm{im}\, \partial_{k+1}$, so that the closures of these two subspaces of $(\ell^2)^N$ are the same.

2.8.2. *Betti numbers for bimodule maps.* Let (A,τ) be a tracial $*$-algebra. Let $F:(A\otimes A^o)^n\to(A\otimes A^o)^m$ be a left $A\otimes A^o$-module map (or, equivalently, an A,A-bimodule map). Then F is given by a matrix

$$F=\begin{pmatrix} F_{11} & \cdots & F_{1n}\\ \vdots & \ddots & \vdots \\ F_{m1} & \cdots & F_{mn}\end{pmatrix},$$

where $F_{ij}\in A\otimes A^o$ and the action is given by right multiplication.
Let $M=W^*(A)$ in the GNS representation associated to τ.
The right multiplication by F_{ij} admits a unique continuous extension to $L^2(M\bar{\otimes}M^o)$. Thus F admits a unique continuous extension to a left $M\bar{\otimes}M^o$-module map from $(L^2(M\bar{\otimes}M^o))^n$ to $(L^2(M\bar{\otimes}M^o))^m$, which we denote by $F^{(2)}$.
We note that $\ker F\subset\overline{\ker F}\subset\ker F^{(2)}$ (where $\bar{\cdot}$ refers to closure in L^2-norm). By analogy with the group case, we make the following definition (compare [Lüc97], Definition 5.1).

Definition 2.10. The L^2 Betti number of F is the Murray-von Neumann dimension

$$\beta^{(2)}(F)=\dim_{M\bar{\otimes}M^o}\frac{\ker F^{(2)}}{\overline{\ker F}}=\dim_{M\bar{\otimes}M^o}\ker F^{(2)}-\dim_{M\bar{\otimes}M^o}\overline{\ker F}.$$

Lemma 2.11. *One has*

$$\dim_{M\bar{\otimes}M^o}\frac{\ker F^{(2)}}{\overline{\ker F}^{L^2}}=\dim_{M\bar{\otimes}M^o}\frac{\ker F^{vN}}{(M\bar{\otimes}M^o)\ker F},$$

where F^{vN} is the restriction of $F^{(2)}$ to $(M\bar{\otimes}M^o)^n\subset(L^2(M\bar{\otimes}M^o))^n$, and $(M\bar{\otimes}M^o)\cdot\ker F$ denotes the saturation of $\ker F$ *under the action of $M\bar{\otimes}M^o$.*

The proof is almost verbatim the argument on the bottom of page 158 and top of page 159 of [Lüc98], see also Theorem 5.4 of [Lüc97].

Remark 2.12. Note that F^{vN} is exactly the map

$$1\otimes F:(M\bar{\otimes}M^o)\otimes_{A\otimes A^o}(A\otimes A^o)^n\to(M\bar{\otimes}M^o)\otimes_{A\otimes A^o}(A\otimes A^o)^m,$$

if we identify $(M\bar{\otimes}M^o)\otimes_{A\otimes A^o}(A\otimes A^o)$ with $M\bar{\otimes}M^o$. Thus in particular,

$$\beta^{(2)}(F)=\dim_{M\bar{\otimes}M^o}\frac{\ker(1\otimes F)}{(M\bar{\otimes}M^o)\cdot\ker F}.$$

2.8.3. *Homological algebra interpretation.* Let $F:(A\otimes A^o)^n\to(A\otimes A^o)^m$ be a bimodule map, as above. Put $V=(A\otimes A^o)^n$, $W=(A\otimes A^o)^m$.
Consider the exact sequence

$$V\xrightarrow{F}W\to\operatorname{im}F\to 0.$$

Since the $A\otimes A^o$- left modules V and W are projective (in fact, free), this sequence is the beginning of a projective resolution of $\operatorname{im}F$. More precisely, one can choose projective modules $V_1,V_2,\dots$ and morphisms $F_1,F_2,\dots$ so that the following sequence is exact:

$$\cdots\to V_2\xrightarrow{F_2}V_1\xrightarrow{F_1}V\xrightarrow{F}W\to\operatorname{im}F\to 0.$$

Note that $\operatorname{im} F_1 = \ker F$. Consider the differential complex

$$\cdots \to (M\bar{\otimes}M^o) \otimes_{A\otimes A^o} V_1 \xrightarrow{1\otimes F_1} (M\bar{\otimes}M^o) \otimes_{A\otimes A^o} V \xrightarrow{1\otimes F} (M\bar{\otimes}M^o) \otimes_{A\otimes A^o} W \to \cdots .$$

By definition,

$$\operatorname{Tor}_1^{A\otimes A^o}(\operatorname{im} F, M\bar{\otimes}M^o) = \frac{\ker(1\otimes F)}{\operatorname{im}(1\otimes F_1)}.$$

Since $\operatorname{im}(1\otimes F_1) = (M\bar{\otimes}M^o)\cdot \operatorname{im} F_1 = (M\bar{\otimes}M^o)\ker F$, we conclude that

$$\beta^{(2)}(F) = \dim_{M\bar{\otimes}M^o}(\operatorname{Tor}_1^{A\otimes A^o}(\operatorname{im} F, M\bar{\otimes}M^o)). \tag{2.16}$$

2.8.4. *Examples of Betti numbers.* The following statement gives many examples of bimodule maps over von Neumann algebras, for which the L^2 Betti numbers are non-zero. For a group module map $f : \mathbb{C}\Gamma^n \to \mathbb{C}\Gamma^m$, denote by $f^{(1)}$ its extension to $\ell^1(\Gamma)^n$. Denote by $\beta^{(2,1)}(f)$ the dimension

$$\beta^{(2,1)}(f) = \dim_{L(\Gamma)}(\ker f^{(2)}) - \dim_{L(\Gamma)} \overline{\ker f^{(1)}}.$$

Theorem 2.13. *Let Γ be a discrete group, $n, m < \infty$ and let $f : \mathbb{C}\Gamma^n \to \mathbb{C}\Gamma^m$ be a Γ-left module map given by a matrix with entries $f_{ij} \in \mathbb{C}\Gamma$. Let $A = M = L(\Gamma)$.*
Let $F_{ij} = \Delta(f_{ij}) \in M\bar{\otimes}M^o$ be the images of f_{ij} under the canonical diagonal inclusion $\Delta(\gamma) = (\gamma, (\gamma^{-1})^o)$ of $\mathbb{C}\Gamma$ into $M \otimes M^o = L(\Gamma) \otimes L(\Gamma^o)$. Let $F : (M \otimes M^o)^n \to (M \otimes M^o)^m$ be given by the matrix whose entries are right multiplications by F_{ij}. Then,

$$\beta^{(2)}(f) \geq \beta^{(2)}(F) \geq \beta^{(2,1)}(f)$$

Note that the statement is not automatic, since we are comparing

$$\operatorname{Tor}_1^{\Gamma}(\operatorname{im} f, L(\Gamma)) \qquad \text{with} \qquad \operatorname{Tor}_1^{L(\Gamma)\otimes L(\Gamma^o)}(\operatorname{im} F, L(\Gamma)\bar{\otimes}L(\Gamma^o))$$

and not

$$\operatorname{Tor}_1^{\Gamma}(\operatorname{im} f, L(\Gamma)) \qquad \text{with} \qquad \operatorname{Tor}_1^{\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o}(\operatorname{im} F, L(\Gamma)\bar{\otimes}L(\Gamma^o))$$

Let Δ_* be the induction functor from left Γ-modules to left $M \otimes M^o$-modules associated to the morphism $\Delta : \mathbb{C}\Gamma \mapsto M \otimes M^o$,

$$\Delta_*(X) := (M \otimes M^o) \otimes_{\mathbb{C}\Gamma} X$$

where $M \otimes M^o$ is viewed as a right $\mathbb{C}\Gamma$-module using Δ, then the $M \otimes M^o$-module $(M \otimes M^o)^n$ is induced from $\mathbb{C}\Gamma^n$ while F is induced from f, or in short $F = 1 \otimes f$.

Proof. It is sufficient to prove that $\dim_{M\bar{\otimes}M^o} \ker F^{(2)} = \dim_{L(\Gamma)} \ker f^{(2)}$, $\dim_{M\bar{\otimes}M^o} \overline{\ker F} \geq \dim_{L(\Gamma)} \overline{\ker f}$ and $\dim_{M\bar{\otimes}M^o} \overline{\ker F} \leq \dim_{L(\Gamma)} \overline{\ker f^{(1)}}$. Here $f^{(1)}$ stands for the extension of f to $\ell^1(\Gamma)$.
The morphism Δ preserves the trace and extends to an inclusion of von Neumann algebras $\Delta : L(\Gamma) \mapsto M\bar{\otimes}M^o$. Denote by F^* the "adjoint" of F (with i,j-th entry F_{ji}^*). Then $\ker F^{(2)} = \ker((F^{(2)})^* F^{(2)})$. Regard $T = (F^{(2)})^* F^{(2)}$ as an element of the algebra of $n \times n$ matrices over $M\bar{\otimes}M^o$. Then the dimension of the kernel of $F^{(2)}$ is precisely the non-normalized

trace, computed in $M_{n\times n}(M\bar{\otimes}M^o)$, of the spectral projection P of T corresponding to the eigenvalue 0. But $F=\Delta(f)$, and hence $P=\Delta(p)$ where p is the spectral projection p corresponding to the eigenvalue 0 of the element $t=(f^{(2)})^*f^{(2)}\in M_{n\times n}(L(\Gamma))$. The trace of p is exactly $\dim_{L(\Gamma)}\ker f^{(2)}$. Thus

$$\dim_{M\bar{\otimes}M^o}\ker F^{(2)}=\dim_{L(\Gamma)}\ker f^{(2)}$$

Consider now the orthogonal projection $E:L^2(M\bar{\otimes}M^o)\to\Delta(L^2(M))$. It defines a conditional expectation $E:M\bar{\otimes}M^o\to M$ where we identify ΔM with M. Note that if $\eta\in M\otimes M^o\subset L^2(M\bar{\otimes}M^o)$, then $E(\eta)\in\ell^1(\Gamma)$. To prove this, note first that it is sufficient to prove that $E(\eta)\in\ell^1(\Gamma)$ whenever η is a simple tensor of the form $\xi\otimes\zeta$, $\xi\in L^2(M)$, $\zeta\in L^2(M^o)$. For $\gamma\in\Gamma$, let u_γ be the corresponding unitary in M. Let $\xi=\sum\alpha_\gamma u_\gamma$ and $\zeta=\sum\beta_\gamma u^o_{\gamma^{-1}}$ (where the sums are in L^2 sense). Then $E(\xi\otimes\zeta)=\sum\alpha_\gamma\,\beta_\gamma\,\Delta(u_\gamma)\sim\sum\alpha_\gamma\,\beta_\gamma\,u_\gamma$. Since the sequences $\{\alpha_\gamma\}$ and $\{\beta_\gamma\}$ are in $\ell^2(\Gamma)$, their product lies in $\ell^1(\Gamma)$, and $E(\eta)\in\ell^1(\Gamma)$.
Denote by E_γ the map $\eta\mapsto E((u_{\gamma^{-1}}\otimes 1)\cdot\eta)$. Then any $\eta\in L^2(M\bar{\otimes}M^o)$ is the L^2-sum $\eta=\sum(u_\gamma\otimes 1)\cdot E_\gamma(\eta)$. We extend E and E_γ (componentwise) to maps of direct sums of $L^2(M\bar{\otimes}M^o)$ and denote them with the same letter. Since E is a conditional expectation and F acts by right multiplication by the $\Delta(f_{ij})$ one has $E\circ F^{(2)}=f^{(2)}\circ E$ and also

$$E_\gamma\circ F^{(2)}=f^{(2)}\circ E_\gamma\quad\forall\gamma\in\Gamma$$

Let now $\eta\in\ker F$. Then $\eta\in(M\otimes M^o)^n$, and $E_\gamma(\eta)\in(\ell^1(\Gamma))^n$, for all γ. Denote by $f^{(1)}$ the restriction of $f^{(2)}$ to $(\ell^1(\Gamma))^n$. Then $f^{(1)}\circ E_\gamma(\eta)=E_\gamma\circ F^{(2)}(\eta)=0$, so that $E_\gamma(\eta)\in\overline{\ker f^{(1)}}$. Denote by q the projection in $M_{n\times n}(M)$ corresponding to the invariant subspace $\overline{\ker f^{(1)}}$. Let $Q=\Delta(q)$ be the image of q under the inclusion map of $M_{n\times n}(M)\subset M_{n\times n}(M\bar{\otimes}M^o)$ induced by Δ. Since $E_\gamma(\eta)\in\ker f^{(1)}$ for all γ, it follows easily that $\eta=\sum(u_\gamma\otimes 1)\cdot E_\gamma(\eta)$ is in the range of Q. Thus we have proved that $\ker F$ is contained in the range of Q, so that $\dim_{M\bar{\otimes}M^o}\overline{\ker F}\le\dim_{M\bar{\otimes}M^o}\operatorname{im}Q=\dim_M\operatorname{im}q=\dim_M\overline{\ker f^{(1)}}$.

Finally, if we are given a finite sequence $t_\gamma\in\ker f$, then $\sum(u_\gamma\otimes 1)\cdot t_\gamma\in\ker F$. Thus the induced module from $\overline{\ker f}$ is contained in $\overline{\ker F}$. This shows that $\dim_M\overline{\ker f}\le\dim_{M\bar{\otimes}M^o}\overline{\ker F}$. Thus we have

$$\dim_M\overline{\ker f}\le\dim_{M\bar{\otimes}M^o}\overline{\ker F}\le\dim_M\overline{\ker f^{(1)}}.$$

□

It would be interesting to know exactly when $\beta^{(2)}(F)=\beta^{(2)}(f)$. Note that by the results of [Min], if Γ is a "combable group" (in particular, a finitely generated hyperbolic group), and $f:C_n(X)\to C_{n-1}(X)$ is the boundary homomorphism of a contractible chain complex with a cocompact free action of Γ, then $\beta^{(2)}(f)=\beta^{(2,1)}(f)$. Indeed, it it proved in [Min] that any element in $\ker f^{(1)}$ can be approximated in ℓ^1 (and hence ℓ^2) by elements from $\ker f$. This implies the following fact:

Theorem 2.14. *Let Γ be a discrete combable group acting freely and co-compactly on a contractible chain complex $C_*(X)$ of a topological space X. Let $D_*=(L(\Gamma)\otimes L(\Gamma^o))\otimes_\Gamma$*

$C_*(X)$, where Γ acts on $L(\Gamma)\otimes L(\Gamma^o)$ by Δ. Then for each k,

$$\begin{aligned}\dim_{L(\Gamma)\bar{\otimes}L(\Gamma^o)} H_k((L(\Gamma)\bar{\otimes}L(\Gamma^o))\otimes_{L(\Gamma)\otimes L(\Gamma^o)} D_*(X)) &= \dim_{L(\Gamma)} H_k(L(\Gamma)\otimes_\Gamma C_*(X)) \\ &= \beta_k^{(2)}(\Gamma).\end{aligned}$$

This theorem is of interest in conjunction with equation (2.11), since $D_*(X)$ (is homotopic to a a sub-complex that) occurs among the approximating sub-complexes of the bar resolution of $L(\Gamma)$.

2.9. **Dual definition of Betti numbers for bimodule maps.** For the remainder of this section, we shall concentrate on bimodule maps over a von Neumann algebra; i.e., we assume that $A = M$ is a von Neumann algebra with a fixed trace τ.
Let $f : (M\otimes M^o)^n \to (M\otimes M^o)^m$ be a left $M\otimes M^o$-module map. We are interested in the size of the kernel $\ker f$ in $(M\otimes M^o)^n$. As before, denote by f^{vN} and $f^{(2)}$, respectively, the extensions of f to $(M\bar{\otimes}M^o)^n$ and $L^2(M\bar{\otimes}M^o)^n$.
Let us consider the algebraic tensor product $FR = L^2(M)\otimes L^2(M^o)$ as a subset of $L^2(M)\bar{\otimes}L^2(M^o) = L^2(M\bar{\otimes}M^o)$ in the natural way. Note that $M\otimes M^o \subset FR$.
Note also that in the identification Ψ of $L^2(M\bar{\otimes}M^o)$ with the space $HS = HS(L^2(M))$ of Hilbert-Schmidt operators on $L^2(M)$ (see (1.1)), the set FR corresponds precisely to the subset of HS consisting of finite-rank operators.
We begin with a lemma, which shows that it does not matter for the purposes of L^2-closure whether we compute the kernel of $f^{(2)}$ in $(M\otimes M^o)^n$ or FR^n.

Lemma 2.15. $\ker f^{(2)}\cap FR^n$ *has the same L^2-closure in $L^2(M\bar{\otimes}M^o)^n$ as* $\ker f = \ker f^{(2)}\cap (M\otimes M^o)^n$. *Thus*

$$\beta^{(2)}(f) = \dim_{M\bar{\otimes}M^o}\ker f^{(2)} - \dim_{M\bar{\otimes}M^o}(\overline{\ker f^{(2)}\cap FR^n}) \tag{2.17}$$

Proof. We view elements of $L^2(M)$ as unbounded operators (of left multiplication) on $L^2(M)$, affiliated with M.
Given any finite subset $K\subset L^2(M)$, and $\epsilon > 0$ there exists a projection $e\in M$ such that

$$e\,\xi\in M\,,\quad \|e\xi-\xi\|<\epsilon\,,\quad \forall\xi\in K$$

where $e\xi\in M$ means that the a priori unbounded operator of left multiplication by $e\xi$ is bounded. This is proved for a single ξ using the polar decomposition $\xi = bu$ and a suitable spectral projection of the unbounded self-adjoint operator b. One controls moreover the trace $\tau(1-e)<\epsilon$. Taking the intersection f of the projections $e_\xi, \xi\in K$ one gets $\tau(1-f)<n\epsilon$, $n = \mathrm{card}(K)$. This gives a sequence of projections such that $f_k\,\xi$ is bounded $\forall\xi\in K$ and $f_k\to 1$ in L^2 and hence strongly, which gives the answer. This shows that for any element $\xi\in FR^n$ there exists a projection $e\in M$ such that $(e\otimes e^o)\,\xi\in(M\otimes M^o)^n$ and $\|(e\otimes e^o)\,\xi-\xi\|<\epsilon$. Since $\ker f^{(2)}\cap FR^n$ is a left module over $M\otimes M^o$ the conclusion of the lemma follows. ☐

As consequence, we have the following description of the dimension of the kernel of a left $M\otimes M^o$-module map $f:(M\otimes M^o)^n\to(M\otimes M^o)^m$. Extend f to a map (still denoted by f) from $FR^n\to FR^m$ as in the previous Lemma. Identify FR with the set of finite-rank operators on $L^2(M)$ using the identification Ψ. Let B be the von Neumann algebra of all bounded operators on $L^2(M)$. Denote by $\langle\cdot,\cdot\rangle$ the canonical pairing between FR^n and B^n given by

$$\langle (T_1,\dots,T_n),(S_1,\dots,S_n)\rangle = \sum_{j=1}^n \mathrm{Tr}(T_jS_j).$$

Denote by $f^{\,t}$ the map

$$f^{\,t}: B^m \to B^n$$

uniquely determined by

$$\langle f^{\,t}(T),S\rangle = \langle T, f(S)\rangle.$$

The map $f^{\,t}$ is well-defined and is in fact given by right multiplication by the matrix $f^{\,t}_{ji}$ where for $h=\sum m_i\otimes n_i^o \in M\otimes M^o$ one lets

$$h^{\,t} := \sum n_i \otimes m_i^o\,.$$

The right multiplication by $a\otimes b^o\in M\otimes M^o$ acts in the obvious way on $(M\otimes M^o)^n$ and becomes, after the identification Ψ, using (1.2)

$$T\cdot(a\otimes b^o) = Ja^*J\,T\,Jb^*J\,,\quad \forall T\in B\,. \tag{2.18}$$

Lemma 2.16. *We have*

$$\dim_{M\bar{\otimes}M^o}((M\bar{\otimes}M^o)\cdot\ker f) = n - \dim_{M\bar{\otimes}M^o}(\overline{f^{\,t}(B^m)}^w \cap HS^n),$$

where the closure is taken with respect to the weak operator topology. In particular,

$$\beta^{(2)}(f) = \dim_{M\bar{\otimes}M^o}\left(\frac{\overline{f^{\,t}(HS^m)}^w\cap HS^n}{\overline{f^{\,t}(HS^m)}^{HS}}\right).$$

Proof. We note that FR is the dual of $B(H)$ with respect to the weak topology. By duality, $\overline{f^{\,t}(B^m)}^w$ is the annihilator of $\ker f\subset FR^n$. Let us show that $\overline{f^{\,t}(B^m)}^w\cap HS$ is the annihilator of $\overline{\ker f}^{HS}$ in HS^n. The answer then follows by duality in HS^n whose dimension over $M\bar{\otimes}M^o$ is equal to n.
Note that the two pairings are compatible. By continuity of the pairing in HS^n, $\overline{f^{\,t}(B^m)}^w\cap HS^n$ is perpendicular to $\overline{\ker f}^{HS}$.
Since the Hilbert-Schmidt topology is stronger than the weak topology on HS, the subspace $\overline{f^{\,t}(B^m)}^w\cap HS^n$ is already closed in the Hilbert-Schmidt topology.
Assume that $\xi\in HS^n$ belongs to $(\overline{\ker f}^{HS})^\perp$. Then $\xi\perp\ker f$ and viewing $\xi\in HS^n\subset B^n$, as an element of B^n we find using the compatibility of the pairings that ξ is in the co-kernel of f, so that $\xi\in\overline{f^{\,t}(B^m)}^w$ and $\xi\in\overline{f^{\,t}(B^m)}^w\cap HS^n$ as claimed.
Finally note that $\overline{f^{\,t}(HS^m)}^w=\overline{f^{\,t}(B^m)}^w$, because HS is weakly-dense in B and $f^{\,t}$ is weakly continuous. ∎

3. First Betti number and Δ.

In this section, we concentrate on the first L^2-Betti number.

3.1. $\beta_1^{(2)}$ as a limit.

3.1.1. *Sub-complexes associated to a set of generators.* We recall that all L^2-Betti numbers can be represented as limits, as described by equation (2.11). We particularize to the first Betti number.
Let M be a von Neumann algebra with a faithful trace-state τ.
Let $F = \{X_1, \ldots, X_n\}, X_j \in M$ be a self-adjoint set of elements in M; that is, we assume that $X^* \in F$ whenever $X \in F$. Assume further that F generates M as a von Neumann algebra. Let

$$C_1(F) = (M \otimes M^o) \otimes \operatorname{span} F \cong (M \otimes M^o)^{\dim \operatorname{span} F}$$

and consider

$$\partial_F : C_1(F) \to M \otimes M^o,$$

given by

$$\partial_F(a \otimes b^o \otimes X) = a\,X \otimes b^o - a \otimes (X\,b)^o, \qquad a \otimes b^o \in M \otimes M^o,\ X \in F. \tag{3.1}$$

Then

$$\ker \partial_F \to C_1(F) \xrightarrow{\partial_F} M \otimes M^o \to M \to 0 \tag{3.2}$$

is a sub-complex of the bar resolution of M. The sequence above is not exact, and the quotient of $M \otimes M^o$ by the image of ∂_F is the left $M \otimes M^o$-module $(M \otimes M^o) \otimes_{A \otimes A^o} A$, where A is the algebra generated by F. When viewed as a bimodule over M this left $M \otimes M^o$-module can be identified by the map $x \otimes y^o \mapsto x \otimes y$ with $M \otimes_A M$, where $(m \otimes n^o) \cdot (x \otimes_A y) = m\,x \otimes_A y\,n$. We shall thus use the notation,

$$M \otimes_A M := (M \otimes M^o) \otimes_{A \otimes A^o} A$$

The sequence

$$\ker \partial_F \to C_1(F) \xrightarrow{\partial_F} M \otimes M^o \to M \otimes_A M \to 0 \tag{3.3}$$

is exact. Applying the induction functor $(M \bar{\otimes} M^o) \otimes_{M \otimes M^o}$ one gets the map

$$1 \otimes \partial_F : (M \bar{\otimes} M^o) \otimes \operatorname{span} F \to M \bar{\otimes} M^o,$$

given as in (3.1) by right multiplication by $X \otimes 1 - 1 \otimes X^o$. Then the first homology of the induced complex from (3.3) (or (3.2)) is given by

$$H(F) = \frac{\ker(1 \otimes \partial_F)}{(M \bar{\otimes} M^o) \cdot \ker \partial_F}.$$

In other words,

Lemma 3.1. *Let A be the algebra generated by F. Then*

$$H(F) = \operatorname{Tor}_1^{M \otimes M^o}(M \otimes_A M, M \bar{\otimes} M^o).$$

In particular, $H(F)$ depends only on the inclusion $A \subset M$ and not on F.

We let $\beta(F) = \dim_{M \bar{\otimes} M^o} H(F)$ so that by Lemma 2.11 one has

$$\beta(F) = \beta^{(2)}(\partial_F)$$

Note that if $F \subset F'$, then there is a natural inclusion map $i_{F',F} : C_1(F) \to C_1(F')$. This map induces a homomorphism

$$(i_{F',F})_* : H(F) \to H(F').$$

For $F \subset F'$ two finite subsets, let

$$H(F : F') = \frac{i_{F',F}(\ker(1 \otimes \partial_F))}{i_{F',F}(\ker(1 \otimes \partial_F)) \cap (M \bar{\otimes} M^o) \cdot \ker \partial_{F'}} = (i_{F',F})_* H(F).$$

Note that $i_{F',F}(\ker(1 \otimes \partial_F)) \cap (M \bar{\otimes} M^o) \cdot \ker \partial_{F'}$ is exactly the intersection of $(M \bar{\otimes} M^o) \otimes \operatorname{span} F$ with $(M \bar{\otimes} M^o) \cdot \ker \partial_{F'}$.
Then (2.11) implies that

$$\beta_1^{(2)}(M) = \sup_F \inf_{F' \supset F} \dim_{M \bar{\otimes} M^o} H(F : F').$$

We are thus led to the natural question of the computation of

$$\beta(F : F') = \dim_{M \bar{\otimes} M^o} H(F : F')$$

and, in particular, of $\beta(F) = \dim_{M \bar{\otimes} M^o} H(F)$ (which corresponds to the case that $F = F'$). Note in particular that

$$\beta(F : F') \leq \beta(F)$$

(since the dimension of the image via $(i_{F',F})_*$ is not larger that the dimension of the domain), and also that

$$\beta(F : F') \leq \beta(F') \tag{3.4}$$

(since the image via $(i_{F',F})_*$ is a sub-module of $H(F')$).
Let $F = (X_1, \ldots, X_n)$, $F' = F \cup (Y_1, \ldots, Y_m)$, and assume for simplicity that $n = \dim \operatorname{span} F$. Denote by $\partial_F^{(2)}$ the extension of ∂_F to $(L^2(M) \bar{\otimes} L^2(M^o))^n \cong L^2(M \bar{\otimes} M^o) \otimes \operatorname{span} F$ obtained by continuity. By Lemma 2.15, equation (2.17), we have that

$$\beta(F) = \beta^{(2)}(\partial_F) = \dim_{M \bar{\otimes} M^o} \ker \partial_F^{(2)} - \dim_{M \bar{\otimes} M^o} \overline{\ker \partial_F^{(2)} \cap FR^n}.$$

Lemma 3.2. *Let $F \subset F'$ be self-adjoint sets of elements in M, as before. If F generates M as a von Neumann algebra, then*

$$\dim_{M \bar{\otimes} M^o} i_{F',F} \ker \partial_F^{(2)} = n - (1 - \beta_0^{(2)}(M, \tau)),$$

and

$$\begin{aligned} \beta(F) &= n - (1 - \beta_0^{(2)}(M, \tau)) - \dim_{M \bar{\otimes} M^o} \overline{(\ker \partial_F^{(2)} \cap FR^n)}, \\ \beta(F : F') &= n - (1 - \beta_0^{(2)}(M, \tau)) - \dim_{M \bar{\otimes} M^o} \overline{\ker \partial_{F'}} \cap i_{F',F} \ker \partial_F^{(2)}. \end{aligned}$$

Proof. We just need to prove the first statement. The inclusion map $i_{F',F} : C_1(F) \to C_1(F')$ is injective, so that we need only to consider the case that $F = F'$. As explained in the proof of Proposition 2.7, $\beta_0^{(2)}(M, \tau)$ is $1 - \dim_{M \bar{\otimes} M^o} (\overline{\operatorname{im}(\partial_1^{(2)})})$ where $\partial_1^{(2)}$ comes from the bar resolution.

Considering the kernel and cokernel of the morphism

$$M\bar{\otimes}M^o \otimes F \xrightarrow{\partial_F^{(2)}} M\bar{\otimes}M^o, \tag{3.5}$$

it is enough to show that

$$\overline{\operatorname{im}(\partial_F^{(2)})} = \overline{\operatorname{im}(\partial_1^{(2)})}\,.$$

By construction $\overline{\operatorname{im}(\partial_1^{(2)})}$ is the strong closure in $M\bar{\otimes}M^o$ of the left ideal L generated by

$$V = \{n\otimes 1 - 1\otimes n^o : n\in M\}.$$

We use the map $\Psi : M\bar{\otimes}M^o \to HS = HS(L^2(M))$ (of (1.1)); one has by (1.2) or (2.18)

$$\Psi(x(n\otimes 1 - 1\otimes n^o)) = [Jn^*J,\, \Psi(x)]\,, \quad \forall x\in M\bar{\otimes}M^o\,, n\in M. \tag{3.6}$$

We adopt the following notation for any bounded operator T in $L^2(M)$,

$$\sigma(T) := T^\sigma = JT^*J\,, \quad \forall T\in B\,. \tag{3.7}$$

This gives an antiautomorphism of B which restricts to the canonical antiisomorphism $\sigma : M \to M'$. We thus see that the closure of L in $L^2(M\bar{\otimes}M^o)$ can be identified with the subspace

$$\overline{[HS, M']},$$

which is the closure of the linear span of commutators of M' with HS.
Similarly the closure of $\operatorname{im}(\partial_F^{(2)})$ is the subspace

$$\overline{[HS, F^\sigma]}\,, \quad F^\sigma := \sigma(F).$$

We just need to show that $[HS, F^\sigma]$ is dense in $[HS, M']$ in the HS-topology. Then the algebra A generated by F is $*$-strongly dense in M by hypothesis. For fixed $T\in HS$ the map $x\mapsto [T,x]$ is continuous on bounded sets, from B endowed with the strong topology to HS. Thus $[HS,\sigma(A)]$ is dense in $[HS, M']$.
It remains to sow that $[HS,\sigma(A)] = [HS, F^\sigma]$, which follows from the indentities,

$$\begin{aligned}[T, X_1X_2\dots X_n] &= [X_2\dots X_n T,\, X_1] + [X_3\dots X_n T X_1,\, X_2]\\ &+ \dots + [X_{j+1}\dots X_n T X_1\dots X_{j-1},\, X_j]\\ &+ \dots + [T X_1 \dots X_{n-1},\, X_n]\end{aligned}$$

and the fact that HS is a two sided ideal. □

3.1.2. $\Delta(F)$ *and* $\Delta(F:F')$. It is thus of interest to consider the quantities:

$$\begin{aligned}\Delta(F) &= n - \dim_{M\bar{\otimes}M^o} \overline{(\ker\partial_F^{(2)} \cap FR^n)} = n - \dim_{M\bar{\otimes}M^o} \overline{\ker\partial_F},\\ \Delta(F:F') &= n - \dim_{M\bar{\otimes}M^o} \overline{\ker\partial_{F'}} \cap i_{F',F}\ker\partial_F^{(2)},\\ \Delta(M,\tau) &= \sup_{F \text{ s.t. } M=W^*(F)}\ \inf_{F'\supset F} \Delta(F;F'),\end{aligned} \tag{3.8}$$

where in the last equation we require that F generates M.

Explicitly, if $F = (X_1, \dots, X_n)$, $F' = F \cup (Y_1, \dots, Y_m)$, we have:

$$(3.9) \qquad \begin{aligned} \Delta(F) &= n - \dim_{M\bar{\otimes}M^o} \overline{\{(T_1, \dots, T_n) \in FR^n : \sum [T_j, X_j^\sigma] = 0\}}^{HS}, \\ \Delta(F : F') &= n - \dim_{M\bar{\otimes}M^o} \Big(HS^n \oplus 0 \\ &\quad \cap \overline{\{(T_1, \dots, T_n, S_1, \dots, S_m) \in FR^{n+m}} \\ &\quad \overline{: \sum [T_j, X_j^\sigma] + \sum [S_j, Y_j^\sigma] = 0\}}^{HS} \Big), \end{aligned}$$

where we used the map $\Psi : M\bar{\otimes}M^o \to HS = HS(L^2(M))$. Note that the X_j, Y_k are moved to the commutant M' of M by the map σ, as follows from (3.6), it is thus clear that the subspaces of HS^n involved in the above equations are M-bimodules. In either equation above, FR^n can be replaced by $\Psi(M \otimes M^o)^n \subset FR^n$.
Furthermore, we have by Lemma 3.2:

$$(3.10) \qquad \beta_1^{(2)}(M, \tau) = \Delta(M, \tau) - (1 - \beta_0^{(2)}(M, \tau)).$$

Note that if $F = (X_1, \dots, X_n)$, then $\partial_F : FR^n = FR \otimes \operatorname{span} F \to FR$ is given by

$$\partial_F(T_1, \dots, T_n) = -\sum [T_i, X_i^\sigma].$$

The transpose of ∂_F is the map

$$\partial_F^t : B(L^2(M)) \to B(L^2(M))^n$$

given by

$$(3.11) \qquad \partial_F^t(D) = ([D, X_1^\sigma], \dots, [D, X_n^\sigma]).$$

In view of Lemma 2.16, we have the following description of $\Delta(F)$:

$$(3.12) \qquad \Delta(F) = \dim_{M\bar{\otimes}M^o} \overline{\partial_F^t(B(L^2(M))}^w \cap HS^n.$$

Similarly, if $F' = F \cup \{Y_1, \dots, Y_m\}$, then

$$(3.13) \qquad \Delta(F : F') = \dim_{M\bar{\otimes}M^o} \pi_n(\overline{\partial_{F'}^t(B(L^2(M))}^w \cap HS^{n+m}),$$

where $\pi_n : HS^{n+m} \to HS^n$ denotes the orthogonal projection onto the first n coordinates.

3.2. **Properties of Δ.** We record the following properties of Δ:

Theorem 3.3. *Let $X_1, \dots, X_n \in (M, \tau)$ be a fixed self-adjoint set of elements. Then we have:*
(a) $\Delta(X_1, \dots, X_n) \le n$.
(b) $\Delta(X_1, \dots, X_n)$ depends only on the pair $(A, \tau|_A)$, where A is the algebra generated by $X_1, \dots, X_n$
(c) Let Γ be a finitely generated group, and let $X_1, \dots, X_n \in L(\Gamma)$ be a family of unitaries associated to a symmetric set of generators of Γ. Then

$$\Delta(X_1, \dots, X_n) \le \beta_1^{(2)}(\Gamma) - \beta_0^{(1)}(\Gamma) + 1.$$

If in addition Γ is combable, we have that

$$\Delta(X_1, \dots, X_n) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1.$$

(d) For all $m < n$,

$$\Delta(X_1, \ldots, X_n) \le \Delta(X_1, \ldots, X_m) + \Delta(X_{m+1}, \ldots, X_n).$$

(e) Let $1 < m < n$, and assume that the families $X_1, \ldots, X_m$, $X_{m+1}, \ldots, X_n$ are free. Then

$$\Delta(X_1, \ldots, X_n) = \Delta(X_1, \ldots, X_m) + \Delta(X_{m+1}, \ldots, X_n).$$

Proof. (a) follows immediately from the definition of Δ.
(b) Let A be the algebra generated by $X_1, \ldots, X_n$, and let N be the von Neumann algebra generated by A inside of M. Let $F = (X_1, \ldots, X_n)$. By the obvious variant of Lemma 3.2 for non generating sets, we find that

$$\Delta(F) = \beta_1^{(2)}(F) + (1 - \beta_0^{(2)}(N)).$$

where $\beta_1^{(2)}(F)$ is computed inside (M, τ). Moreover, by Lemma 3.1, we have that

$$\beta_1^{(2)}(F) = \dim_{M\bar{\otimes}M^o} \operatorname{Tor}_1^{M\otimes M^o}(M \otimes_A M, M\bar{\otimes}M^o).$$

Since the functors

$$M \otimes_N \cdot, \qquad \cdot \otimes_{N^o} M^o, \qquad M\bar{\otimes}M^o \otimes_{N\bar{\otimes}N^o}$$

are flat [Lüc97], it follows that

$$\operatorname{Tor}_1^{M\otimes M^o}(M \otimes_A M, M\bar{\otimes}M^o) = M\bar{\otimes}M^o \otimes_{N\bar{\otimes}N^o} \operatorname{Tor}_1^{N\otimes N^o}(N \otimes_A N, N\bar{\otimes}N^o).$$

Finally, since

$$\dim_{M\bar{\otimes}M^o}(M\bar{\otimes}M^o \otimes_{N\bar{\otimes}N^o} W) = \dim_{N\bar{\otimes}N^o} W,$$

we find that

$$\Delta(F) = \dim_{N\bar{\otimes}N^o} \operatorname{Tor}_1^{N\otimes N^o}(N \otimes_A N, N\bar{\otimes}N^o) + 1 - \beta_0^{(2)}(N),$$

which depends only on A and $\tau|_A$.
(c) The inequality follows from Theorem 2.13. The equality in the combable case follows from Theorem 2.14.
(d) Let $F_1 = (X_1, \ldots, X_m)$ and $F_2 = (X_{m+1}, \ldots, X_n)$, $F = F_1 \cup F_2$. Let $V_i = \operatorname{span} F_i$, $C_i = (M\bar{\otimes}M^o) \otimes V_i$, $i = 1, 2$. Put $C = (M\bar{\otimes}M^o) \otimes \operatorname{span}(V_1, V_2)$. Consider

$$\partial_{F_i} : C_i \to M \otimes M^o$$

given by

$$\partial_{F_i}(a \otimes b^o \otimes x) = a\,x \otimes b^o - a \otimes (x\,b)^o.$$

Then

$$(\ker \partial_{F_1}) \oplus (\ker \partial_{F_2}) \subset \ker \partial_F \cap C.$$

Thus

$$\dim_{M\bar{\otimes}M^o} \overline{\ker \partial_F} \ge \dim_{M\bar{\otimes}M^o} \overline{\ker \partial_{F_1}} + \dim_{M\bar{\otimes}M^o} \overline{\ker \partial_{F_2}}.$$

In view of (b), this implies the desired inequality for Δ.
(e) Let $M_1 = W^*(X_1, \ldots, X_m)$ and $M_2 = W^*(X_{m+1}, \ldots, X_n)$. By Remark 13.2(e) of [Voi99a], there exist operators D_1, D_2 in $B(L^2(M))$ so that

$$[D_i, M_k] = \{0\}, \quad [D_i, m] = [m, \Psi(1 \otimes 1)], \tag{3.14}$$

for all $i \neq k$ and $m \in M_i$. These operators are denoted by T_j in [Voi99a] and are called a dual system to $M_1, M_2, \mathbb{C}1$. It is worth mentioning in conjunction with Def. 13.1 of [Voi99a] that a single algebra A always has a dual system relative to $\mathbb{C}1$, namely the operator of orthogonal projection onto $\mathbb{C}1$ in $L^2(A)$.
One can in fact explicitly describe these operators. Denote by H_i^0 the space $L^2(M_i) \ominus \mathbb{C}1$. Then since $M = M_1 * M_2$,

$$L^2(M) = \mathbb{C}1 \oplus \bigoplus_k \bigoplus_{i_1 \neq i_2, i_2 \neq i_3, \dots i_{k-1} \neq i_k} H_{i_1}^0 \otimes \cdots \otimes H_{i_k}^0.$$

We refer the reader to [VDN92] for more details and the definition of the action of M_j on this space. The operator D_k is then given by

$$D_k 1 = 1$$

and

$$D_k \xi_1 \otimes \cdots \otimes \xi_r = \begin{cases} 0, & \xi_r \in H_k^o \\ \xi_1 \otimes \cdots \otimes \xi_r, & \text{otherwise.} \end{cases}$$

Let $F = (X_1, \dots, X_n)$. Assume now that

$$\sum_{i=1}^n T_i \otimes X_i \in \ker \partial_F, \qquad T_i \in M \otimes M^o.$$

Then

$$\sum_{i=1}^n [\Psi(T_i), X_i^\sigma] = 0.$$

Let k be equal to 1 or 2, and write $I_1 = \{1, \dots, m\}$, $I_2 = \{m+1, \dots, n\}$. Then for all $a, b \in M$,

$$\begin{aligned} 0 &= \sum_i \mathrm{Tr}([\Psi(T_i), X_i^\sigma]\, a\, D_k^\sigma\, b) \\ &= \sum_i \mathrm{Tr}(\Psi(T_i)[X_i^\sigma,\, a\, D_k^\sigma\, b]) \\ &= \sum_i \mathrm{Tr}(\Psi(T_i)\, a\, [X_i^\sigma,\, D_k^\sigma]\, b) \\ &= \sum_{i \in I_k} \mathrm{Tr}(\Psi(T_i)[\Psi(a \otimes b^o), X_i^\sigma]) \\ &= -\sum_{i \in I_k} \mathrm{Tr}([\Psi(T_i), X_i^\sigma]\Psi(a \otimes b^o)). \end{aligned}$$

where we used (3.14) and the equality $\Psi(1 \otimes 1)^\sigma = \Psi(1 \otimes 1)$.
It follows (since $a, b \in M$ were arbitrary) that

$$\sum_{i=1}^m [\Psi(T_i), X_i^\sigma] = \sum_{i=m+1}^n [\Psi(T_i), X_i^\sigma] = 0.$$

It follows that $\sum_{i=1}^m T_i \otimes X_i \in \ker \partial_F$, and $\sum_{i=m+1}^n T_i \otimes X_i \in \ker \partial_F$.

Let $F_1 = (X_1, \dots, X_m)$, $F_2 = (X_{m+1}, \dots, X_n)$. If we denote by V_j the span of F_j and let $V = \mathrm{span}(V_1, V_2)$, then we have shown that

$$\ker \partial_F \subset \ker \partial_F \cap M \otimes M^o \otimes V_1 + \ker \partial_F \cap M \otimes M^o \otimes V_2,$$

so that

$$\ker \partial_F \subset \ker \partial_{F_1} + \ker \partial_{F_2}.$$

Thus

$$\dim_{M \bar{\otimes} M^o}(\overline{\ker \partial_F}) \leq \sum_{k=1}^{2} \dim_{M \bar{\otimes} M^o}(\overline{\ker \partial_{F_k}}).$$

From this and part (b) we conclude that

$$\Delta(X_1, \dots, X_n) \geq \Delta(X_1, \dots, X_m) + \Delta(X_{m+1}, \dots, X_n).$$

Since we have proved the opposite inequality in part (c), the desired equality now follows. □

3.3. **Δ and diffuse center.**

Lemma 3.4. *Let $F = (X, X_1, \dots, X_n)$, and assume that $[X, X_j] = 0$ for all j. Assume furthermore that the spectrum of X is diffuse. Then $\Delta(F) = 1$.*

Proof. Let $M = W^*(F)$. Since $\Delta(F) = \beta_1^{(2)} + (1 - \beta_0^{(2)}(M))$, we find that

$$\Delta(F) \geq 1 - \beta_0^{(2)}(M).$$

Since M contains a diffuse von Neumann algebra (namely, $W^*(X)$), it follows from Proposition 2.7 that $\beta_0^{(2)}(M) = 0$. Thus $\Delta(F) \geq 1$.
For the opposite inequality, let FR be the ideal of finite-rank operators on $L^2(M)$, and assume that $Q_1, \dots, Q_n \in FR^n$ are arbitrary. Let

$$\begin{aligned} T_j &= [Q_j, X^\sigma], \\ T &= -\sum_{j=1}^{n} [Q_j, X_j^\sigma]. \end{aligned}$$

Then since $[X_j^\sigma, X^\sigma] = 0$ for all j, we have that

$$[T, X^\sigma] + \sum_{j=1}^{n} [T_j, X_j^\sigma] = \sum_{j=1}^{n} -[[Q_j, X_j^\sigma], X^\sigma] + [[Q_j, X^\sigma], X_j^\sigma] = 0$$

by the Jacobi identity. Thus the image of the map

$$FR^n \ni (Q_1, \dots, Q_n) \mapsto (T, T_1, \dots, T_n)$$

lies inside $\ker \partial_F$ (we identify as usual $M \otimes M^o$ with a subset of FR via the map Ψ; see Lemma 2.15). It follows that the closure of $\ker \partial_F$ in the Hilbert-Schmidt norm contains the image of the map

$$\phi : HS^n \ni (Q_1, \dots, Q_n) \mapsto (-\sum_{j=1}^{n} [Q_j, X_j^\sigma], [Q_1, X^\sigma], \dots, [Q_n, X^\sigma]).$$

Since X has diffuse spectrum, the commutant of $W^*(X^\sigma)$ in $B(L^2(M))$ does not intersect compact (and hence Hilbert-Schmidt) operators. Thus the map ϕ is injective. Hence by Luck's results on additivity of dimension for weakly exact sequences [Lüc98] we conclude that

$$\dim_{M\bar{\otimes}M^o} \overline{\ker\partial_F}^{HS} \geq \dim_{M\bar{\otimes}M^o} \operatorname{im}\phi = \dim_{M\bar{\otimes}M^o} HS^n = n.$$

Thus

$$\Delta(F) = n + 1 - \dim_{M\bar{\otimes}M^o} \overline{\ker\partial_F}^{HS} \leq 1.$$

We conclude that $\Delta(F) = 1$. □

Corollary 3.5. *If (M,τ) is a von Neumann algebra, and M has a diffuse center, then $\beta_1^{(2)}(M,\tau) = 0$ and $\Delta(M,\tau) = 1$.*

Proof. Let X be a generator of the center of M. Then for any finite subset F of self-adjoint elements of M, we have that

$$\Delta(F \cup \{X\}) = 1,$$

by Lemma 3.4. Hence if F generates M, $F' \supset F$ and $X \in F'$, then by (3.4)

$$\begin{aligned}\Delta(F : F') &= \beta_1^{(2)}(F : F') + (1 - \beta_0^{(2)}(M))\\ &\leq \beta_1^{(2)}(F') + (1 - \beta_0^{(2)}(M))\\ &= \Delta(F') = 1.\end{aligned}$$

Thus for any F generating M, we have that

$$\inf_{F'\supset F} \Delta(F : F') \leq \Delta(F : F \cup \{X\}) = 1,$$

so that $\Delta(M,\tau) \leq 1$. Since M contains a diffuse von Neumann algebra (namely, $W^*(X)$), it follows that $\Delta(M,\tau) = 1$, and that $\beta_1^{(2)}(M) = 0$. □

4. Δ and free entropy dimension.

4.1. **Free entropy dimension.** The properties of $\Delta(X_1,\ldots,X_n)$ seem very similar to those enjoyed by the various versions of Voiculescu's free entropy dimension. We therefore are interested in connections between the two quantities.

4.1.1. *Non-microstates entropy dimension.* We consider the free entropy dimensions defined in terms of the non-microstates free entropy and the non-microstates free Fisher information. Let $S_1,\ldots,S_n$be a free semicircular family, free from $(X_1,\ldots,X_n)$. Then set

$$\delta^*(X_1,\ldots,X_n) = n - \liminf_{\varepsilon\to 0} \frac{\chi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)}{\log\varepsilon^{1/2}}$$

and

$$\delta^\star(X_1,\ldots,X_n) = n - \liminf_{\varepsilon\to 0} \varepsilon\,\Phi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n).$$

Note that $\delta^\star$ is obtained from δ^* by formally applying L'Hopital's rule to the limit. We will also use the microstates free entropy dimension δ and δ_0, which were introduced in [Voi94, Voi96]. Here

$$\Phi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)=\sum_{i=1}^n\|\xi_i^\varepsilon\|_2^2,$$

where $\xi_i^\varepsilon\in L^2(W^*(X_1,\ldots,X_n))$ are the conjugate variables

$$\xi_i^\varepsilon=J(X_i+\sqrt{\varepsilon}S_i:X_1+\sqrt{\varepsilon}S_1,\ldots,\widehat{X_i+\sqrt{\varepsilon}S_i},\ldots,X_n+\sqrt{\varepsilon}S_n)$$

(here $\hat{\cdot}$ denotes omission).
Let

$$E_\varepsilon=E_{W^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)}$$

be the unique conditional expectation. By [Voi98], one has:

$$\xi_i^\varepsilon=\frac{1}{\sqrt{\varepsilon}}E_\varepsilon(S_i).$$

Thus

$$\delta^\star=n-\liminf_{\varepsilon\to0}\sum_{i=1}^n\|E_\varepsilon(S_i)\|_2^2.$$

Lemma 4.1. $\delta^\star(X_1,\ldots,X_n)\geq\delta^*(X_1,\ldots,X_n)$.

Proof. Assume that $\delta^\star(X_1,\ldots,X_n)<n-C$. Thus

$$\liminf_{\varepsilon\to0}\varepsilon\,\Phi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)>C.$$

Then for some $\varepsilon_0>0$ and all $0<\varepsilon<\varepsilon_0$, we have that

$$\varepsilon\,\Phi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)\geq C,$$

so that

$$\Phi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)\geq\frac{C}{\varepsilon}.$$

Thus for all $0<\varepsilon<\varepsilon_0$,

$$\begin{aligned}\frac{1}{2}\int_\varepsilon^{\varepsilon_0}\Phi^*(X_1+\sqrt{t}S_1,\ldots,X_n+\sqrt{t}S_n)\,dt &\geq \frac{1}{2}\int_\varepsilon^{\varepsilon_0}\frac{C}{t}dt\\ &= C(\log\varepsilon_0^{1/2}-\log\varepsilon^{1/2}).\end{aligned}$$

Now $\chi_\varepsilon=\chi^*(X_1+\sqrt{\varepsilon}S_1,\ldots,X_n+\sqrt{\varepsilon}S_n)$ is given by (cf [Voi98])

$$\begin{aligned}\chi_\varepsilon &= \tfrac{1}{2}\int_0^\infty\left(\frac{n}{1+t}-\Phi^*(X_1+\sqrt{t+\varepsilon}\,S_1,\ldots,X_n+\sqrt{t+\varepsilon}\,S_n\right)dt\\ &= \frac{1}{2}\int_\varepsilon^\infty\left(\frac{n}{1+t-\varepsilon}-\Phi^*(X_1+\sqrt{t}S_1,\ldots,X_n+\sqrt{t}S_n\right)dt\\ &\leq K+\frac{1}{2}\int_\varepsilon^{\varepsilon_0}\left(\frac{n}{1+t-\varepsilon}-\Phi^*(X_1+\sqrt{t}S_1,\ldots,X_n+\sqrt{t}S_n\right)dt,\end{aligned}$$

for some constant K depending only on ε_0 and $X_1, \dots, X_n$.
Thus
$$\chi_\varepsilon \le K + \frac{n}{2}\log(1+\varepsilon_0-\varepsilon) + C(\log\varepsilon^{1/2} - \log\varepsilon_0^{1/2})$$
Since for small ε, $\log\varepsilon$ is negative, it follows that
$$\liminf_{\varepsilon\to 0} \frac{\chi^*(X_1+\sqrt{\varepsilon}S_1,\dots,X_n+\sqrt{\varepsilon}S_n)}{\log\varepsilon^{1/2}} \ge \liminf_{\varepsilon\to 0}\frac{C\log\varepsilon^{1/2}}{\log\varepsilon^{1/2}} = C.$$
Thus $\delta^*(X_1,\dots,X_n) \le n - C$. Since C is arbitrary, we get that $\delta^*(X_1,\dots,X_n) \le \delta^\star(X_1,\dots,X_n)$. □

4.2. **The inequality $\Delta \ge \delta^\star$.** In preparation for the next theorem, we need to set up some notation.
Let $S_1,\dots,S_n$ be a free semicircular system, free from $(X_1,\dots,X_n)$. Let $X_j(\varepsilon) = X_j + \sqrt{\varepsilon}\,S_j$, $j=1,\dots,n$. Let also $M_\varepsilon = W^*(X_1(\varepsilon),\dots,X_n(\varepsilon))$, $N = W^*(X_1,\dots,X_n,S_1,\dots,S_n)$, $H = L^2(N)$. (We recall in Appendix II the details of this standard construction (cf. [Voi98])) Thus $M_\varepsilon \subset N \subset B(H)$. Let E_ε be the orthogonal projection from H onto $L^2(M_\varepsilon)$. We denote by the same symbol the conditional expectation from N onto M_ε. Note that $P_1 = E_\varepsilon P_1 = P_1 E_\varepsilon$.
Let $D_j(\varepsilon)$, $j=1,\dots,n$, be a dual system to $(X_1(\varepsilon),\dots,X_n(\varepsilon))$ on $B(H)$. That is, we require that

$$D_j(\varepsilon) = E_\varepsilon D_j(\varepsilon) E_\varepsilon\,, \quad [D_j(\varepsilon), X_i(\varepsilon)] = \delta_{ij}P_1 \tag{4.1}$$

Such a system is always possible to find: one can set

$$D_j(\varepsilon) = E_\varepsilon \frac{1}{\sqrt{\varepsilon}} Q_j E_\varepsilon \tag{4.2}$$

where Q_j is a right creation operator (cf. Appendix II). Note that one has the property that $\|D_j(\varepsilon)\|_\infty \le 1/\sqrt{\varepsilon}$.
Identify now $M\otimes M^o$ via the map Ψ with a subspace of the space of finite-rank operators on H. Let $T_1,\dots,T_n \in \Psi(M\otimes M^o)$ be so that $\sum[T_i, X_i^\sigma(\varepsilon)] = 0$. To be explicit, let $T_i = \sum_k a_k^i\, P_1\, b_k^i$, $a_k^i, b_k^i \in M$.
We first need a few lemmas.

Lemma 4.2. *Let $\delta > 0$. Then there exist $x_k^i(\varepsilon), y_k^i(\varepsilon) \in M_\varepsilon$ and $\varepsilon_0 > 0$ such that,*
$$\|T_i'(\varepsilon) - T_i\|_{HS} \le \|T_i'(\varepsilon) - T_i\|_1 < \delta\,, \quad \forall\varepsilon < \varepsilon_0$$
where
$$T_i'(\varepsilon) = \sum_k x_k^i(\varepsilon)\, P_1\, y_k^i(\varepsilon)\,.$$

Proof. (of Lemma). It is sufficient to prove the statement for single rank-one operator $T_{a,b} = a\,P_1\,b$. Note that $\|T_{a,b}\|_1 = \sup_{\|S\|_\infty = 1}|\langle a, Sb\rangle| = \|a\|_2\|b\|_2$. We can assume that $\|a\|_2 = \|b\|_2 = 1$. Choose non-commutative polynomials p and q so that
$$\|p(X_1,\dots,X_n) - a\|_2 \le \delta/4,\ \|q(X_1,\dots,X_n) - b\|_2 < \delta/4.$$
Let $\varepsilon_0 > 0$ so that whenever $X_j' \in N$ and $\|X_j - X_j'\|_\infty < 2\sqrt{\varepsilon_0}$, we have
$$\|p(X_1,\dots,X_n) - p(X_1',\dots,X_n')\|_\infty < \delta/4,\ \|q(X_1,\dots,X_n) - q(X_1',\dots,X_n')\|_\infty < \delta/4.$$

Set $x(\varepsilon) = p(X_1(\varepsilon), \ldots, X_n(\varepsilon))$, $y(\varepsilon) = q(X_1(\varepsilon), \ldots, X_n(\varepsilon))$. Let $0 < \varepsilon < \varepsilon_0$. Then $\| x(\varepsilon) - a\|_2 \leq \delta/2$ and $\|y(\varepsilon) - b\|_2 \leq \delta/2$ which gives the answer. □

The following lemma is implicit in [Voi98], but we restate it for convenience.

Lemma 4.3. *Let N be von Neumann algebra, and let τ be a faithful normal trace on N. Let $H = L^2(N, \tau)$, and let J be the Tomita conjugation associated to N. Denote by P_1 the orthogonal projection onto $1 \in H$.*
Let $Q \in B(H)$ and $Z \in N$. Then

$$\mathrm{Tr}(P_1[Q, Z^*]) = \langle (JQ^*J - Q)1, Z1\rangle.$$

Proof. Using $\langle J\,\xi,\,\eta\rangle = \langle J\,\eta,\,\xi\rangle$ we get:

$$\begin{aligned} \mathrm{Tr}(P_1[Q, Z^*]) &= \langle QZ^*1, 1\rangle - \langle Z^*Q1, 1\rangle \\ &= \langle Z^*, Q^*1\rangle - \langle Q1, Z\rangle \\ &= \langle JQ^*J1, Z\rangle - \langle Q1, Z\rangle \\ &= \langle (JQ^*J - Q)1, Z\rangle, \end{aligned}$$

which is the desired identity. □

Theorem 4.4. $\Delta(X_1, \ldots, X_n) \geq \delta^\star(X_1, \ldots, X_n)$.

Proof. Let $T_1^j, \ldots, T_n^j$, $j = 1, \ldots, n$ be in $\Psi(M \otimes M^o)$ and such that $\sum_i [T_i^j, X_i^\sigma] = 0$ for all j. Let

$$T_i^j = \sum_k a_k^{ij}\, P_1\, b_k^{ij}\,.$$

Then using (4.1), $\sum_i [T_i^j, X_i^\sigma] = 0$, and $\sigma(P_1) = P_1$, we get,

$$\begin{aligned} \sum_j \langle T_j^j, P_1\rangle &= \sum_{ij} \mathrm{Tr}(T_i^j\,[X_i^\sigma(\varepsilon), D_j^\sigma(\varepsilon)]) \\ &= \sum_{ij} \mathrm{Tr}([T_i^j,\, X_i^\sigma(\varepsilon)] D_j^\sigma(\varepsilon)) \\ &= \sum_{ij} \mathrm{Tr}([T_i^j,\, \sqrt{\varepsilon}\, S_i^\sigma]\, D_j^\sigma(\varepsilon)) \\ &= \sum_{ij} \mathrm{Tr}(T_i^j [S_i^\sigma,\, \sqrt{\varepsilon}\, D_j^\sigma(\varepsilon)]). \end{aligned}$$

Now, fix $\kappa > 0$ and let $\delta = \kappa/8n^2$. Choose ε_0 and $T'_{ij}(\varepsilon)$ as in Lemma 4.2, so that

$$\|T_i^j - T'_{ij}(\varepsilon)\|_1 < \delta. \tag{4.3}$$

Then for all $\varepsilon < \varepsilon_0$, since $\|[\sqrt{\varepsilon} D_j(\varepsilon), S_i]\|_\infty \leq 4$, we find that

$$|\sum_j \langle T_j^j, P_1\rangle| \leq \kappa/2 + |\sum_{ij} \mathrm{Tr}(T'_{ij}(\varepsilon)[\sqrt{\varepsilon}\, D_j^\sigma(\varepsilon), S_i^\sigma])|. \tag{4.4}$$

Since

$$T'_{ij}(\varepsilon) = \sum x_k^{ij}(\varepsilon)\, P_1\, y_k^{ij}(\varepsilon)\,, \qquad x_k^{ij}(\varepsilon), y_k^{ij}(\varepsilon) \in M_\varepsilon$$

and $P_1 = E_\varepsilon P_1 E_\varepsilon$, we have that

$$T'_{ij}(\varepsilon) = E_\varepsilon T'_{ij}(\varepsilon) E_\varepsilon.$$

Let $\xi_i(\varepsilon) = E_\varepsilon(\frac{1}{\sqrt{\varepsilon}} S_i)$ (cf.[Voi98]) then $\xi_i(\varepsilon) \in M_\varepsilon$ and $\|\sqrt{\varepsilon}\,\xi_i(\varepsilon)\|_\infty \le \|S_i\|_\infty \le 2$, in fact

$$E_\varepsilon S_i E_\varepsilon = \sqrt{\varepsilon}\,\xi_i(\varepsilon)$$

Note that $E^\sigma_\varepsilon = E_\varepsilon$ and $\sqrt{\varepsilon}\, D^\sigma_j(\varepsilon) = E_\varepsilon Q^\sigma_j E_\varepsilon$ by (4.2). One has

$$\begin{aligned}(4.5)\qquad \sum_{ij} \mathrm{Tr}(T'_{ij}(\varepsilon)[\sqrt{\varepsilon}\, D^\sigma_j(\varepsilon), S^\sigma_i]) &= \sum_{ij} \mathrm{Tr}(T'_{ij}(\varepsilon)[Q^\sigma_j, E_\varepsilon S^\sigma_i E_\varepsilon]) \\ &= \sum_{ij} \mathrm{Tr}(T'_{ij}(\varepsilon)[Q^\sigma_j, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma])\end{aligned}$$

We thus get, using (4.3), the fact that a^{ij}_k , b^{ij}_k commute with $\xi_i(\varepsilon)^\sigma$ and the inequality $\|\sqrt{\varepsilon}\,\xi_i(\varepsilon)\|_\infty \le \|S_i\|_\infty \le 2$:

$$\begin{aligned}|\sum_{ij} \mathrm{Tr}(T'_{ij}(\varepsilon)[Q^\sigma_j, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma])| &\le \kappa/2 + |\sum_{ij} \mathrm{Tr}(T^j_i[Q^\sigma_j, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma]) \\ &= \kappa/2 + |\sum_{ij} \mathrm{Tr}(P_1\ [\sum_k\ b^{ij}_k\, Q^\sigma_j\, a^{ij}_k\ ,\ \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma])| \\ &= \kappa/2 + |\sum_{ij} \langle (Y_{ij} - JY^*_{ij}J)1, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma\rangle_H|,\end{aligned}$$

where $Y_{ij} = \sum_k b^{ij}_k\, Q^\sigma_j\, a^{ij}_k$, and the last equality is by Lemma 4.3.
Combining this with (4.4) and (4.5) we get

$$(4.6)\qquad |\sum_j \langle T^j_j, P_1\rangle| \le \kappa + |\sum_{ij} \langle (Y_{ij} - JY^*_{ij}J)1, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma\rangle_H|.$$

Let $\eta_{ij} = (Y_{ij} - JY^*_{ij}J)1 \in H$. Computing explicitly, we get that (cf. Appendix II)

$$(4.7)\qquad \eta_{ij} = -\sum_k b^{ij}_k S_j a^{ij}_k$$

and in particular $\|\sum_j \eta_{ij}\|^2_2 = \sum_j \|T^j_i\|^2_2$, since the subspaces MS_jM are orthogonal for $j = 1, \ldots, n$, and the map $\sum x_k \otimes y_k \to \sum x_k S_j y_k$ is an isometry from $L^2(M)\bar{\otimes} L^2(M)$ into $L^2(N)$, for each j (cf. Appendix II). We thus conclude that

$$\begin{aligned}|\sum_j \langle T^j_j, P_1\rangle| &\le \kappa + |\sum_i \langle \sum_j \eta_{ij}, \sqrt{\varepsilon}\,\xi_i(\varepsilon)^\sigma\rangle| \\ &\le \kappa + (\sum_i \|\sum_j \eta_{ij}\|^2)^{1/2}(\varepsilon \sum_i \|\xi_i(\varepsilon)\|^2)^{1/2} \\ &= \kappa + (\sum_{ij} \|T^j_i\|^2_2)^{1/2}(\varepsilon\Phi^*(X_1(\varepsilon), \ldots, X_n(\varepsilon)))^{1/2}.\end{aligned}$$

since the free Fisher information is defined as $\Phi^*(X_1(\varepsilon), \ldots, X_n(\varepsilon)) = \sum_{i=1}^n \|\xi_i(\varepsilon)\|_2^2$. Passing to $\liminf_{\varepsilon\to 0}$ and noticing that κ is arbitrary finally gives us:

$$|\sum_j \langle T_j^j, P_1\rangle| \le (n - \delta^\star(X_1, \ldots, X_n))^{1/2} \left(\sum_{ij} \|T_i^j\|_2^2\right)^{1/2}. \tag{4.8}$$

The conclusion of the proof of the theorem now follows from the next lemma (cf. [Shl03b, Lemma 2.9]) applied to the von Neumann algebra $M\bar{\otimes}M^o$ and the subspace K of $L^2(M\bar{\otimes}M^o)^n = HS^n$ closure of the space $\{(T_1, \ldots, T_n) \in \Psi(M\otimes M^o)^n : \sum[T_i, X_i^\sigma] = 0\}$. □

Lemma 4.5. *Let N be a finite von Neumann algebra with a faithful normal trace τ. Let n be a finite integer, and let $H = L^2(N,\tau)^n$ viewed as a left module over N. Denote by $\Omega \in L^2(N,\tau)$ the GNS vector associated to τ.*
Let $K \subset H$ be a closed N-invariant subspace of H. Endow $M_{n\times n}(L^2(N))$ with the norm

$$\|h\|_{M_n}^2 = \sum_{ij=1}^n \|h_{ij}\|^2.$$

Let $A(K) = \{T \in M_{n\times n}(N) : TH \subset K\} \cong K^n$. Then we have:

$$\dim_N K = \mathrm{Sup}|\langle T, I\rangle|^2/\|T\|^2\,, \quad T \in A(K),$$

where $I \in M_n(H)$ denotes the matrix $I_{ij} = \delta_{ij}\Omega$

Proof. (of Lemma). We identify the commutant N' of N acting on H with the algebra of $n\times n$ matrices $M_n(N)$. Endow this algebra with the non-normalized trace Tr, defined by the property that $\mathrm{Tr}(1) = n$, where $1 \in M_n(N)$ denote the identity matrix. Let $e_K \in N'$ be the orthogonal projection from H onto K. Then

$$\dim_N K = \mathrm{Tr}(e_K).$$

Now, $L^2(M_n(N), \mathrm{Tr}) = M_n(H)$ isometrically. Moreover, the orthogonal projection of I onto $A(K)$ is $e_K \in A(K)$, since $1-e_K$ is orthogonal to $A(K) = e_K\, M_n(N)$. The above supremum is thus reached for $T = e_K$ and its value is $\mathrm{Tr}(e_K)$ which gives the result. □

4.2.1. *Some consequences for Δ.*

Corollary 4.6. *We have*

$$\Delta(X_1, \ldots, X_n) \ge \delta^\star(X_1, \ldots, X_n) \ge \delta^*(X_1, \ldots, X_n) \ge \delta(X_1, \ldots, X_n) \ge \delta_0(X_1, \ldots, X_n).$$

This is immediate from the preceding discussion and the work of Biane, Guionnet and Capitaine [BCG03].
The following corollary gives a strong indication that the first L^2-Betti number of a free group factor does not vanish (compare with equations (3.10) and (3.8)).

Corollary 4.7. *Let $F = (X_1, \ldots, X_n)$ be a self-adjoint finite subset of M, and assume that F generates M.*
Assume that the microstates free entropy $\chi(X_1, \ldots, X_n)$ is finite.
Then for any seld-adjoint subset F' of M, we have

$$\Delta(F \cup F') \ge n.$$

Proof. Let $F' = (Y_1, \ldots, Y_n)$. Then

$$\Delta(F \cup F') \geq \delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \geq n$$

where the second inequality follows from [Voi94]. □

It is of course of interest if one has $\Delta = \delta^\star$. In conjunction with this, we note the following. Let $F = (X_1, \ldots, X_n)$ be a finite self-adjoint subset of M. Consider as in (3.11)

$$\partial_F^t : B(L^2(M)) \to B(L^2(M))^n$$

given by

$$\partial_F^t(D) = ([D, X_1^\sigma], \ldots, [D, X_n^\sigma]).$$

Theorem 4.8. *One has*

$$\dim_{M \bar{\otimes} M^o} \overline{\partial_F^t(B(L^2(M)) \cap HS^n}^{HS} \leq \delta^*(X_1, \ldots, X_n)$$

$$\delta^\star(X_1, \ldots, X_n) \quad \leq \quad \dim_{M \bar{\otimes} M^o} \overline{\partial_F^t(B(L^2(M))}^{w} \cap HS^n = \Delta(F).$$

Proof. The first inequality is the statement of Corollary 2.12 in [Shl03b]. The second inequality is the statement of Theorem 4.4, together with the "dual" description of $\Delta(F)$ given in equation (3.12). □

4.2.2. *Some consequences for free entropy dimension.* Let $C(\Gamma)$ be the cost of a discrete group Γ in the sense of [Gab00].

Corollary 4.9. *Let Γ be a finitely generated group with a symmetric set of generators $\gamma_1, \ldots, \gamma_n$. Denote by $u_i = \lambda(\gamma_i) \in L(\Gamma)$ the corresponding unitaries in the left regular representation. Let $X_i = u_i + u_i^*$, $Y_i = i(u_i - u_i^*)$. Then*

$$\delta^*(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1 \leq C(\Gamma).$$

Proof. The inequality between the cost and $\beta_1 - beta_0 + 1$ is due to Gaboriau [Gab02]. The rest of the inequalities follow immediately from the corresponding estimate for Δ. □

Corollary 4.10. *Let Γ be a discrete group with Kazhdan's property (T). Let $\gamma_1, \ldots, \gamma_n$ be a symmetric set of generators of Γ, and let $u_j = \lambda(\gamma_j) \in L(\Gamma)$ be the associated unitaries in the left regular representation. Let $X_i = u_i + u_i^*$, $Y_i = i(u_i - u_i^*)$. Then*

$$\delta_0(\Gamma) \leq \delta(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \leq \delta^\star(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \leq 1.$$

If moreover $L(\Gamma)$ is diffuse, one has

$$\delta^*(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \delta^\star(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = 1.$$

If $L(\Gamma)$ is diffuse and moreover $L(\Gamma)$ can be embedded into the ultrapower of the hyperfinite II_1-factor, one has

$$\delta_0(\Gamma) = 1.$$

Proof. The upper estimates are a consequence of the fact that if Γ has property (T), then $\beta_1^{(2)}(\Gamma) = 0$ (see e.g. [CG86, BV97]). Thus

$$\Delta(\Gamma) \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1 = 1 - \beta_0^{(2)}(\Gamma) \leq 1.$$

The lower estimate for δ^* is a consequence of [Shl03b, Theorem 2.13]. The corresponding estimate for δ_0 is a consequence of hyperfinite monotonicity of [Jun01]. $\square$

4.3. $\Delta(F)$ **and** $\Delta(F : F')$**.** The results of the previous section are insufficient to give a lower bound for $\Delta(M, \tau)$ and thus for $\beta_1^{(2)}(M, \tau)$. We show, however, that under certain smoothness conditions on the families F and F', $\Delta(F, F') \geq \Delta(F)$.

Theorem 4.11. *Let $F = (X_1, \ldots, X_n)$ be a self-adjoint family of generators of M and let $F' = F \cup (Y_1, \ldots, Y_m)$. Let $D_1, \ldots, D_n$ be a dual system to $X_1, \ldots, X_n$ in the sense of* [Voi98]*; thus $D_j \in B(L^2(M))$ satisfy*

$$[D_j, X_i] = \delta_{ji} P_1,$$

where P_1 denotes the projection onto the trace vector in $L^2(M)$. Assume that $[D_j, Y_i]$ is a Hilbert-Schmidt operator for all i and j. Then

$$\Delta(X_1, \ldots, X_n : X_1, \ldots, X_n, Y_1, \ldots, Y_m) = \Delta(X_1, \ldots, X_n) = n.$$

Proof. Note that for each j, the $n+m$-tuple

$$(0, \ldots, P_1, \ldots, 0, [Y_1, D_j]^\sigma, \ldots, [Y_m, D_j]^\sigma)$$

(P_1 in the j-th place) lies in $\partial^t_{F \cup F'}(B(L^2(M)))$, in the notation of (3.11). Thus

$$\xi_j = (0, \ldots, P_1, \ldots, 0)$$

(P_1 in the j-th place) lies in

$$K = \pi_n(\overline{\partial^t_{F \cup F'}(B(L^2(M))}^w \cap HS^{n+m}),$$

in the notation of (3.13). Since $(\xi_1, \ldots, \xi_n)$ clearly densely generate HS^n as an M, M-bimodule, it follows that the dimension of K over $M \bar{\otimes} M^o$ is exactly n. Thus $\Delta(F : F') = n$. Applying this to the case that $m = 0$ gives also the estimate for $\Delta(F : F) = \Delta(F)$. $\square$

An important case of existence of a dual system is when $X_1, \ldots, X_n$ are free semicircular variables; see [Voi98].

5. Appendix I: Abelian von Neumann algebras.

The following theorem is the analog of [Lüc98, Theorem 5.1], which makes one suspect that its statement should hold more generally if A is hyperfinite. We were unable to prove this, however. If the statement holds for A hyperfinite, it would be interesting if it can be used as a characterization of hyperfinite algebras (see Remark 5.13 in [Lüc98]).

Theorem 5.1. *Let A be a commutative von Neumann algebra and τ a normal faithful trace on A.*
(i) Let $f : (A \otimes A^o)^n \to (A \otimes A^o)^m$ be a left $A \otimes A^o$-module map, then $\beta^{(2)}(f) = 0$.

(ii) Let W be an arbitrary $A \otimes A^o$-module. Then for all $p \geq 1$,

$$\dim_{A\bar{\otimes}A^o} \mathrm{Tor}_p^{A\otimes A^o}(W,\, A\bar{\otimes}A^o) = 0,$$

Let us first prove a simple lemma,

Lemma 5.2. *Let $f \in A \otimes A^o$, then the spectral projection p of $f^* f$ corresponding to* $\ker f$ *is the supremum of the projections $e \leq p : e \in A \otimes A^o$.*

Proof. We can assume A is diffuse and identify A with $L^\infty([0,1])$, τ with the Lebesgue measure λ and $L^2(A)$ with $L^2([0,1])$. We drop the distinction between A and A^o.
Let $f = \sum g_i \otimes h_i$ and consider f as the function

$$f(x,y) = \sum_{i=1}^{k} g_i(x)\, h_i(y)\,, \quad x\,, y \in [0,1].$$

Then the projection $p \in A\bar{\otimes}A^o = L^\infty([0,1] \times [0,1])$ is given by the zero set

$$Z = \{(x,y) : f(x,y) = 0\}.$$

Recall that a point $z \in Z$ is called a point of density of Z if the proportion of Z in squares $S = I \times J$, (I and J intervals of equal length) with center z tends to 1 when their size tends to 0. By Lebesgue's a.e. differentiability theorem, the set Z differs by a set of measure zero from its set of points of density. We thus only need to prove the following, with k as above,

Claim 5.3. Let $I, J \subset [0,1]$ be intervals, and $\delta < k^{-1}$ with

$$\frac{\lambda^{\times 2}((I \times J) \cap Z)}{\lambda^{\times 2}(I \times J)} > 1 - \delta^2.$$

Then there are measurable subsets $E \subset I$, $F \subset J$, such that $E \times F \subset Z$ and

$$\frac{\lambda^{\times 2}(E \times F)}{\lambda(I \times J)} \geq (1 - k\,\delta)^2.$$

To prove the claim, let $g : I \to \mathbb{C}^k$, $h : J \to \mathbb{C}^k$ be given by $(g(x))_i = g_i(x)$, $h(x))_j = h_j(x)$ so that,

$$f(x,y) = g(x) \cdot h(y)\,, \quad \forall x\,, y \in [0,1],$$

where $\cdot$ denotes the standard scalar product on $\mathbb{C}^k$.
Let

$$E := \{x \in I : \lambda\{y : (x,y) \in Z\} > (1-\delta)\lambda(J)\}.$$

Then by Fubini's Theorem $\lambda(E) > (1-\delta)\lambda(I)$. Denote by $V(x)$ the subspace of $\mathbb{C}^k$ spanned by $g(x)$. Let $V = \mathrm{span}(V(x) : x \in E)$. Since the dimension of V is at most k, we can choose $x_1, \ldots, x_l \in E$, $l \leq k$, so that $V = \mathrm{span}(g(x_1), \ldots, g(x_l))$. For each $1 \leq j \leq l$, the set F_j of $y \in J$ for which $h(y)$ is perpendicular to $g(x_j)$ (i.e. $(x_j, y) \in Z$) has measure at least $(1-\delta)\lambda(J)$. Thus the measure of $F = \bigcap F_j$ is at least $(1 - l\delta)\lambda(J) \geq (1 - k\,\delta)\lambda(J)$. But then for all $y \in F$ and $x \in E$, $g(x) \in V$ and $h(y) \perp V$, so that $f(x,y) = 0$. It follows that $E \times F \subset Z$.

□

Proof. (of Theorem 5.1). First the above lemma implies (i) for $n = m = 1$. Indeed let $f \in A \otimes A^o$, and p the spectral projection of $f^* f$ corresponding to $\ker f$. The subspace $(A \otimes A^o) \cdot p$ is dense in $\ker f^{(2)}$. Thus since p is a strong limit of projections $e_j \in A \otimes A^o$, $p\, e_j = e_j\, p = e_j$, one gets $(A \otimes A^o) \cdot e_j \subset \ker(f)$ and the required density of $\ker f$ in $\ker f^{(2)}$.
Let now n and m be arbitrary, and reduce to $n = m$ by e.g. replacing f with $f^* f$. Let $F(x, y)$ be the matrix with entries $f_{ij}(x, y)$ and,

$$F\begin{pmatrix} i_1 \; i_2 \; \cdots \; i_k \\ j_1 \; j_2 \; \cdots \; j_k \end{pmatrix}$$

the $k \times k$ minor of F obtained by keeping the $i_1, \ldots, i_k$-th rows and $j_1, \ldots, j_k$-th columns of F. Let $Z\binom{i_1 \cdots i_k}{j_1 \cdots j_k}$ be the zero set of $F\binom{i_1 \cdots i_k}{j_1 \cdots j_k}$. Since $F\binom{i_1 \cdots i_k}{j_1 \cdots j_k}$ is a polynomial expression in the entries of F, it belongs to $A \otimes A^o$.
Let $t \in [0, 1]^2$ be such that the minors $F\binom{1}{1}$, ..., $F\binom{1 \cdots r}{1 \cdots r}$ are all non-zero, while the minors $F\binom{1 \cdots r+1}{1 \cdots r+1}$, ..., $F\binom{1 \cdots n}{1 \cdots n}$ are zero.
In this case, the equation $F\xi = 0$, $\xi = (\xi_1, \ldots, \xi_n)$ after performing Gaussian elimination has the form

$$\begin{aligned} a_{11}^{(0)}\xi_1 + a_{12}^{(0)}\xi_2 + \cdots + \cdots + a_{1n}^{(0)}\xi_n &= 0 \\ a_{22}^{(1)}\xi_2 + \cdots + \cdots + a_{2n}^{(1)}\xi_n &= 0 \\ \cdots \\ a_{rr}^{(r-1)}\xi_r + \cdots + a_{rn}^{(r-1)}\xi_n &= 0 \end{aligned}$$

where

$$a_{ik}^{(p)} = \frac{F\binom{1 \cdots p \; i}{1 \cdots p \; k}}{F\binom{1 \cdots p}{1 \cdots p}}$$

(see pp. 24–25 in [Gan60]). Thus a basis for the null space of F consists of the vectors $\eta^{(k,r)}$, $k = 1, \ldots, n - r$, with coordinates

$$\begin{aligned} \eta_t^{(k,r)} &= 0, \quad t > r, \; t \neq k + r \\ \eta_r^{(k,r)} &= -\frac{a_{rk}^{(r-1)}}{a_{rr}^{(r-1)}}, \\ \eta_{r-1}^{(k,r)} &= -\frac{a_{r-1\;k}^{(r-2)} + a_{r-1\;r}^{(r-2)}\eta_r^{(k)}}{a_{r-1\;r-1}^{(r-2)}} \\ &\cdots \\ \eta_1^{(k,r)} &= -\frac{a_{1k}^{(0)} + a_{12}^{(0)}\eta_2^{(k)} + \cdots + a_{1r}^{(0)}\eta_r^{(k)}}{a_{11}^{(0)}}. \end{aligned}$$

If we set $\xi^{(k,r)}$ to be the product of $\eta^{(k,r)}$ by a sufficiently high power of the (nonzero) expression

$$F\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdots F\begin{pmatrix} 1 \cdots r \\ 1 \cdots r \end{pmatrix},$$

we get that for each k, the $\xi_j^{(k,r)}$ are polynomials in the entries of F, and the vectors $\xi^{(k,r)}$ span the kernel of F.
The polynomial expressions $\xi_j^{(k,r)}$ in the entries of F make sense without any assumptions on F. If $F\binom{1\cdots r+1}{1\cdots r+1}$, ..., $F\binom{1\cdots n}{1\cdots n}$ are zero, the vectors $\xi^{(k,r)}$ lie in the kernel of F (although they will no longer span the kernel unless $F\binom{1}{1}$, ..., $F\binom{1\cdots r}{1\cdots r}$ are all nonzero).
By construction $\xi_j^{(k,r)} \in A \otimes A^o$ (as polynomial functions in F). Since $\xi^{(k,r)} \in \ker F(z)$ for all z such that $F\binom{1\cdots r+1}{1\cdots r+1}$, ..., $F\binom{1\cdots n}{1\cdots n}$ are zero, using Lemma 5.2 we therefore get that,

$$\zeta^{(k,r)}(F) = \xi^{(k,r)} \chi_{\left\{z : F\binom{1\cdots r+1}{1\cdots r+1}(z) = \cdots = F\binom{1\cdots n}{1\cdots n}(z) = 0\right\}} \in \overline{\ker f}$$

(here χ denotes the characteristic function of the given set).
Applying this result to the matrix $F^{\sigma,\sigma'}$ obtained from F by permuting rows via a permutation σ and columns via a permutation σ', we obtain vectors

$$\zeta^{(k,r,\sigma,\sigma')} = \sigma^{-1}(\xi^{(k,r)}(F^{\sigma,\sigma'})) \in \overline{\ker f}\,.$$

For each z, let r be the rank of $F(z)$, we can find σ, σ' so that the $\binom{1}{1}$, ..., $\binom{1\cdots r}{1\cdots r}$-minors of $F^{\sigma,\sigma'}$ are non-zero and the $\binom{1\cdots r+1}{1\cdots r+1}$, ..., $\binom{1\cdots n}{1\cdots n}$ minors are zero. Thus $\{\zeta^{(k,r,\sigma,\sigma')}(z) : 1 \le k \le n-r\}$ (and hence $\{\zeta^{(k,r,\sigma,\sigma'}(z)\}_{k,r,\sigma,\sigma'}$ span the kernel of F at z. It thus follows that $\ker f$ is dense in $\ker f^{(2)}$ which proves (i).

Finally the proof of (ii) follows verbatim the argument of Luck (Th. 5.1 [Lüc98]).

□

Corollary 5.4. *Let A be an abelian von Neumann algebra. Then for all $k \geq 1$,*

$$\beta_k^{(2)}(A) = 0.$$

Proof. By definition,

$$\beta_k^{(2)} = \dim_{A\bar{\otimes}A^o} \operatorname{Tor}_k^{A\otimes A^o}(A, A\bar{\otimes}A^o),$$

which is zero by the main result of this section. □

6. Appendix II: Dual Systems.

We recall in this appendix the construction of the dual system in the framework of section 4, and give the details of the proof of (4.7).
Let M be the von Neumann algebra generated by $X_1, \ldots, X_n$, and let $\Omega \in L^2(M)$ be the trace vector. We start by explicitly constructing the standard form of the von Neumann algebra N obtained by adjoining the free semicircular variables $S_1, \ldots, S_n$.

Consider the vector space $V = L^2(M)\otimes L^2(M)\oplus\cdots\oplus L^2(M)\otimes L^2(M) = (L^2(M)\otimes L^2(M))^n$, and let

$$H = L^2(M) \oplus V \oplus (V \otimes_M V) \oplus (V \otimes_M V \otimes_M V) \oplus \cdots$$

Note that $V \otimes_M V \cong (L^2(M) \otimes L^2(M) \otimes L^2(M))^{n^2}$.

Then M acts on H both on the right and on the left in the obvious way, acting on the leftmost or rightmost tensor copy of V each time. Denote by ϕ_i the inclusion map from $L^2(M) \otimes L^2(M)$ into V, which places $L^2(M) \otimes L^2(M)$ as the i-th direct summand.
Let ω_i be the i-th copy of $1 \otimes 1$ in V. Denote by L_i and R_i the following operators on H:

$$\begin{aligned} L_i m &= \phi_i(1 \otimes m), \quad \forall m \in L^2(M) \\ L_i v_1 \otimes \cdots \otimes v_n &= \omega_i \otimes v_1 \otimes \cdots \otimes v_n, \quad \forall v_j \in V \\ R_i m &= \phi_i(m \otimes 1), \quad \forall m \in L^2(M) \\ R_i v_1 \otimes \cdots \otimes v_n &= v_1 \otimes \cdots \otimes v_n \otimes \omega_i, \quad \forall v_j \in V. \end{aligned}$$

Formally, these are the left and right tensor multiplications by ω_i.
It is not hard to check that if we denote by λ the left action of M on H, then we have

$$L_i^* \lambda(m) L_j = \delta_{ij} \tau(m), \qquad \forall m \in M. \tag{6.1}$$

Similarly, if we denote by ρ the right action of M on H, then we have

$$R_i^* \rho(m) R_j = \delta_{ij} \tau(m), \qquad \forall m \in M.$$

In particular, $L_i^* L_i = R_i^* R_i = 1$, and these operators have norm one.
Furthermore,

$$[R_i, \lambda(m)] = [R_i^*, \lambda(m)] = 0, \qquad \forall m \in M.$$

Consider on $B(H)$ the vector state $\psi = \langle \Omega, \cdot\, \Omega \rangle$, where $\Omega \in L^2(M)$ is regarded as a vector in H. By a result from [Shl] it follows that if we let

$$S_i = L_i + L_i^*,$$

then $S_1, \ldots, S_n$ are a family of free semicircular variables, free in $(B(H), \psi)$ from $\lambda(M)$. Furthermore, the von Neumann algebra generated by M (which we identify with $\lambda(M)$) and $S_1, \ldots, S_n$ is in standard form, and the operator J is given by

$$\begin{aligned} J(v_1 \otimes \cdots \otimes v_n) &= v_n^s \otimes \cdots \otimes v_1^s, \\ J|_{L^2(M)} &= J_M, \end{aligned}$$

where $s(x \otimes y) = Jy \otimes Jx$, and $J_M$is the Tomita conjugation associated to M.
Moreover, one has

$$[R_i, S_j] = \delta_{ij} P_\Omega,$$

which means that $Q_j = R_j$ is the desired conjugate system to $S_1, \ldots, S_n$ relative to $X_1, \ldots, X_n$, i.e., it satisfies:

$$[R_i, X_j] = 0, \qquad [R_i, S_j] = \delta_{ij} P_\Omega.$$

This way, if we set $D_j(\varepsilon) = E_{W^*(X_1(\varepsilon), \ldots, X_n(\varepsilon))} \frac{1}{\sqrt{\varepsilon}} Q_j E_{W^*(X_1(\varepsilon), \ldots, X_n(\varepsilon))}$, we get that

$$[D_j(\varepsilon), X_i(\varepsilon)] = \delta_{ij}\, P_\Omega$$

which thus gives the desired dual system.

Moreover, one sees that $J(\sigma(a)Q_j\sigma(b))^*J1 = 0$, since $\sigma(a)^*J1$ lies in $L^2(M) \subset H$ and $Q_j^*L^2(M) = 0$. On the other hand,

$$\begin{aligned}\sigma(a)Q_j\sigma(b)\cdot 1 &= \sigma(a)Q_j\cdot b = \sigma(a)\cdot(\phi_j(b\otimes 1))\\ &= J\lambda(a^*)J\cdot(\phi_j(b\otimes 1))\\ &= J\lambda(a^*)\cdot\phi_j(1\otimes b^*)\\ &= J\phi_j(a^*\otimes b^*)\\ &= \phi_j(b\otimes a)\\ &= (bS_ja).\end{aligned}$$

which gives (4.7).

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