

Hecke Algebras, Type III Factors and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory

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In this paper, we construct a natural C^* -dynamical system whose partition function is the Riemann ζ function. Our construction is general and associates to an inclusion of rings (under a suitable finiteness assumption) an inclusion of discrete groups (the associated $ax + b$ groups) and the corresponding Hecke algebras of bi-invariant functions. The latter algebra is endowed with a canonical one parameter group of automorphisms measuring the lack of normality of the subgroup. The inclusion of rings $\mathbb{Z} \subset \mathbb{Q}$ provides the desired C^* -dynamical system, which admits the ζ function as partition function and the Galois group $\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ of the cyclotomic extension \mathbb{Q}^{cycl} of \mathbb{Q} as symmetry group. Moreover, it exhibits a phase transition with spontaneous symmetry breaking at inverse temperature $\beta = 1$ (cf. [Bos-C]). The original motivation for these results comes from the work of B. Julia [J] (cf. also [Spe]).

§1. Description of the system and its phase transition

Let us first recall the general discussion of quantum statistical mechanics and phase transitions. A quantum statistical system is given by

- (α) The C^* -algebra of observables: A ;
- (β) The time evolution $(\sigma_t)_{t \in \mathbb{R}}$, which is a one-parameter group of automorphisms of A .

An equilibrium, or KMS state, at inverse temperature β on (A, σ_t) is a state φ on A which fulfills the KMS_β condition with respect to σ_t , namely for any $x, y \in A$, there exists a bounded holomorphic function, (continuous on the closed strip), $F_{x,y}(z)$, $0 \leq \text{Im } z \leq \beta$ such that

$$\begin{aligned} F_{x,y}(t) &= \varphi(x\sigma_t(y)) & \forall t \in \mathbb{R} \\ F_{x,y}(t + i\beta) &= \varphi(\sigma_t(y)x) & \forall t \in \mathbb{R}. \end{aligned}$$

In the simplest case where $A = M_N(\mathbb{C})$ is the algebra of $N \times N$ matrices, any one parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$ of A is of the form

$$\sigma_t(x) = e^{itH} x e^{-itH} \quad \forall x \in A, t \in \mathbb{R}$$

for some selfadjoint element $H = H^* \in A$. Then for any $\beta \in [0, \infty[$, one has a *unique* KMS_β state for σ_t , and it is given by

$$\varphi_\beta(x) = \frac{\text{Trace}(e^{-\beta H} x)}{\text{Trace}(e^{-\beta H})} \quad \forall x \in A.$$

Note here that the additive normalization of H can be fixed by requiring that $H \geq 0$ and $0 \in \text{Sp}H$. Then the normalization factor $\text{Trace}(e^{-\beta H})$ is called the partition function of (A, σ_t) . The following formula holds:

$$\text{Log Trace}(e^{-\beta H}) = \sup_{\varphi} (S(\varphi) - \beta \varphi(H))$$

where φ varies over all states on A and $S(\varphi)$ is the entropy of the state

$$S(\varphi) = -\text{Trace}(\rho \text{Log } \rho) \quad \text{for} \quad \varphi = \text{Trace}(\rho \cdot).$$

More generally, if (A, σ_t) is any C^* -dynamical system and φ is a KMS_β such that the weak closure of A is a factor of type I, then the above discussion applies and the free energy $S(\varphi) - \beta \varphi(H)$ is well defined.

In a situation slightly more involved than that where $A = M_N(\mathbb{C})$, namely for *systems without interaction*, it is still true that for any $\beta \in [0, \infty[$ there exists a unique KMS_β state. More precisely one has the following immediate:

PROPOSITION 1. *Let $A = \bigotimes_{\nu \in I} A_\nu$ be an infinite tensor product of matrix algebras $A_\nu = M_{n_\nu}(\mathbb{C})$, and $\sigma_t = \bigotimes_{\nu \in I} \sigma_t^\nu$ a product time evolution. Then for any $\beta \geq 0$, there exists a unique KMS_β state φ_β for (A, σ_t) , and one has $\varphi_\beta = \bigotimes_{\nu} \varphi_{\beta, \nu}$, where $\varphi_{\beta, \nu}$ is the unique KMS_β state for (A_ν, σ_t^ν) .*

For interesting systems *with interaction*, one expects in general that for *large* temperature, i.e. small β , the disorder will be predominant so that there will exist only one KMS_β state. For low enough temperatures, some order should set in and allow for the existence of various thermodynamical phases, i.e., of various KMS_β states. It is a very important general fact of the C^* -algebraic formulation of quantum statistical mechanics that, for a given β , every KMS_β state decomposes *uniquely* as a statistical superposition of *extreme* KMS_β states:

PROPOSITION 2. ([Br-R] [H]) *Let (A, σ_t) be a C^* -dynamical system and $\beta \in [0, \infty[$. Then the space of KMS_β states is a compact convex Choquet simplex. The extreme points are the KMS_β factor states and, when they are of type I, yield a well defined free energy.*

For a careful discussion of the link between extreme KMS_β states and thermodynamical phases, we refer the reader to [H].

As a simple (classical) example illustrating the coexistence of phases at low temperature one can think of the phase diagram for water and vapor or better for the ferromagnet. In the latter example when the temperature T is greater than the *critical* temperature T_c , of the order of $10^3 K$, the disorder dominates, while for $T < T_c$ the individual magnets tend to align with each other, which in the classical 3-dimensional set-up yields a set of thermodynamical homogeneous phases parametrized by the 2-dimensional sphere of directions in 3 space.

This example serves to illustrate the phenomenon of *spontaneous symmetry breaking*: The group $SO(3)$ of rotations in \mathbb{R}^3 is a symmetry group of the dynamical system one starts with, and for large T , $T > T_c$, the equilibrium state is unique and hence invariant under rotation. For small T however, $T < T_c$, the group $SO(3)$ acts non trivially on the set of thermodynamical phases and the choice of an equilibrium state breaks the symmetry.

The C^* -algebraic formulation of this is straightforward. One has a (compact) group G of automorphisms of the C^* -algebra A which commutes with the time evolution

$$\alpha_g \in \text{Aut } A, \quad \forall g \in G, \quad \alpha_g \sigma_t = \sigma_t \alpha_g \quad \forall t \in \mathbb{R}.$$

Such a group obviously acts on the compact convex space of KMS_β states and hence on its extreme points.

We shall now describe (the precise motivation will be explained below) a C^* -dynamical system intimately related to the distribution of prime numbers and exhibiting the above behavior of spontaneous symmetry breaking.

The C^* -algebra A is a *Hecke algebra*, which contains the algebra of usual Hecke operators of number theory, i.e., those related to Hecke correspondences for lattices in \mathbb{C} ([Sh], [Ser₁]). The latter algebra is commutative and is essentially the algebra of composition of double cosets

$$\gamma \in GL(2, \mathbb{Z}) \backslash GL(2, \mathbb{Q}) / GL(2, \mathbb{Z}).$$

More generally, given a discrete group Γ and a subgroup Γ_0 which is *almost normal*, namely which satisfies the condition

“The orbits of Γ_0 acting on the left on Γ/Γ_0 are *finite*”,

one defines the Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$ as the convolution algebra of (\mathbb{C} -valued for our purposes) functions with finite support on $\Gamma_0 \backslash \Gamma / \Gamma_0$. More specifically, given two such functions $f, f' \in \mathcal{H}(\Gamma, \Gamma_0)$, their convolution is

$$(f * f')(\gamma) = \sum_{\gamma_1 \in \Gamma_0 \backslash \Gamma} f(\gamma \gamma_1^{-1}) f'(\gamma_1) \quad \forall \gamma \in \Gamma.$$

In this formula f and f' are viewed as Γ_0 -bi-invariant functions on Γ with finite support in $\Gamma_0 \backslash \Gamma / \Gamma_0$.

To complete \mathcal{H} to a C^* -algebra we just close it in norm in the following *regular representation* of \mathcal{H} in $\ell^2(\Gamma_0 \backslash \Gamma)$ (cf. [Bi]).

PROPOSITION 3. *Let $\Gamma_0 \subset \Gamma$ be an almost normal subgroup of the discrete group Γ . Then the following formula defines an (involutive) representation λ of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0 \backslash \Gamma)$:*

$$(\lambda(f)\xi)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f(\gamma\gamma_1^{-1}) \xi(\gamma_1) \quad \forall \gamma \in \Gamma_0 \backslash \Gamma, \quad \forall f \in \mathcal{H}.$$

One checks that $\lambda(f)$ is bounded for any $f \in \mathcal{H}(\Gamma, \Gamma_0)$. The involution on \mathcal{H} such that

$$\lambda(f^*) = \lambda(f)^* \quad \forall f \in \mathcal{H}$$

is given by the following equality:

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0.$$

Thus we let A be the C^* -algebra norm closure of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0 \backslash \Gamma)$. A good notation for it, compatible with the discrete group case is

$$\overline{\mathcal{H}}(\Gamma, \Gamma_0) = C_r^*(\Gamma, \Gamma_0).$$

Let us now define the one parameter group of automorphisms $\sigma_t \in \text{Aut } A$. We first need to introduce notation. Since each Γ_0 orbit on Γ / Γ_0 is finite we shall let, for $\gamma \in \Gamma$

$$L(\gamma) = \text{cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma / \Gamma_0$$

$$R(\gamma) = \text{cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma_0 \backslash \Gamma.$$

Thus by construction $L(\gamma) \in \mathbb{N}^*$, $R(\gamma) \in \mathbb{N}^*$, $R(\gamma) = L(\gamma^{-1})$, L and R are both Γ_0 -bi-invariant functions.

PROPOSITION 4. *Let $\Gamma_0 \subset \Gamma$ be an almost normal subgroup of the discrete group Γ . There exists a unique one parameter group of automorphisms $\sigma_t \in \text{Aut}(C_r^*(\Gamma, \Gamma_0))$ such that*

$$(\sigma_t(f))(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma) \quad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0.$$

In fact, as we shall see later, σ_{-t} is the restriction of the modular automorphism group σ_t^φ for the state on $M = \lambda(\mathcal{H})''$ given by the unit vector corresponding to the coset $\Gamma_0 \in \Gamma_0 \backslash \Gamma$.

Let us now consider the Hecke algebra \mathcal{H} for the groups:

$$\Gamma = P_{\mathbb{Q}}^+, \quad \Gamma_0 = P_{\mathbb{Z}}^+$$

where P is the group of 2×2 matrices $P = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} ; aa^{-1} = a^{-1}a = 1 \right\}$ and the $+$ indicates that we restrict to $a > 0$. One checks that $P_{\mathbb{Z}}^+$ is almost normal in $P_{\mathbb{Q}}^+$ (cf. Lemma 13).

We shall now describe the phase transition with spontaneous symmetry breaking for the dynamical system corresponding to $\Gamma = P_{\mathbb{Q}}^+, \Gamma_0 = P_{\mathbb{Z}}^+$. Let us denote by ψ_{β} the following function on the group \mathbb{Q}/\mathbb{Z} . Given $n = \frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$, with $a, b \in \mathbb{Z}$, with a relatively prime to $b > 0$, one lets

$$b = \prod_{p \in \mathcal{P}} p^{k_p}$$

be the prime factor decomposition of b and one sets

$$\psi_{\beta}(n) = \prod_{p \in \mathcal{P}, k_p \neq 0} p^{-k_p \beta} (1 - p^{\beta-1})(1 - p^{-1})^{-1}.$$

The inclusion of the unipotent subgroup

$$\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} ; n \in \mathbb{Q} \right\} \subset P_{\mathbb{Q}}^+$$

defines an imbedding $\mathbb{Q}/\mathbb{Z} \subset \Gamma_0 \backslash \Gamma / \Gamma_0$ and injective morphisms of involutive algebras

$$\mathbb{C}[\mathbb{Q}/\mathbb{Z}] \subset \mathcal{H}(\Gamma, \Gamma_0)$$

and

$$C^*(\mathbb{Q}/\mathbb{Z}) \subset C_r^*(\Gamma, \Gamma_0).$$

The main result of this paper is

THEOREM 5. *Let (A, σ_t) be the C^* -dynamical system associated to the almost normal subgroup $P_{\mathbb{Z}}^+$ of $P_{\mathbb{Q}}^+$. Then*

- (a) *For $0 < \beta \leq 1$ there exists a unique KMS_{β} state φ_{β} on (A, σ_t) . Its restriction to $C^*(\mathbb{Q}/\mathbb{Z}) \subset C_r^*(\Gamma, \Gamma_0)$ is given by the above function of positive type ψ_{β} on \mathbb{Q}/\mathbb{Z} . Each φ_{β} is a factor state and the associated factor is the hyperfinite factor of type III_1 , R_{∞} .*
- (b) *For $\beta > 1$ the KMS_{β} states on (A, σ_t) form a simplex whose extreme points $\varphi_{\beta, \chi}$ are parametrized by the complex imbeddings $\chi : \mathbb{Q}^{\text{cycl}} \rightarrow \mathbb{C}$ of the subfield*

\mathbb{Q}^{cycl} of \mathbb{C} generated by the roots of unity and whose restrictions to $C^*(\mathbb{Q}/\mathbb{Z})$ are given by the following formula:

$$\varphi_{\beta, \chi}(\gamma) = \zeta(\beta)^{-1} \sum_{n=1}^{\infty} n^{-\beta} \chi(\gamma)^n.$$

These states are type I_{∞} factor states.

(c) The partition function is the Riemann zeta function.

The normalization factor is the inverse of the Riemann ζ function evaluated at β . In other words the critical temperature here is $T_c = 1$ and at low temperature ($\beta > 1$) the phases of the system are parametrized by all possible embeddings of $K = \mathbb{Q}^{\text{cycl}}$ in the field of complex numbers.

As we shall see below the Galois group $G = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ does act naturally as a group of automorphisms of $C_r^*(\Gamma, \Gamma_0)$ commuting with the time evolution $(\sigma_t)_{t \in \mathbb{R}}$, and the spontaneous symmetry breaking occurs for $\beta > 1$.

Before we begin the proof of Theorem 5, we shall explain how the above C^* -dynamical system is related to the distribution of prime numbers.

§2. Bosonic second quantization and prime numbers as a subset of \mathbb{R}

It is a saying of E. Nelson that first quantization is a mystery while second quantization is a *functor*. In the bosonic case this functor \mathbf{S} , from the category of Hilbert spaces to itself, assigns to every Hilbert space \mathcal{H} the new Hilbert space $\mathbf{S}\mathcal{H}$ given by

$$\mathbf{S}\mathcal{H} = \bigoplus_{n=0}^{\infty} S^n \mathcal{H}$$

where $S^n \mathcal{H}$ is the n^{th} symmetric power of \mathcal{H} endowed with the following inner product:

$$\langle \xi_1 \cdots \xi_n, \eta_1 \cdots \eta_n \rangle = \sum_{\sigma} \prod_{i=1}^n \langle \xi_i, \eta_{\sigma(i)} \rangle \quad \forall \xi_j, \eta_j \in \mathcal{H}$$

(see for instance [G]). Given an operator T in \mathcal{H} (more generally $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$), the operator $\mathbf{S}T$ in $\mathbf{S}\mathcal{H}$ is given by

$$(\mathbf{S}T)(\xi_1 \cdots \xi_n) = (T\xi_1)(T\xi_2) \cdots (T\xi_n) \quad \forall \xi_i \in \mathcal{H}.$$

Even if T is bounded, $\mathbf{S}T$ is not bounded in general but if T is selfadjoint (unbounded) so is $\mathbf{S}T$. Thus we shall work with such operators. One has the following formula:

$$\text{Trace}(\mathbf{S}T) = \frac{1}{\det(1 - T)} \quad (*)$$

which makes good sense if $\|T\| < 1$ and $T \in \mathcal{L}^1(\mathcal{H})$.

The problem we shall now consider is the following: to give a simple characterization of selfadjoint operators T in \mathcal{H} whose spectrum is the subset $\mathcal{P} \subset \mathbb{R}$ formed of all prime numbers, each with multiplicity one:

$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\} \subset \mathbb{R}.$$

The corresponding problem for the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers, or $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, is easier, and was solved by Dirac's paper [Dir] which inaugurated quantum field theory. In that case the solution is simply that there exists an operator a such that

$$aa^* - a^*a = 1, \quad a^*a = T.$$

(For \mathbb{N}^* one requires that aa^* be equal to T .) Let us now state the result for the subset $\mathcal{P} \subset \mathbb{R}$:

LEMMA 6. *Let T be a selfadjoint operator in a Hilbert space \mathcal{H} ; then (counting multiplicities)*

$$\text{Spectrum } T = \mathcal{P} \Leftrightarrow \text{Spectrum } \mathbf{S}T = \mathbb{N}^*.$$

Proof. Let us first assume that $\text{Spectrum } \mathbf{S}T = \mathbb{N}^*$. Then, as quite generally $\text{Spec } T \subset \text{Spec } \mathbf{S}T$, (using the inclusion $\mathcal{H} \subset \mathbf{S}\mathcal{H}$) we see that $\Sigma = \text{Spec}(T) \subset \mathbb{N}^*$. Let us show that $\mathcal{P} \subset \Sigma$. Indeed let $p \in \mathcal{P}$ not be in Σ . Then since $\Sigma \subset \mathbb{N}^*$, one has $p \notin \Sigma^n$ for any n (with $\Sigma^n = \{k_1 k_2 \cdots k_n; k_j \in \Sigma\}$). This shows that $p \notin \text{Spec}(\mathbf{S}T) = \bigcup \Sigma^n$, whence a contradiction. Thus $\mathcal{P} \subset \Sigma$. If $k \in \Sigma \setminus \mathcal{P}$, then as $k \in \mathcal{P}^n$ for some $n > 1$, this would mean that k is not a simple eigenvalue for $\mathbf{S}T$. Thus $\mathcal{P} = \Sigma$. The converse is obvious from Euclid's unique factorization theorem, but we shall fix the corresponding notations: we let $\mathcal{H}_1 = \ell^2(\mathcal{P})$ be the Hilbert space with basis $(\varepsilon_p)_{p \in \mathcal{P}}$, and we identify it to the one particle subspace of $\mathbf{S}\mathcal{H}_1 = \ell^2(\mathbb{N}^*)$, the Hilbert space of square integrable sequences of complex numbers, with canonical basis $(\varepsilon_n)_{n \in \mathbb{N}^*}$. We shall denote by T the operator:

$$T : \ell^2(\mathcal{P}) \rightarrow \ell^2(\mathcal{P}) ; T\varepsilon_p = p\varepsilon_p \quad \forall p \in \mathcal{P}$$

and by $\mathbf{S}T$ the corresponding operator

$$\mathbf{S}T : \ell^2(\mathbb{N}^*) \rightarrow \ell^2(\mathbb{N}^*) ; (\mathbf{S}T)\varepsilon_n = n\varepsilon_n \quad \forall n \in \mathbb{N}^*.$$

We shall let $H = \log(\mathbf{S}T)$. It is the generator of a one parameter unitary group $U_t = \exp(itH) = T^{it}$, whose role is made clear by the following special case of formula (*) which is the Euler product formula for the Riemann ζ function:

$$\text{For } \text{Re } s > 1; \zeta(s) = \text{Trace}(\mathbf{S}T)^s = \frac{1}{\det(1 - T^s)}.$$

The meaning of Lemma 6 is that the subset $\mathcal{P} \subset \mathbb{R}$ has a neat definition provided one is ready to use the formalism of *bosonic quantum field theory*. That formalism includes *the algebra* of creation and annihilation operators, respectively $a^*(\xi)$ and $a(\eta)$, for $\xi, \eta \in \mathcal{H}$, given by

$$\begin{aligned} a^*(\xi)\xi_1 \cdots \xi_n &= \xi\xi_1 \cdots \xi_n \quad \forall \xi_j \in \mathcal{H} \\ a(\eta) &= (a^*(\eta))^*. \end{aligned}$$

It also includes the *time evolution*, in Heisenberg's picture, given by

$$\sigma_t(x) = U_t x U_t^* = e^{itH} x e^{-itH} \quad \forall t \in \mathbb{R}.$$

In our case the corresponding C^* -algebra in $S\mathcal{H} = \ell^2(\mathbb{N}^*)$ and time evolution are given by the following:

PROPOSITION 7.

- (a) For every $p \in \mathcal{P}$, let μ_p be the isometry in $\ell^2(\mathbb{N}^*)$ given by the polar decomposition of the creation operator associated to the unit vector $\varepsilon_p \in \mathcal{H}$. The C^* -algebra $C^*(\mathbb{N}^*)$ generated by the μ_p 's is the same as that generated by the isometries μ_n , $n \in \mathbb{N}^*$, defined by

$$\mu_n \varepsilon_k = \varepsilon_{kn} \quad \forall k \in \mathbb{N}^*.$$

- (b) This C^* -algebra is the infinite tensor product

$$C^*(\mathbb{N}^*) = \bigotimes_{p \in \mathcal{P}} \tau_p$$

where each τ_p is the C^* -algebra generated by μ_p and is the Toeplitz C^* -algebra.

- (c) The equality $\sigma_t(x) = e^{itH} x e^{-itH}$, $\forall x \in C^*(\mathbb{N}^*)$, $t \in \mathbb{R}$ where $H = \log(ST)$, defines a one parameter group of automorphisms of $C^*(\mathbb{N}^*)$ given as

$$\sigma_t = \bigotimes_{p \in \mathcal{P}} \sigma_{t,p}; \quad \sigma_{t,p}(\mu_p) = p^{it} \mu_p \quad \forall t \in \mathbb{R}.$$

Proof. (a) By construction μ_p is the one-sided shift in the Hilbert space $S \mathbb{C} \varepsilon_p$ tensored by the identity in each of the Hilbert spaces $S \mathbb{C} \varepsilon_q$, $q \neq p$ in the decomposition

$$S\mathcal{H} = \bigotimes_{q \in \mathcal{P}} S\mathbb{C} \varepsilon_q.$$

In terms of the basis (ε_n) of $S\mathcal{H}$ one thus has

$$\mu_p \varepsilon_n = \varepsilon_{pn} \quad \forall n \in \mathbb{N}$$

so that (a) follows.

(b) We recall that the Toeplitz C^* -algebra τ is the C^* -algebra defined by a unique generator u satisfying the relation $u^*u = 1$. If u is any non-unitary isometry in a (separable) Hilbert space, the smallest C^* -algebra containing u is isomorphic to τ . This C^* -algebra is nuclear so that the finite tensor products $\bigotimes_{p \leq n} \tau_p$ are unambiguously defined. The C^* -algebra $\bigotimes_{p \in \mathcal{P}} \tau_p$ is their inductive limit. Now for each p the isometry μ_p generates τ_p in $S \mathbb{C} \varepsilon_p$ and since the finite tensor products $\bigotimes_{p \leq n} \tau_p$ are faithfully represented in \mathcal{H} , we get (b).

(c) follows from a direct computation.

The C^* -dynamical system thus obtained is not very interesting because it is without interaction (see Proposition 8 (a)). Nevertheless the unique associated KMS_β states will be useful later and are given by the following corollary of Proposition 1 and of the Araki–Woods classification of ITPFI.

PROPOSITION 8.

(a) For every $\beta > 0$, there exists a unique KMS_β state on $(C^*(\mathbb{N}^*), \sigma_t)$. It is the infinite tensor product

$$\varphi_\beta = \bigotimes_{p \in \mathcal{P}} \varphi_{\beta,p}$$

where $\varphi_{\beta,p}$ is the unique KMS_β state, on $(\tau_p, \sigma_{t,p})$ for $\sigma_{t,p}$. The eigenvalue list of $\varphi_{\beta,p}$ is

$$\{(1 - p^{-\beta})p^{-n\beta} ; n \in \mathbb{N}\}.$$

(b) For $\beta > 1$, the state φ_β is of type I_∞ and is given by

$$\varphi_\beta(x) = \zeta(\beta)^{-1} \text{Trace}(e^{-\beta H} x) \quad \forall x \in C^*(\mathbb{N}^*).$$

(c) For $\beta = 1$, the state φ_β is a factor state of type III_1 given by

$$\varphi_\beta(x) = \text{Trace}_\omega(e^{-H} x) \quad \forall x \in C^*(\mathbb{N}^*)$$

where Trace_ω is the Dixmier trace.

(d) For $0 < \beta \leq 1$, φ_β is a factor state of type III_1 and the associated factor is the factor R_∞ of Araki–Woods.

Statement (d) for $\beta = 1$ is due to B. Blackadar ([Bl]). We refer to [Co] IV.2 for the definition of the Dixmier trace, whose general properties make it clear that the equality (c) defines a KMS_1 state.

Proof. (a) Let us first show that there exists a unique KMS_β state on τ_p for the group $\sigma_{t,p}$. We let τ_p be the C^* -algebra in $\ell^2(\mathbb{N})$ generated by the one-sided shift S , while $\sigma_{t,p}(S) = p^{it}S$ is the one parameter group of automorphisms. First let $\varphi_{\beta,p}$ be the restriction to τ_p of the state on $\mathcal{L}(\ell^2(\mathbb{N}))$ given by

$$\varphi(A) = \sum_{n \in \mathbb{N}} A_{n,n} \nu(n)$$

$$\text{where } \nu(n) = p^{-n\beta} \left(\sum_{m=0}^{\infty} p^{-m\beta} \right)^{-1}.$$

One checks that $\varphi_{\beta,p}$ is a KMS_β state on τ_p . Conversely, let φ be a KMS_β state on τ_p . Then the KMS_β condition shows that φ vanishes on any eigenvector $A \in \tau_p$,

$$\sigma_{t,p}(A) = \lambda^{it} A \quad \forall t \in \mathbb{R}, \quad \text{provided } \lambda \neq 1.$$

It also shows that $\varphi(SS^*) = \varphi(S^* \sigma_{-i\beta}(S)) = p^{-\beta} \varphi(S^*S) = p^{-\beta}$ so that $\varphi(1 - SS^*) = 1 - p^{-\beta}$. More generally, one has for any $k, \ell \in \mathbb{N}$:

$$\begin{aligned} \varphi(S^k S^{*\ell}) &= 0 & \text{if } k \neq \ell \\ &= p^{-k\beta} & \text{if } k = \ell. \end{aligned}$$

This shows the uniqueness of $\varphi = \varphi_{\beta,p}$ on the ideal \mathcal{K} of compact operators, $\mathcal{K} \subset \tau_p$.

Thus the difference $\psi = \varphi - \varphi_{\beta,p}$ vanishes on \mathcal{K} and is a continuous linear form on the quotient C^* -algebra $\tau_p/\mathcal{K} = C(S^1)$. We saw that $\varphi(S^n) = \varphi_{\beta,p}(S^n) = 0$ for any $n > 0$ and similarly for $(S^*)^n$ so that $\psi = 0$ and $\varphi = \varphi_{\beta,p}$.

The uniqueness of φ_β then follows by a general argument for tensor products: Let (A, σ_t^A) and (B, σ_t^B) be C^* -dynamical systems and φ a KMS_β state on $(A \otimes B, \sigma_t^A \otimes \sigma_t^B)$. Then for any $b \in B$ the functional on A

$$\varphi_b(a) = \varphi(a \otimes b)$$

is KMS_β on A . Indeed for any x, y in A one has

$$\begin{aligned} \varphi(\sigma_t^A(x) y \otimes b) &= \varphi(\sigma_t^{A \otimes B}(x \otimes 1)(y \otimes b)) \\ \varphi(y \sigma_t^A(x) \otimes b) &= \varphi((y \otimes b) \sigma_t^{A \otimes B}(x \otimes 1)). \end{aligned}$$

(b) One uses the finiteness of $\text{Trace}(e^{-\beta H})$ for $\beta > 1$.

(c) By construction one has an infinite tensor product of type I factors with eigenvalue list given by

$$\lambda_{p,n} = p^{-n\beta}(1 - p^{-\beta});$$

thus the assertion follows from [A-W].

(d) One directly checks the KMS₁ condition, and the same argument used for (c) shows that φ_1 is of type III₁.

§3. Products of trees and the non-commutative Hecke algebra

In this section we shall relate the C^* -dynamical system $(C^*(\mathbb{N}^*), \sigma_t)$, of section 2 with basic number theory notions [We₂] and get the Hecke dynamical system of Theorem 5.

Let P be the $ax + b$ group, i.e., the group of triangular 2×2 matrices of the form $\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, with a invertible. We view it as an algebraic group over \mathbb{Z} , i.e., as a functor $A \rightarrow P_A$ from commutative rings to groups. It plays an important role in the elementary classification of locally compact (commutative and non discrete) fields (cf. [We₂]). Indeed given such a field K , then the group $G = P_K$ is a *locally compact* group, and as such, it has a module

$$\delta : G \rightarrow \mathbb{R}_+^*,$$

defined by the lack of invariance of a left Haar measure dg on G under right translations:

$$d(gk) = \delta(k)dg \quad \forall k \in G \quad (1)$$

(or equivalently $d(g^{-1}) = \delta(g)^{-1}dg$ as measures on G).

This module $\delta : P_K \rightarrow \mathbb{R}_+^*$ is 1 on the additive group and its restriction to the multiplicative group (extended by 0 on $K \setminus K^*$) yields a proper continuous multiplicative map

$$\text{mod}_K : K \rightarrow \mathbb{R}_+. \quad (2)$$

In fact, if dx denotes the Haar measure on the (locally compact) additive group K , we have

$$d(ax) = \text{mod}_K(a) dx \quad \forall a \in K^*.$$

Moreover the open sets $\{k \in K; \text{mod}_K(k) < \varepsilon\}$ form a basis of neighborhoods of 0. The image of δ is a closed subgroup of \mathbb{R}_+^* and except for the case of the archimedean fields \mathbb{R} or \mathbb{C} , this closed subgroup is discrete equal to $\lambda^{\mathbb{Z}}$ for some $\lambda \in]0, 1[$ whose inverse $q = \lambda^{-1}$ is called the *module* of K . The function on $K \times K$ $d(x, y) := \text{mod}_K(x - y)$ is then a ultrametric distance giving back the topology of K ([We₂]). In other words, we have:

PROPOSITION 9. (cf. [We₂]) *Let K be a non-discrete commutative locally compact field, $K \neq \mathbb{R}$ or \mathbb{C} . Then there exists a prime p such that $\text{mod}_K(p) < 1$. Call R , R^* and P the subsets of K respectively given by*

$$\begin{aligned} R &= \{x \in K ; \text{mod}_K(x) \leq 1\}, \\ R^* &= \{x \in K ; \text{mod}_K(x) = 1\}, \\ J &= \{x \in K ; \text{mod}_K(x) < 1\}. \end{aligned}$$

Then R is the unique maximal compact subring of K ; R^ is the group of invertible elements of R ; J is the unique maximal ideal of R , and there is $\pi \in J$ such that $J = \pi R = R\pi$. The topology on the topological group K is the unique (ultrametric) topology such that the ideals $\pi^n R$ form a basis of neighborhood of 0. Moreover, the residue field $k = R/J$ is a finite field of characteristic p ; if q is the number of its elements, the image of K^* in \mathbb{R}_+^* under mod_K is the subgroup of \mathbb{R}_+^* generated by q .*

Given $x \in K$ the integer $v(x)$ such that $\text{mod}_K(x) = q^{-v(x)}$ is called the *valuation* of x .

As a basic example the field \mathbb{Q}_p of p -adic numbers is defined (given any prime number p), as the completion of the field \mathbb{Q} of rational numbers for the distance function:

$$d(x, y) = |x - y|_p \quad (3)$$

where for $x \in \mathbb{Q}$, $x = p^n \frac{a}{b}$ (with n, a, b integers and a, b relatively prime to p) one sets

$$|x|_p = p^{-n}. \quad (4)$$

The maximal subring R of $K = \mathbb{Q}_p$ is the ring \mathbb{Z}_p of p -adic integers and the residual field $k = R/J$ is the finite field \mathbb{F}_p .

One obtains in this way, together with the inclusion $\mathbb{Q} \subset \mathbb{R}$, all the inclusions $\mathbb{Q} \subset K$ of the field of rational numbers as a dense subfield of a local field K . Such inclusions (or rather in general, equivalence classes of completions) are called *places* and to distinguish the real place $\mathbb{Q} \subset \mathbb{R}$ from the others, the latter ones are called *finite places*.

Putting together the inclusions of \mathbb{Q} in its completions $\mathbb{Q}_v = K$ parametrized by the places of \mathbb{Q} , one gets a single inclusion of \mathbb{Q} in the *locally compact* commutative ring of *adeles* which is the restricted product of the fields \mathbb{Q}_v . More specifically this ring is the product $\mathbb{R} \times \mathcal{A}$ where the ring \mathcal{A} of *finite adeles* is obtained as follows:

- (a) Elements x of \mathcal{A} are arbitrary families $(x_p)_{p \in \mathcal{P}}$, with $x_p \in \mathbb{Q}_p$, such that $x_p \in \mathbb{Z}_p$ for all p but a finite number.
- (b) $(x + y)_p = x_p + y_p$; $(xy)_p = x_p y_p$ define the addition and product in \mathcal{A} .

- (c) Finally \mathcal{A} has the unique topology of a locally compact ring such that the subring

$$\mathcal{R} = \prod_{p \in \mathcal{P}} \mathbb{Z}_p$$

is open and closed and inherits its compact product topology.

We shall now relate the C^* -dynamical system $(C^*(\mathbb{N}^*), \sigma_t)$ of section 2 with the locally compact ring \mathcal{A} of finite adeles.

We just need to recall that, given any locally compact group G with modular function $\delta : G \rightarrow \mathbb{R}_+^*$, one has a natural one parameter group of automorphisms σ_t of $C^*(G)$ given by the following formula valid on $L^1(G)$:

$$(\sigma_t(f))(g) = \delta(g)^{-it} f(g) \quad \forall g \in G, t \in \mathbb{R}. \quad (5)$$

This automorphism group defines also an automorphism group of the reduced C^* -algebra $C_r^*(G)$, and the group σ_{-t} is the modular automorphism group of the Plancherel weight on $C^*(G)$.

PROPOSITION 10. *Let \mathcal{A} be the ring of finite adeles over \mathbb{Q} , and \mathcal{R} its maximal (open) compact subring. Let G be the locally compact group $G = P_{\mathcal{A}}$, and $e \in C^*(G)$ the characteristic function of the open and compact subgroup $P_{\mathcal{R}} \subset P_{\mathcal{A}}$. Then:*

- (1) *One has $e = e^* = e^2$, and the reduced C^* -algebra $C^*(G)_e = \{x \in C^*(G) ; ex = xe = x\}$ is canonically isomorphic to the C^* -algebra $C^*(\mathbb{N}^*)$ of section 2.*
- (2) *One has $\sigma_t(e) = e \quad \forall t \in \mathbb{R}$, and the restriction of σ_t to the reduced C^* -algebra $C^*(\mathbb{N}^*)$ is the time evolution of section 2.*

We think of the characteristic function of $P_{\mathcal{R}}$ as an element of $L^1(G, dg) \subset C^*(G)$, with dg the unique left Haar measure which gives measure 1 to $P_{\mathcal{R}}$. The group G is solvable and hence amenable so that there is no distinction between $C^*(G)$ and the reduced C^* -algebra $C_r^*(G)$.

The proof of Proposition 10 reduces immediately to a *local* statement, namely, if $K = \mathbb{Q}_p$ and $R \subset K$ is the maximal compact subring, the C^* -dynamical system $(C^*(P_K), \sigma_t)$ given by (5), reduced by the projection e_p defined as the characteristic function of P_R , is isomorphic to the Toeplitz C^* -algebra τ_p , with the time evolution $\sigma_{t,p}$ of Proposition 7 (c).

To check this one uses the isomorphisms

$$C^*(P_K) \cong C^*(K) \rtimes K^* \cong C_0(K) \rtimes K^*$$

given by the identification of the additive group K with its Pontrjagin dual (cf. [We₂]; we use in the sequel the same normalizations for the Haar measure and the Fourier transform on K as in *loc.cit.*, namely the additive Haar measure of $R = \mathbb{Z}_p$ is 1, and the identification of K with its Pontrjagin dual is such that

$R^\perp = R$). Then to e_p corresponds the element of the crossed product given by $\int_{R^*} 1_R U_g dg$. The reduced C^* -algebra $C^*(P_K)_{e_p}$ is then generated by the isometry μ_p , $\mu_p \in C^*(P_{\mathbb{Q}_p})_{e_p}$, given by the following L^1 function:

$$\mu_p \left(\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \right) = 1 \text{ if } b \in R, \text{ val}(a) = 1, \text{ and is equal to 0 otherwise.} \quad (6)$$

Let us check that μ_p is an isometry, i.e., that $\mu_p^* \mu_p = e_p$. The left Haar measure on P_K is given by $d \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} = db d^*a$ where d^*a is the multiplicative Haar measure normalized so that $\int_{R^*} d^*a = 1$. The module δ of the locally compact group P_K is given by $\delta \left(\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \right) = |a|$. The adjoint μ_p^* of μ_p is given by the function

$$\mu_p^*(g) = \overline{\mu_p(g^{-1})} \delta(g^{-1}). \quad (7)$$

Thus the convolution $\mu_p^* * \mu_p$ is given by the integral

$$(\mu_p^* * \mu_p)(g) = \int_{P_K} \overline{\mu_p(g_1)} \mu_p(g_1 g) dg_1. \quad (8)$$

This vanishes unless $g \in P_R$ as can be seen using $g = g_1^{-1} g_2$ for $\mu_p(g_i) = 1$. With $g_i = \begin{bmatrix} 1 & b_i \\ 0 & a_i \end{bmatrix}$ one gets $g_1^{-1} g_2 = \begin{bmatrix} 1 & -a_1^{-1}b_1 \\ 0 & a_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & b_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & b_2 - a_2 a_1^{-1} b_1 \\ 0 & a_1^{-1} a_2 \end{bmatrix} \in P_R$. Moreover the integral $\int_{P_K} \mu_p(g) dg$ is equal to 1 so that one gets $\mu_p^* * \mu_p = e_p$.

One has $\sigma_t(\mu_p)(g) = 0$ unless $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, $\text{val}(a) = 1$, $b \in R$, and for such g 's one has $\sigma_t(\mu_p)(g) = \delta(g)^{-it} \mu_p(g) = |a|^{-it} \mu_p(g) = p^{it} \mu_p(g)$. Thus $\sigma_t(\mu_p) = p^{it} \mu_p$.

We may also write the state $\varphi_{\beta,p}$ on $C^*(P_K)_{e_p}$ in terms of bi-invariant functions. One obtains the following identity for any $f \in C^*(P_K)_{e_p}$:

$$\varphi_{\beta,p}(f) = f \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + \left(\sum_{k \geq 0} p^{k(1-\beta)} f \left[\begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right] \right) (1 - p^{\beta-1}). \quad (9)$$

(Observe that the elements of $C^*(P_K)$ may be identified with some P_R -bi-invariant functions in $L^2(P_K)$, and therefore to some locally constant functions on P_K ; in particular they have well defined values on points of P_K .) According to Proposition 8 (a) and its proof, to prove (9) we only need to check that the linear functional on $C_c(P_K)_{e_p}$, defined by the right-hand side of (9), satisfies the KMS $_\beta$ condition; this follows from a straightforward computation.

The C^* -dynamical system $(C^*(P_A), \sigma_t)$ of Proposition 10 is without interaction, exactly as is $(C^*(\mathbb{N}^*), \sigma_t)$ (Proposition 8), and there is an exact analogue of

Proposition 8, which states the existence and uniqueness (up to scale) of KMS_β weights on the above system. One needs to use weights because one is dealing with non-unital C^* -algebras. At the technical side such weights have to be semi-continuous and semi-finite (for the norm topology) (cf. [Com]). It is however instructive to work out the explicit formula for those KMS_β weights. Using the natural isomorphism of the Pontrjagin dual $\hat{\mathcal{A}}$ of the additive group \mathcal{A} with itself (cf. [We₂]), one gets an isomorphism

$$C^*(P_{\mathcal{A}}) = C_0(\mathcal{A}) \rtimes \mathcal{A}^* \quad (10)$$

where the multiplicative group \mathcal{A}^* of finite ideles acts by homotheties in the locally compact space \mathcal{A} . The KMS_β weight on $C^*(P_{\mathcal{A}})$, σ_t is then the dual weight of the following measure μ_β on \mathcal{A} :

$$\mu_\beta(f) = \zeta(\beta)^{-1} \int_{\mathcal{A}^*} |j|^\beta f(j) d^*j. \quad (11)$$

Here d^*j is the Haar measure on the multiplicative group \mathcal{A}^* , $j \rightarrow |j|$ is the module, and the formula makes sense as such for $\beta > 1$, and by analytic continuation for $0 < \beta < 1$ (cf. [T], [We₁], [We₂]).

It is clear that to obtain a C^* -dynamical system *with interaction* we need to use not only the locally compact ring \mathcal{A} but also the fundamental inclusion

$$\mathbb{Q} \subset \mathcal{A}. \quad (12)$$

We shall use the corresponding inclusion $P_{\mathbb{Q}} \subset P_{\mathcal{A}} = G$ and the action of $P_{\mathcal{A}}$ on the C^* -module $\mathcal{E} = C^*(G)e$ over $C^*(\mathbb{N}^*)$ given by the isomorphism of Proposition 10: $C^*(G)_e = C^*(\mathbb{N}^*)$. Indeed, given any C^* -algebra B and (self adjoint) projection $e \in B$, the space $\mathcal{E} = Be = \{x \in B ; xe = x\}$ is in a natural way a (right) C^* -module over the reduced C^* -algebra $B_e = \{x \in B ; ex = xe = x\}$. Thus we let

$$\langle \xi, \eta \rangle = \xi^* \eta \in B_e, \quad \forall \xi, \eta \in \mathcal{E} = Be \quad (13)$$

$$\xi a \in \mathcal{E}, \quad \forall \xi \in \mathcal{E}, \quad a \in B_e.$$

This C^* -module has moreover a natural left B -module structure, given by $(b, \xi) \rightarrow b\xi \in \mathcal{E}$, $\forall b \in B$, $\xi \in \mathcal{E}$.

In our case $\mathcal{E} = C^*(G)e$ is a space of functions on G which are invariant by right multiplications by elements of $P_{\mathcal{R}} \subset G$, or in other words, it is a space of functions on the homogeneous space

$$\Delta = G/P_{\mathcal{R}}. \quad (14)$$

This space Δ is by construction the restricted product of the spaces

$$\Delta_p = P_{\mathbb{Q}_p}/P_{\mathbb{Z}_p} \quad (15)$$

relative to the base point given by $P_{\mathbb{Z}_p}$.

PROPOSITION 11. *The homogeneous space $\Delta_p = P_{\mathbb{Q}_p}/P_{\mathbb{Z}_p}$ over the group $P_{\mathbb{Q}_p} \subset GL(2, \mathbb{Q}_p)$ is naturally isomorphic to the (set of vertices of the) tree of $SL(2, \mathbb{Q}_p)$. The group $P_{\mathbb{Q}_p}$ acts by isometries of Δ_p and preserves a point at ∞ .*

Let us recall (cf. [Ser₂]) that the tree of $SL(2, K)$, where K is a local field, is defined in terms of equivalence classes of lattices in a two-dimensional vector space V over K . With the notations of Proposition 9, a lattice $L \subset V$ is a sub R -module of V which is of finite type and generates V as a vector space. The multiplicative group K^* operates on the set of lattices by $(L, x) \rightarrow xL$ for $x \in K^*$, and one lets T be the set of orbits of this action of K^* . Given a lattice $L \subset V$ and a class $\Lambda' \in T$ there exists a unique representative $L' \in \Lambda'$ such that $L' \subset L$ and $L' \not\subset \pi L$ (with π given by Proposition 9). Then $L/L' = R/\pi^n R$ and the integer n , which only depends upon the classes of L and L' , defines a distance d on T , by the equality

$$d(\text{class of } L, \text{ class of } L') = n. \quad (16)$$

Using the set of pairs with mutual distance equal to 1 to define a 1-dimensional simplicial complex, one gets a tree, the tree of $SL(2, K)$, and the above distance is the length of the unique shortest injective path joining two elements of this tree (cf. [Ser₂]). The group $GL(V)$ of automorphisms of the vector space V acts on the set of lattices by

$$(L, g) \rightarrow gL \quad \forall g \in GL(V), \quad (17)$$

and since this action commutes with that of K^* it gives an action, by isometries, of $GL(V)$ on the tree T . Let us identify V with K^2 , $GL(V)$ with $GL(2, K)$, and consider P_K as a subgroup of $GL(2, K) : P_K = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} ; a \in K^*, b \in K \right\}$. Let L_0 be the lattice $R^2 \subset K^2$. Then one checks that P_K acts transitively on T and that the stabilizer of the class of L_0 is P_R . We thus get a canonical identification $T = P_K/P_R$. Taking $K = \mathbb{Q}_p$ yields the conclusion.

PROPOSITION 12.

- (1) *The homogeneous space $G/P_{\mathcal{R}} = \Delta$ equipped with the base point $P_{\mathcal{R}}$ is canonically isomorphic to the restricted product of the trees T_p equipped with the base points $P_{\mathbb{Z}_p}$, and the action of G on Δ is simplicial.*
- (2) *The subgroup $P_{\mathbb{Q}}^+ \subset P_A = G$ acts transitively on Δ , and the isotropy subgroup of the base point $*$ is $P_{\mathbb{Z}}^+$.*

Proof. (1) Since both P_A and P_R are restricted products, the proof of (1) is straightforward.

(2) First \mathbb{Q} is dense in the additive group \mathcal{A} of adeles (with the ∞ place removed), (cf. [Ser₁]).

The subgroup \mathbb{Q}_+^* of \mathcal{A}^* is discrete and one has $\mathcal{A}^* = \mathbb{Q}_+^* \mathcal{R}^*$ where $\mathcal{R}^* = \{(x_p) ; \text{val}(x_p) = 0 \quad \forall p\}$ is a compact subgroup of \mathcal{A}^* (cf. [We₂]).

Consider the exact sequence of groups

$$0 \rightarrow \mathcal{A} \rightarrow P_{\mathcal{A}} \xrightarrow{\rho} \mathcal{A}^* \rightarrow 1. \quad (18)$$

The closure of $P_{\mathbb{Q}}^+$ in $P_{\mathcal{A}}$ is given by $\rho^{-1}(\mathbb{Q}_+^*)$. Thus $\overline{P_{\mathbb{Q}}^+} = \mathcal{A} \rtimes \mathbb{Q}_+^*$. Given $g = \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} \in P_{\mathcal{A}}$ one can write $h = rs$ with $r \in \mathbb{Q}_+^*$, $s \in \mathcal{R}^*$, then

$$g = \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} = \begin{bmatrix} 1 & ns^{-1} \\ 0 & r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \in \overline{P_{\mathbb{Q}}^+} P_{\mathcal{R}}.$$

Thus $g \in \overline{P_{\mathbb{Q}}^+} P_{\mathcal{R}}$ and since $P_{\mathcal{R}}$ is open in $P_{\mathcal{A}}$, we get $g \in P_{\mathbb{Q}}^+ P_{\mathcal{R}}$. Thus $P_{\mathcal{A}} = P_{\mathbb{Q}}^+ P_{\mathcal{R}}$ and $P_{\mathbb{Q}}^+$ acts transitively on Δ .

Finally the isotropy subgroup of the base point $* = P_{\mathcal{R}}$ is given by $P_{\mathbb{Q}}^+ \cap P_{\mathcal{R}} = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} ; n \in \mathbb{Z} \right\} = P_{\mathbb{Z}}^+$.

We can thus identify Δ with $P_{\mathbb{Q}}^+ / P_{\mathbb{Z}}^+$, and we shall now get the Hecke algebra of Theorem 5 from the commutant of the action of $P_{\mathbb{Q}}^+$ in the space of functions on Δ . We need for that purpose to consider the Hilbert space $\ell^2(\Delta) = \ell^2(P_{\mathbb{Q}}^+ / P_{\mathbb{Z}}^+)$. Let $\Gamma = P_{\mathbb{Q}}^+$, $\Gamma_0 = P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$. Let us check first

LEMMA 13. *The action of $\Gamma_0 = P_{\mathbb{Z}}^+$ on Γ / Γ_0 only has finite orbits.*

Proof. Let $g = \begin{bmatrix} 1 & a \\ 0 & k \end{bmatrix} \in P_{\mathbb{Q}}^+$. Then $g \Gamma_0 =$

$$\left\{ g \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} ; n \in \mathbb{Z} \right\} = \left\{ \begin{bmatrix} 1 & n+a \\ 0 & k \end{bmatrix} ; n \in \mathbb{Z} \right\}.$$

One has

$$\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+a+n_1 k \\ 0 & k \end{bmatrix}.$$

When n_1 varies, the $n_1 k$ only take finitely values modulo \mathbb{Z} , their number depending upon the denominator of k . Thus we see that we get indeed a finite orbit of cardinality the denominator of k .

LEMMA 14.

- (a) Any element T of the commutant of Γ acting in $\ell^2(\Gamma/\Gamma_0)$ is characterized by the Γ_0 -bi-invariant function $f_T(g) = \langle T \varepsilon_e, g^{-1} \varepsilon_e \rangle$.
- (b) One has $\sum_{g \in \Gamma_0 \backslash \Gamma} |f_T(g)|^2 < \infty$.
- (c) One has $f_{T^*}(g) = \overline{f_T}(g^{-1})$.
- (d) The Γ_0 -bi-invariant function $f_{T_1 T_2}$ associated to the product of two elements T_1 and T_2 in this commutant is given by

$$f_{T_1 T_2}(g) = \sum_{\Gamma/\Gamma_0} f_{T_1}(gg_1) f_{T_2}(g_1^{-1}).$$

Proof. (a) Each element of Γ acts by permutation of the basis ε_x , $x \in \Gamma/\Gamma_0$ of $\mathcal{H} = \ell^2(\Gamma/\Gamma_0)$. Since T commutes with Γ it is characterized by $T\varepsilon_e$ which is determined by f_T . One has $f_T(g) = \langle Tg \varepsilon_e, \varepsilon_e \rangle$ so that f_T is Γ_0 -bi-invariant.

(b) Obvious.

$$(c) f_{T^*}(g) = \langle T^* \varepsilon_e, g^{-1} \varepsilon_e \rangle = \langle g \varepsilon_e, T \varepsilon_e \rangle = \overline{\langle T \varepsilon_e, g \varepsilon_e \rangle}$$

$$\begin{aligned} (d) \quad f_{T_1 T_2}(g) &= \langle T_1 T_2 \varepsilon_e, g^{-1} \varepsilon_e \rangle = \langle T_2 \varepsilon_e, g^{-1} T_1^* \varepsilon_e \rangle \\ &= \sum_{g_1 \in \Gamma/\Gamma_0} \langle T_2 \varepsilon_e, g_1 \varepsilon_e \rangle \langle g_1 \varepsilon_e, g^{-1} T_1^* \varepsilon_e \rangle \\ &= \sum_{g_1 \in \Gamma/\Gamma_0} f_{T_1}(gg_1) f_{T_2}(g_1^{-1}). \end{aligned}$$

One can write (d) as

$$f_{T_1 T_2}(g) = \sum_{g_1 \in \Gamma/\Gamma_0} f_{T_1}(gg_1^{-1}) f_{T_2}(g_1).$$

Condition (b) shows that $f_T(g) = 0$ unless there exists a finite set $F \subset \Gamma$ with

$$\Gamma_0 g \Gamma_0 \subset F \Gamma_0$$

i.e., if $g \Gamma_0$ belongs to a *finite* orbit of Γ_0 on Γ/Γ_0 . By Lemma 13 we can take a basis of Γ_0 -bi-invariant functions, e_X , where X runs through the double classes $X \in \Gamma_0 \backslash \Gamma/\Gamma_0$, $e_X(g) = 0$ if $g \notin X$, $e_X(g) = 1$ if $g \in X$. For each such class one has two associated finite integers:

$$L(X) = \text{cardinality of the image of } X \text{ in } \Gamma/\Gamma_0$$

(19)

$$R(X) = \text{cardinality of the image of } X \text{ in } \Gamma_0 \backslash \Gamma.$$

In fact to each double class X corresponds the *positive* rational number given by k for any $\begin{bmatrix} 1 & a \\ 0 & k \end{bmatrix} \in X$, and $L(X)$ is the denominator of k , while $R(X)$ is the numerator of k .

PROPOSITION 15.

(a) Let f be a Γ_0 -bi-invariant function on Γ with finite support in $\Gamma_0 \backslash \Gamma / \Gamma_0$. Then there exists a unique element $r(f)$ of the commutant of Γ in $\ell^2(\Gamma / \Gamma_0)$ such that

$$f(g) = \langle r(f) \varepsilon_e, g^{-1} \varepsilon_e \rangle \quad \forall g \in \Gamma.$$

(b) The map r extends to an isomorphism of $C_r^*(\Gamma, \Gamma_0)$ with the C^* -algebra $C = C_{\mathbb{Q}}$ generated by the $r(f)$ in $\ell^2(\Delta)$.

Proof. (a) Let X be a double class, $X \in \Gamma_0 \backslash \Gamma / \Gamma_0$ and e_X the corresponding Γ_0 -bi-invariant function. Let T be the matrix given by

$$\langle T g_1 \varepsilon_e, g_2 \varepsilon_e \rangle = e_X(g_2^{-1} g_1).$$

One has, by Lemma 13, the bounds

$$\sup_{\alpha} \sum_{\beta} |T_{\alpha, \beta}| \leq L(X), \quad \sup_{\beta} \sum_{\alpha} |T_{\alpha, \beta}| \leq R(X)$$

which shows that T defines a bounded operator in $\ell^1(\Delta)$ and in $\ell^\infty(\Delta)$, hence in $\ell^2(\Delta)$. The uniqueness is clear.

(b) follows from Lemma 14 (d) and the definitions of $\mathcal{H}(\Gamma, \Gamma_0)$ and $C_r^*(\Gamma, \Gamma_0)$ (cf. Proposition 3).

Next we let φ be the state on $C_{\mathbb{Q}}$ defined by

$$\varphi(T) = \langle T \varepsilon_e, \varepsilon_e \rangle. \quad (20)$$

LEMMA 16. φ is a KMS_1 state on C with respect to the one parameter group σ_t ,

$$\sigma_t(e_X) = k^{it} e_X, \quad k = \frac{R(X)}{L(X)}.$$

Proof. The product of Γ_0 -bi-invariant functions is given by

$$(f_1 * f_2)(g) = \sum_{g_1 \in \Gamma / \Gamma_0} f_1(g_1) f_2(g_1^{-1} g).$$

One has

$$\varphi(f_1 * f_2) = (f_1 * f_2)(e) = \sum_{g_1 \in \Gamma/\Gamma_0} f_1(g_1) f_2(g_1^{-1})$$

$$\varphi(e_X * f) = \sum_{g_1 \in \Gamma/\Gamma_0} e_X(g_1) f(g_1^{-1})$$

$$\varphi(f * e_X) = \sum_{g_1 \in \Gamma/\Gamma_0} f(g_2) e_X(g_2^{-1}).$$

Let $g \in \Gamma$ with $X = \Gamma_0 g \Gamma_0$ a fixed double class; then as f is Γ_0 -bi-invariant, the two above expressions are multiples of $f(g^{-1})$:

$$\varphi(e_X * f) = L(X) f(g^{-1}), \quad \varphi(f * e_X) = R(X) f(g^{-1}).$$

Thus $\varphi(e_X * f) = \frac{L(X)}{R(X)} \varphi(f * e_X)$. This shows that e_X is an eigenvector for the modular automorphism group of the faithful normal state

$$T \rightarrow \langle T\varepsilon_e, \varepsilon_e \rangle$$

on the weak closure C'' of C acting in $\mathcal{H} = \ell^2(\Gamma/\Gamma_0)$. It thus follows that φ is a KMS_1 state for the group $\sigma_t = \sigma_{-t}^\varphi$. The vector $\varepsilon_e \in \ell^2(\Delta)$ is still separating for the weak closure of $C_{\mathbb{Q}}$ because it is cyclic for $P_{\mathbb{Q}}^+$ by Proposition 12 (2). The subspace \mathcal{H}_r of $\ell^2(\Delta)$ given by

$$\mathcal{H}_r = \overline{C_{\mathbb{Q}} \varepsilon_e}$$

is in the bicommutant of the action of $P_{\mathbb{Q}}^+$ on Δ and we shall compute it.

LEMMA 17. *Let $\xi \in \ell^2(\Delta)$; then $\xi \in \mathcal{H}_r$ iff ξ is fixed by $\Gamma_0 \subset P_{\mathbb{Q}}^+$ acting on the left on $\Delta = P_{\mathbb{Q}}^+/\Gamma_0$.*

Proof. Let $\xi \in \mathcal{H}$ be fixed under Γ_0 . Then the function $f(g) = \langle \xi, \varepsilon_g \rangle$, $g \in \Gamma/\Gamma_0$ is Γ_0 -bi-invariant. To show that $\xi \in \mathcal{H}_r$ we can assume, using an orthogonal decomposition, that f is the characteristic function of a double class, $f = e_X$. We get as above that $e_X^* \varepsilon_e = \sum_{g \in F} g \varepsilon_e$ where $X = F\Gamma_0$ so that all the characteristic functions of double classes belong to \mathcal{H}_r . Conversely if $\xi \in \mathcal{H}_r$ and $\xi = T \varepsilon_e$ for some operator T commuting with $P_{\mathbb{Q}}^+$, then as T commutes with $\Gamma_0 \subset P_{\mathbb{Q}}^+$ and $g \varepsilon_e = \varepsilon_e \quad \forall g \in \Gamma_0$, one gets that ξ is fixed by Γ_0 .

§4. Presentation of the C^* -algebra $C_{\mathbb{Q}} = C_r^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$

Let us first consider the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, \Gamma_0)$ where $\Gamma = P_{\mathbb{Q}}^+$, $\Gamma_0 = P_{\mathbb{Z}}^+$ are as above. It is an involutive algebra over \mathbb{C} with a linear basis e_X , indexed by the double cosets $X \in \Gamma_0 \backslash \Gamma / \Gamma_0$. We shall use the following notations:

- (α) For $n \in \mathbb{N}^*$, $\mu_n = n^{-1/2} e_{X_n}$ where X_n is the double class of the element $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ of $P_{\mathbb{Q}}^+$.
- (β) For $\gamma \in \mathbb{Q}/\mathbb{Z}$, $e(\gamma) = e_{X^\gamma}$ where X^γ is the double class of the element $\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ of $P_{\mathbb{Q}}^+ / P_{\mathbb{Z}}^+$.

PROPOSITION 18. *The elements $\mu_n, e(\gamma)$, $n \in \mathbb{N}^*$, $\gamma \in \mathbb{Q}/\mathbb{Z}$ generate the involutive algebra \mathcal{H} and the following relations give a presentation of \mathcal{H} .*

- (a) $\mu_n^* \mu_n = 1 \quad \forall n$
- (b) $\mu_{nm} = \mu_n \mu_m \quad \forall n, m$
- (c) $\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{if } (n, m) = 1$
- (d) $e(\gamma)^* = e(-\gamma)$, $e(\gamma_1 + \gamma_2) = e(\gamma_1) e(\gamma_2) \quad \forall \gamma, \gamma_1, \gamma_2$
- (e) $e(\gamma) \mu_n = \mu_n e(n\gamma) \quad \forall n, \forall \gamma$
- (f) $\mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta=\gamma} e(\delta) \quad \forall n, \forall \gamma.$

Proof. We first have to check that the relations (a)–(f) are fulfilled using the definitions of the product and involution in \mathcal{H} , namely:

$$f_1 * f_2(g) = \sum_{g_1 \in \Gamma / \Gamma_0} f_1(g_1) f_2(g_1^{-1}g) \quad (1)$$

$$f^*(g) = \overline{f(g^{-1})}. \quad (2)$$

For $n \in \mathbb{N}^*$, the right class $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \Gamma_0$ is already a double class: X_n . This shows that for any Γ_0 -bi-invariant function $f \in \mathcal{H}$:

$$n^{1/2}(\mu_n * f)(g) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & n^{-1} \end{bmatrix} g\right) \quad \forall g \in \Gamma. \quad (3)$$

Similarly, the right class $\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \Gamma_0$, $\gamma \in \mathbb{Q}/\mathbb{Z}$ is already a double class X^γ so that

$$(e(\gamma) * f)(g) = f\left(\begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} g\right) \quad \forall g \in \Gamma \quad (4)$$

and using the equality $\Gamma_0 \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} = X^\gamma$,

$$(f * e(\gamma))(g) = f \left(g \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} \right) \quad \forall g \in \Gamma. \quad (5)$$

Using (3), (4), (5) one proves directly the relations (b), (d) and (e). The left multiplication by μ_n^* is given by

$$n^{1/2}(\mu_n^* * f)(g) = \sum_{k=0}^{n-1} f \left(\begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} g \right) \quad \forall g \in \Gamma \quad (6)$$

where the bi-invariance of f shows that $f \left(\begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} g \right)$ only depends on k modulo n , since

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} = \begin{bmatrix} 1 & k + nb \\ 0 & n \end{bmatrix}.$$

The equality (a) follows directly from (6). Let n, m be integers such that $(n, m) = 1$; using (3) one gets that the bi-invariant function $n^{1/2}m^{1/2} \mu_n^* \mu_m^*$ is the characteristic function of the Γ_0 double coset: $\begin{bmatrix} 1 & \mathbb{Z}/m \\ 0 & n/m \end{bmatrix}$. Using (6) one gets that

$$m^{1/2} \mu_m^* * \mu_n(g) = \sum_{k=0}^{m-1} \mu_n \left(\begin{bmatrix} 1 & k \\ 0 & m \end{bmatrix} g \right).$$

Let $g = \begin{bmatrix} 1 & \beta \\ 0 & \alpha \end{bmatrix} \in \Gamma$. Then the $(k+1)$ -st term on the right-hand side vanishes unless $\begin{bmatrix} 1 & k \\ 0 & m \end{bmatrix} g \in \begin{bmatrix} 1 & \mathbb{Z} \\ 0 & n \end{bmatrix}$, i.e., unless $\alpha = n/m$, $\beta + \frac{nk}{m} \in \mathbb{Z}$. Thus the left-hand side vanishes unless $\beta \in \mathbb{Z}/m$ and is equal to $n^{-1/2}$ if $\beta \in \mathbb{Z}/m$ since, as $(n, m) = 1$, only one value of k will contribute to the sum. This proves the relation (c). To prove (f) one combines (3) and (4) which gives

$$n^{1/2}(\mu_n^* * e(\gamma) * f)(g) = f \left(\begin{bmatrix} 1 & -\gamma n^{-1} \\ 0 & n^{-1} \end{bmatrix} g \right) \quad \forall g \in \Gamma$$

which one applies to $f = \mu_n^*$. One has $f(g) = n^{-1/2}$ if $g \in \begin{bmatrix} 1 & \mathbb{Z}/n \\ 0 & 1/n \end{bmatrix}$ and $f(g) = 0$ otherwise. This shows that $(\mu_n^* * e(\gamma) * \mu_n^*)(g)$ vanishes unless $g = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ with

$$\begin{bmatrix} 1 & -\gamma n^{-1} \\ 0 & n^{-1} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} 1 & \mathbb{Z}/n \\ 0 & n^{-1} \end{bmatrix}$$

i.e., $n\beta = \gamma$ modulo n . Since this equation has n solutions $\beta \in \mathbb{Q}/\mathbb{Z}$ one gets (f).

We have thus proved the relations (a)–(f). Conversely let \mathcal{A} be an involutive algebra generated by elements (μ_n) , (e_γ) satisfying (a)–(f). We shall show that the monomials of the form

$$t_{n,m,\gamma} = \mu_n e(\gamma) \mu_m^*, \quad n, m \in \mathbb{N}^*, \quad (n, m) = 1, \quad \gamma \in \mathbb{Q}/\mathbb{Z}$$

form a set of generators of the vector space \mathcal{A} . It is enough for this to express the adjoint and the products of such monomials as elements of their linear span \mathcal{L} . First, if we continue to denote by $t_{n,m,\gamma}$ the expression $\mu_n e(\gamma) \mu_m^*$ when n, m are not relatively prime but have $(n, m) = q > 1$, we can write

$$t_{n,m,\gamma} = \mu_{n/q} \mu_q e(\gamma) \mu_q^* \mu_{m/q}^*$$

and use (f) to express it as an element of \mathcal{L} . It is clear that $(t_{n,m,\gamma})^* = t_{m,n,-\gamma}$ so that $\mathcal{L} = \mathcal{L}^*$. Let us now compute the product: $t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2}$. Let $q = (m_1, n_2)$, then $\mu_{m_1}^* \mu_{n_2} = \mu_{m_1/q}^* \mu_q^* \mu_q \mu_{n_2/q} = \mu_{m_1/q}^* \mu_{n_2/q} = \mu_{n_2/q} \mu_{m_1/q}^*$ using (a), (b), (c). Thus $t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2} = \mu_{n_1} e(\gamma_1) \mu_{n_2/q} \mu_{m_1/q}^* e(\gamma_2) \mu_{m_2}^*$. Using (e) and its adjoint one thus obtains

$$t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2} = \mu_{n_1 n_2 / q} e\left(\frac{n_2}{q} \gamma_1 + \frac{m_1}{q} \gamma_2\right) \mu_{m_1 m_2 / q}^*$$

which is a $t_{n,m,\gamma}$ with (n, m) not necessarily 1 and can be expressed as above as an element of \mathcal{L} .

We have shown that the linear span \mathcal{L} of the $t_{n,m,\gamma}$ is an involutive algebra and hence that $\mathcal{L} = \mathcal{A}$. In the algebra \mathcal{H} the $t_{n,m,\gamma}$ are given by

$$t_{n,m,\gamma} = (nm)^{-1/2} e_X, \quad X = \text{double class of } \begin{bmatrix} 1 & \gamma \\ 0 & n/m \end{bmatrix}. \quad (7)$$

Thus they are linearly independent in \mathcal{H} and this is enough to conclude that (a)–(f) is a presentation of \mathcal{H} .

It is crucial that the above presentation of \mathcal{H} is defined over the field \mathbb{Q} of rational numbers, since this will allow for a natural action of the Galois group G of the cyclotomic extension \mathbb{Q}^{cycl} of \mathbb{Q} on certain representations of \mathcal{H} which we shall construct later in Section 6.

A result similar to Proposition 18 holds for the C^* -algebra $C_{\mathbb{Q}} = C_r^*(\Gamma, \Gamma_0)$ and due to the amenability of $\Gamma = P_{\mathbb{Q}}^+$, the nuance between $C^*(\Gamma, \Gamma_0)$, the universal C^* -algebra generated by the (μ_n, e_γ) with the above relations, and $C_r^*(\Gamma, \Gamma_0)$ does not arise. One has $C^*(\Gamma, \Gamma_0) = C_r^*(\Gamma, \Gamma_0)$.

PROPOSITION 19. Let π be an involutive representation of \mathcal{H} as operators in a Hilbert space \mathcal{H} . Then π extends uniquely by continuity to a representation of $C_r^*(\Gamma, \Gamma_0) = C_{\mathbb{Q}}$.

Proof. The relations (a) and (d) show that $\pi(\mu_n)$ is an isometry and $\pi(e(\gamma))$ a unitary. Thus one has

$$\|\pi(\mu_n e(\gamma) \mu_m^*)\| \leq 1 \quad \forall n, m, \gamma.$$

This shows that $\pi(f)$ is bounded for any $f \in \mathcal{H}$, with

$$\|\pi(f)\| \leq \|f\|_1$$

where

$$\|f\|_1 = \sum_{\gamma \in \Gamma/\Gamma_0} \delta(\gamma)^{-1/2} |f(\gamma)|, \quad \delta(\gamma) = \frac{L(\gamma)}{R(\gamma)}.$$

It follows then that the following equality defines a norm on \mathcal{H} whose completion is a C^* -algebra:

$$\|f\|_{\max} = \sup\{\|\pi(f)\| ; \pi \in \text{Rep } \mathcal{H}\} \quad (8)$$

where $\text{Rep } \mathcal{H}$ is the class of all involutive representations of \mathcal{H} in a fixed separable Hilbert space.

Let us now show that

$$\|f\|_{\max} = \|f\|_{C_r^*(\Gamma, \Gamma_0)} \quad \forall f \in \mathcal{H}, \quad (9)$$

which is a statement of *amenability* of the pair (Γ, Γ_0) . Let us first prove (9) for elements of the group ring $f \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$, i.e., linear combinations of the e_γ , $\gamma \in \mathbb{Q}/\mathbb{Z}$. Using the representation of \mathcal{H} in $\ell^2(\Gamma_0 \backslash \Gamma)$ one has

$$\|f\|_{\max} \geq \|f\|_{C_r^*} \quad \forall f \in \mathcal{H}. \quad (10)$$

Thus we only need to prove the other inequality. The amenability of the group \mathbb{Q}/\mathbb{Z} shows that

$$\|\pi(f)\| \leq \|f\|_{C_r^*(\mathbb{Q}/\mathbb{Z})} \quad \forall f \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}] \quad (11)$$

for any unitary representation π of \mathbb{Q}/\mathbb{Z} .

Observe that in the representation of \mathcal{H} in $\ell^2(\Gamma_0 \backslash \Gamma)$ the restriction to $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ defines a faithful representation of $C_r^*(\mathbb{Q}/\mathbb{Z}) = C^*(\mathbb{Q}/\mathbb{Z})$. Indeed the restriction of the action of \mathbb{Q}/\mathbb{Z} to the orbit of $\varepsilon_0 = \Gamma_0$, is isomorphic to the regular representation of \mathbb{Q}/\mathbb{Z} , so the result follows. This proves (9) for elements of $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ and allows us to view $C^*(\mathbb{Q}/\mathbb{Z})$ as a C^* -subalgebra of the C^* completion $C^*(\Gamma, \Gamma_0)$ of \mathcal{H} for the norm (8). Let us identify the dual of the group \mathbb{Q}/\mathbb{Z} with the additive group

$\mathcal{R} \subset \mathcal{A}$ using the equality $\mathbb{Q}/\mathbb{Z} = \mathcal{A}/\mathcal{R}$ and the identification of the additive group \mathcal{A} with its Pontrjagin dual. Consider now the locally compact groupoid \mathcal{G} defined as follows. As a locally compact space \mathcal{G} is the following subset of $\mathcal{R} \times \mathbb{Q}_+^*$,

$$\mathcal{G} = \{(b, a) \in \mathcal{R} \times \mathbb{Q}_+^* ; ab \in \mathcal{R}\} \quad (12)$$

which may be identified with a countable union of open and closed subsets of \mathcal{R} . One has $\mathcal{G}^{(0)} = \mathcal{R} \times 1 = \mathcal{R}$ and the range and source maps are

$$r(b, a) = ab, \quad s(b, a) = b \quad (13)$$

while the composition is given by

$$(b_1, a_1) \circ (b_2, a_2) = (b_2, a_1 a_2). \quad (14)$$

By construction the fibers \mathcal{G}^x and \mathcal{G}_x , $x \in \mathcal{R}$ of r and s are discrete countable sets and the C^* -algebras $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$ of this locally compact groupoid make good sense. They are the completions of the convolution algebra $C_c(\mathcal{G})$ of continuous functions with compact support on \mathcal{G} ,

$$(f_1 * f_2)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f_1(\gamma_1) f_2(\gamma_2) \quad (15)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad (16)$$

under the following respective norms:

$$\|f\|_{\max} = \text{Sup}\{\|\pi(f)\| ; \pi \in \text{Rep } \mathcal{G}\} \quad (17)$$

$$\|f\|_r = \text{Sup}_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\| \quad (18)$$

where λ_x is the left regular representation of f in $\ell^2(\mathcal{G}^x)$ given by:

$$(\lambda_x(f)\xi)(\gamma) = \sum_{r(\gamma_1)=x} f(\gamma_1) \xi(\gamma_1^{-1}\gamma). \quad (19)$$

Now since the group \mathbb{Q}_+^* is amenable it follows that the locally compact groupoid \mathcal{G} is amenable in the sense of [Ren], so that the norms (17) and (18) coincide.

Next, given a representation π of \mathcal{H} , we get a representation $\tilde{\pi}$ of $C_c(\mathcal{G})$ as follows. We identify \mathcal{H} with a subalgebra of $C_c(\mathcal{G})$ by checking that the following elements $\tilde{e}(\gamma)$, $\tilde{\mu}_n$ of $C_c(\mathcal{G})$ satisfy the presentation of Proposition 18,

$$\begin{aligned} \tilde{e}(\gamma)(b, a) &= 0 && \text{unless } a = 1, \\ \tilde{e}(\gamma)(b, 1) &= \langle b, \gamma \rangle \end{aligned} \quad (20)$$

(where $\langle b, \gamma \rangle$ is the pairing between \mathcal{R} and \mathbb{Q}/\mathbb{Z} given by Pontrjagin duality of abelian groups)

$$\tilde{\mu}_n(b, a) = 0 \quad \text{unless} \quad a = n^{-1}; \quad \tilde{\mu}_n(b, n^{-1}) = 1 \quad \forall b \in \mathcal{R}. \quad (21)$$

One checks directly the relations (a)–(f) of the presentation of \mathcal{H} using in particular the equality

$$\langle n\gamma, b \rangle = \langle \gamma, nb \rangle \quad \forall \gamma \in \mathbb{Q}/\mathbb{Z}, \quad \forall b \in \mathcal{R} \text{ such that } nb \in \mathcal{R}. \quad (22)$$

Let then π be an involutive representation of \mathcal{H} . We have shown above that the restriction of π to the group ring of \mathbb{Q}/\mathbb{Z} extends to a representation of $C^*(\mathbb{Q}/\mathbb{Z}) = C_r^*(\mathbb{Q}/\mathbb{Z})$. It thus follows using the $\tilde{\mu}_n$, which together with $C(\mathcal{R})$ generate $C_c(\mathcal{G})$, that π extends to a representation $\tilde{\pi}$ of the convolution algebra $C_c(\mathcal{G})$ of the locally compact groupoid \mathcal{G} . The amenability of \mathcal{G} thus yields

$$\|\pi(f)\| \leq \sup_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\| \quad \forall f \in \mathcal{H}. \quad (23)$$

The homomorphism $h: \mathcal{G} \rightarrow \mathbb{Q}_+^*$, $h(b, a) = a$, yields for each $x \in \mathcal{R}$, an injection of \mathcal{G}^x in \mathbb{Q}_+^* which allows us to consider the continuous field of Hilbert spaces $\ell^2(\mathcal{G}^x)$, $x \in \mathcal{R}$ as a subfield of the constant field with fiber $\ell^2(\mathbb{Q}_+^*)$. For any $f \in \mathcal{H}$ the map $x \rightarrow \lambda_x(\tilde{f})$ is then strongly continuous with values in $\mathcal{L}(\ell^2(\mathbb{Q}_+^*))$. This can be checked directly for the $\tilde{\mu}_n$ and the elements of $C(\mathcal{R})$.

It follows that for any $f \in \mathcal{H}$ the function on \mathcal{R} given by $x \rightarrow \|\lambda_x(\tilde{f})\|$, is lower semicontinuous in the sense that $\{x; \|\lambda_x(\tilde{f})\| > \alpha\}$ is an open set for any α . Thus

$$\sup_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\| = \text{Ess Sup}_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\| \quad (24)$$

and the right-hand side is equal to $\|f\|_{C_r^*(\Gamma, \Gamma_0)}$ so that the equality (9) follows.

§5. Action of $W \times \mathbb{R}$ on the C^* -algebra $C_{\mathbb{Q}}$

Let $(\sigma_t)_{t \in \mathbb{R}}$ be the action of \mathbb{R} on the C^* -algebra $C_{\mathbb{Q}} = C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ defined in Proposition 4. In terms of double classes X in $\Gamma_0 \backslash \Gamma / \Gamma_0$, $\Gamma = P_{\mathbb{Q}}^+$, $\Gamma_0 = P_{\mathbb{Z}}^+$, one has

$$\sigma_t(e_X) = k^{-it} e_X, \quad k = \frac{R(X)}{L(X)}$$

and for the double class of $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \in P_{\mathbb{Q}}^+$, one has

$$k = a. \quad (1)$$

In terms of the presentation (Proposition 18) of $C_{\mathbb{Q}}$ one has

$$\sigma_t(\mu_n) = n^{it} \mu_n \quad \forall n \in \mathbb{N}^* \quad , \quad \sigma_t(e(\gamma)) = e(\gamma) \quad (2)$$

for any $\gamma \in \mathbb{Q}/\mathbb{Z}$ and any $t \in \mathbb{R}$.

PROPOSITION 20.

- (a) The C^* -subalgebra of $C_{\mathbb{Q}}$ given by the fixed points of σ , $C_{\mathbb{Q}}^{\sigma} = \{x \in C_{\mathbb{Q}} ; \sigma_t(x) = x \quad \forall t \in \mathbb{R}\}$ is the image of $C^*(\mathbb{Q}/\mathbb{Z})$ by the isomorphism associated to the homomorphism $\gamma \in \mathbb{Q}/\mathbb{Z} \rightarrow e(\gamma) \in C_{\mathbb{Q}}$.
- (b) The centralizer of the state φ on $C_{\mathbb{Q}}$ given by Lemma 16, is equal to $C_{\mathbb{Q}}^{\sigma} = C^*(\mathbb{Q}/\mathbb{Z})$.

Proof. (a) By construction $C^*(\mathbb{Q}/\mathbb{Z}) \subset C_{\mathbb{Q}}^{\sigma}$. The action σ of \mathbb{R} on $C_{\mathbb{Q}}$ is almost periodic and diagonal in the linear basis (e_X) of \mathcal{H} . The projection P on the fixed points of σ , given by the almost periodic average of the σ_t , is the identity on the double classes $e_{X\gamma}$, $\gamma \in \mathbb{Q}/\mathbb{Z}$, and vanishes, $P(e_X) = 0$, on the double class of any $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, $a \neq 1$. The conclusion follows since \mathcal{H} is dense on $C_{\mathbb{Q}}$ and P is norm continuous.

(b) follows from (a) and Lemma 16.

We shall now define a natural action by automorphisms of $C_{\mathbb{Q}}$, of the idele class group $W = \mathcal{A}^*/\mathbb{Q}_+^*$. Recall first that we obtained $C_{\mathbb{Q}}$ in Proposition 15 from the commutant of $P_{\mathbb{Q}}^+$ in $\ell^2(\Delta)$ where $\Delta = P_{\mathcal{A}}/P_{\mathcal{R}} = P_{\mathbb{Q}}^+/P_{\mathbb{Z}}^+$. Let us show that $W = \mathcal{A}^*/\mathbb{Q}_+^*$ acts in a natural manner on the commutant of $P_{\mathbb{Q}}^+$ in $\ell^2(\Delta)$. The group $P_{\mathcal{A}}$ acts on Δ and on $\ell^2(\Delta)$ and thus the commutant of $P_{\mathbb{Q}}^+$ is the same as the commutant of its closure $\overline{P_{\mathbb{Q}}^+}$ in $P_{\mathcal{A}}$. One has (cf. Proposition 12), $\overline{P_{\mathbb{Q}}^+} = \mathcal{A} \rtimes \mathbb{Q}_+^*$ which is a *normal subgroup* of $P_{\mathcal{A}}$. Thus the quotient group

$$W = \mathcal{A}^*/\mathbb{Q}_+^* = P_{\mathcal{A}}/\overline{P_{\mathbb{Q}}^+} \quad (3)$$

acts naturally on the commutant $P_{\mathbb{Q}}^{+'}$ of $P_{\mathbb{Q}}^+$ in $\ell^2(\Delta)$, by

$$\theta_u(x) = u x u^* \quad \forall x \in P_{\mathbb{Q}}^{+'}, \quad \forall u \in W. \quad (4)$$

(The choice of the representative $u \in P_{\mathcal{A}}$ of the class of u is irrelevant.) This defines a strongly continuous action of the compact group W on the von Neumann algebra $P_{\mathbb{Q}}^{+'}$.

PROPOSITION 21.

- (a) The action θ of W on $P_{\mathbb{Q}}^{+'}$ leaves the dense C^* -subalgebra $C_{\mathbb{Q}}$ globally invariant and is pointwise norm continuous on $C_{\mathbb{Q}}$.
- (b) The fixed point subalgebra $C_{\mathbb{Q}}^W$ is the C^* -algebra $C^*(\mathbb{N}^*)$ generated by the $\mu_n \in C_{\mathbb{Q}}$.
- (c) The action of W on $C_{\mathbb{Q}}$ preserves the state φ and commutes with the action $(\sigma_t)_{t \in \mathbb{R}}$.

Proof. Let us first show that μ_n , i.e. the right convolution $r(f)$ in $\ell^2(\Gamma/\Gamma_0)$ by $f = n^{-1/2} e_{X_n}$, belongs not only to the commutant of $P_{\mathbb{Q}}^+$ (Proposition 15) but also to the commutant of $P_{\mathcal{A}}$. By construction one has

$$\langle \mu_n^* \varepsilon_e, g \varepsilon_e \rangle = f(g) = n^{-1/2} e_{X_n}(g) \quad \forall g \in \Gamma \quad (5)$$

and since the $g \varepsilon_e$, $g \in \Gamma/\Gamma_0$ form a basis of $\ell^2(\Delta)$, we get

$$\mu_n^* \varepsilon_e = n^{-1/2} g_n \varepsilon_e, \quad g_n = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \in \Gamma. \quad (6)$$

Let us assume that $n = p$ is a prime number and show that $p^{1/2} \mu_p^*$ coincides with the operator in $\ell^2(\Delta) = \bigotimes_q (\ell^2(T_q), *)$ given by $\tilde{t}_p = 1 \otimes t_p \otimes 1 \otimes \dots$, where t_p is the hyperbolic translation of one unit of length towards the point at ∞ in the tree T_p of Proposition 12.

Note that this hyperbolic translation is exactly p to one so that $p^{-1/2} t_p$ is a coisometry on $\ell^2(T_p)$. Now both μ_p^* and t_p commute with the action of $P_{\mathbb{Q}}^+$ on $\ell^2(\Delta)$, and ε_e is cyclic for $P_{\mathbb{Q}}^+$. Thus the equality

$$p^{1/2} \mu_p^* \varepsilon_e = \tilde{t}_p \varepsilon_e$$

implies the equality of operators

$$\mu_p^* = p^{-1/2} \tilde{t}_p. \quad (7)$$

Since \tilde{t}_p belongs to the commutant of $P_{\mathcal{A}}$, by construction, we thus get that $\mu_p \in (P_{\mathcal{A}})'$ and

$$\theta_u(\mu_n) = \mu_n \quad \forall u \in W, \quad n \in \mathbb{N}^*. \quad (8)$$

To prove 21 (a) we just need to show that the action θ of W leaves the C^* -subalgebra $C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ of $C_{\mathbb{Q}}$ globally invariant and is pointwise norm continuous on $C^*(\mathbb{Q}/\mathbb{Z})$. This will follow from the following more precise statement:

LEMMA 22. Let $f \in C(\mathcal{R}) = C^*(\mathbb{Q}/\mathbb{Z}) \subset C_{\mathbb{Q}}$. Then

$$\theta_u(f)(b) = f(ub) \quad \forall b \in \mathcal{R}, u \in \mathcal{R}^* = W.$$

Proof. Let us consider (cf. [We₂] p. 257) the direct product decomposition:

$$\mathcal{A}^* = \mathbb{Q}_+^* \times \left(\prod_p \mathbb{Z}_p^* \right) = \mathbb{Q}_+^* \times \mathcal{R}^*$$

where the morphism $r : \mathcal{A}^* \rightarrow \mathbb{Q}_+^*$ is given by

$$r(z) = \prod_p |z_p|_p^{-1} = \prod_p p^{\text{val}(z_p)}.$$

By construction, r is the identity on \mathbb{Q}_+^* and its kernel is $\prod_p \mathbb{Z}_p^* = \mathcal{R}^*$; thus r is the projection on the first term of this decomposition as a product. We use the second projection to identify W with \mathcal{R}^* .

The equality of abelian groups $\mathcal{A}/\mathcal{R} = \mathbb{Q}/\mathbb{Z}$ shows that the multiplication by an element $u \in \mathcal{R}^*$ defines an automorphism m_u of \mathbb{Q}/\mathbb{Z} and to prove Lemma 22, one just has to show that for any $\gamma \in \mathbb{Q}/\mathbb{Z}$ one has

$$\theta_u(e(\gamma)) = e(m_{u^{-1}} \gamma) \quad \forall u \in \mathcal{R}^*. \quad (9)$$

Since ε_e is separating for $P_{\mathbb{Q}}^+$ it is enough to check that $\theta_u(e(\gamma)) \varepsilon_e = e(m_{u^{-1}} \gamma) \varepsilon_e$, i.e. that

$$u e(\gamma) u^* \varepsilon_e = e(m_{u^{-1}} \gamma) \varepsilon_e. \quad (10)$$

Since ε_e is fixed by $P_{\mathcal{R}} \subset P_{\mathcal{A}}$, it is fixed by the action of $u^* \in \mathcal{R}^* \subset P_{\mathcal{R}}$. Moreover for any $\delta \in \mathbb{Q}/\mathbb{Z}$, the action of $e(\delta)$ on ε_e gives simply $\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \varepsilon_e$. Thus (10) follows from the equality

$$\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \delta u^{-1} \\ 0 & 1 \end{bmatrix}.$$

This completes the proof of Lemma 22 and Proposition 21 (a). To prove Proposition 21 (c) note that the action of \mathcal{R}^* in $\ell^2(\Delta)$ fixes the vector ε_e which proves that the action of W on $C_{\mathbb{Q}}$ preserves the state φ . It obviously commutes with σ_t anyway. Let us prove Proposition 21 (b). As W is a compact group, we can consider the natural projection E of $C_{\mathbb{Q}}$ on $C_{\mathbb{Q}}^W$ given by

$$E(x) = \int_W \theta_u(x) du. \quad (11)$$

By construction E is norm continuous and satisfies

$$E(axb) = a E(x) b \quad \forall x \in C_{\mathbb{Q}}, \quad a, b \in C_{\mathbb{Q}}^W. \quad (12)$$

Since by (8) one has $C^*(\mathbb{N}^*) \subset C_{\mathbb{Q}}^W$, it suffices using the linear basis $\mu_n e(\gamma) \mu_m^*$ of \mathcal{H} , to show that

$$E(e(\gamma)) \in C^*(\mathbb{N}^*) \quad \forall \gamma \in \mathbb{Q}/\mathbb{Z} \quad (13)$$

to conclude that $C^*(\mathbb{N}^*) = C_{\mathbb{Q}}^W$.

Finally to prove (13) note that $E(e(\gamma)) \in C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ is given by a function $f \in C(\mathcal{R})$ such that

$$f(ub) = f(b) \quad \forall u \in \mathcal{R}^*, \quad \forall b \in \mathcal{R}. \quad (14)$$

This equality defines a C^* -subalgebra of $C(\mathcal{R}) = \bigotimes_{p \in \mathcal{P}} C(\mathcal{R}_p)$, which is identical with $C^*(\mathbb{N}^*) \cap C^*(\mathbb{Q}/\mathbb{Z})$. This can be seen locally by showing that

$$f \in C(\mathcal{R}_p), \quad f(ub) = f(b) \quad \forall u \in R_p^*, \quad b \in R_p \quad (15)$$

implies that f is a function of $| \cdot |_p$.

§6. Classification of KMS_{β} states for $\beta > 1$

We shall first construct involutive representations π_{α} of the Hecke algebra \mathcal{H} , labelled by the Galois group $G = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ of the subfield \mathbb{Q}^{cycl} of \mathbb{C} generated by all roots of unity.

Let \mathcal{H} be the Hilbert space $\ell^2(\mathbb{N}^*)$, with its canonical orthonormal basis $(\varepsilon_k)_{k \in \mathbb{N}^*}$.

PROPOSITION 23. *The following equalities define an involutive representation π_1 of the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\mathbb{N}^*)$,*

$$(\alpha) \quad \pi_1(\mu_n) \varepsilon_k = \varepsilon_{nk} \quad \forall n, k \in \mathbb{N}^*$$

$$(\beta) \quad \pi_1(e(\gamma)) \varepsilon_k = \exp(2\pi i k \gamma) \varepsilon_k \quad \forall k \in \mathbb{N}^*, \quad \gamma \in \mathbb{Q}/\mathbb{Z}.$$

Proof. We just need to check that the relations (a)–(f) are fulfilled by the operators $\mu'_n = \pi_1(\mu_n)$ and $e'(\gamma) = \pi_1(e(\gamma))$ defined by (α) , (β) . Since the map $k \rightarrow nk$ from \mathbb{N}^* to \mathbb{N}^* is injective the relation (a) follows and moreover the adjoint $\mu_n'^*$ is given by

$$\mu_n'^* \varepsilon_k = \varepsilon_{k/n} \quad \text{if } n \mid k \quad \text{and } 0 \text{ otherwise.} \quad (1)$$

The relation (b) is obvious. To check (c) note that if $(n, m) = 1$, then $m \mid k \Leftrightarrow m \mid nk$ and when applied to ε_k both $\mu'_n \mu_m^*$ and $\mu_m^* \mu'_n$ vanish or are equal to $\varepsilon_{nk/m}$. The relation (d) is clear as well as (e),

$$e'(\gamma) \mu'_n \varepsilon_k = e'(\gamma) \varepsilon_{nk} = \exp 2\pi i(nk\gamma) \varepsilon_{nk} = \mu'_n e'(n\gamma) \varepsilon_k.$$

Let us check (f). First both sides applied to ε_k vanish unless $n \mid k$. This is clear for the left side by (1), and it is true for the right side because it is of the form

$$e'(\delta_0) \frac{1}{n} \sum_{n\delta=0} e'(\delta), \quad n\delta_0 = \gamma$$

and $\sum_{n\delta=0} e'(\delta) \varepsilon_k = 0$ unless $n \mid k$. Next, when $n \mid k$ so that $k = qn$, one has:

$$\begin{aligned} \mu'_n e'(\gamma) \mu_n^* \varepsilon_k &= \exp(2\pi i q \gamma) \varepsilon_k \\ e'(\delta_0) \varepsilon_k &= \exp(2\pi i k \delta_0) \varepsilon_k = \exp(2\pi i q \gamma) \varepsilon_k \end{aligned}$$

for any $\delta_0 \in \mathbb{Q}/\mathbb{Z}$ such that $n\delta_0 = \gamma$.

PROPOSITION 24.

(1) For $\alpha \in G = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$, the following equalities define an involutive representation π_α of \mathcal{H} in $\ell^2(\mathbb{N}^*)$:

$$(\alpha) \quad \pi_\alpha(\mu_n) \varepsilon_k = \varepsilon_{nk} \quad \forall n, k \in \mathbb{N}^*$$

$$(\beta) \quad \pi_\alpha(e(\gamma)) \varepsilon_k = \alpha(\exp 2\pi i k \gamma) \varepsilon_k, \quad \forall k \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}.$$

(2) For any element $x \in \mathcal{H}(\mathbb{Q})$ of the \mathbb{Q} algebra generated by the μ_n and the $e(\gamma)$, the matrix elements of $\pi_\alpha(x)$ satisfy

$$\langle \pi_\alpha(x) \varepsilon_{k_1}, \varepsilon_{k_2} \rangle = \alpha \langle \pi_1(x) \varepsilon_{k_1}, \varepsilon_{k_2} \rangle \quad \forall k_j.$$

Proof. (1) The above proof of Proposition 23 works with no change. The only important point is to check (d) namely that $\pi_\alpha(e(\gamma))^* = \pi_\alpha(e(-\gamma))$. This follows because the complex conjugation $z \rightarrow \bar{z}$ commutes with any $\alpha \in G$.

(2) By construction the matrix elements of the π_α of the generators satisfy the required relation which is stable under the algebraic operations of matrices involving only finite sums of products.

Let then H be the positive operator in $\ell^2(\mathbb{N}^*)$ corresponding to the time evolution described in Section 2, namely

$$H \varepsilon_n = (\log n) \varepsilon_n \quad \forall n \in \mathbb{N}^*. \quad (3)$$

We already saw in Section 2 that

$$e^{itH} x e^{-itH} = \sigma_t(x) \quad \forall x \in C^*(\mathbb{N}^*), \quad \forall t \in \mathbb{R}. \quad (4)$$

Since it is obvious that H commutes with $\pi_\alpha(y)$ for any $y \in C^*(\mathbb{Q}/\mathbb{Z}) \subset C_\mathbb{Q}$, we thus obtain

$$e^{itH} \pi_\alpha(x) e^{-itH} = \pi_\alpha(\sigma_t(x)) \quad \forall x \in C_\mathbb{Q}, \quad \forall t \in \mathbb{R}. \quad (5)$$

We can now state:

THEOREM 25. *Let $\tilde{\pi}_\alpha$ be the canonical extension of the representation π_α of \mathcal{H} to the C^* -algebra $C_\mathbb{Q}$, and let $\beta > 1$.*

(a) *The following equality defines a KMS_β state on $(C_\mathbb{Q}, \sigma_t)$:*

$$\varphi_{\beta,\alpha}(x) = \zeta(\beta)^{-1} \text{Trace}(\tilde{\pi}_\alpha(x) e^{-\beta H}) \quad \forall x \in C_\mathbb{Q}.$$

(b) *The map $\alpha \rightarrow \varphi_{\beta,\alpha}$ is a homeomorphism of the Galois group G of \mathbb{Q}^{cycl} with the space of extreme points of the Choquet simplex of KMS_β states on $(C_\mathbb{Q}, \sigma_t)$.*

Proof. (a) First by Proposition 19 we know that the representation π_α extends to a representation of $C_\mathbb{Q}$ and the equality (5) together with the finiteness of $\text{Trace}(e^{-\beta H}) = \zeta(\beta)$ gives (a).

(b) Let us fix $\beta > 1$ and show first that the map $\alpha \rightarrow \varphi_{\beta,\alpha}$ is injective. To show this note that each of the representations π_α of $C_\mathbb{Q}$ is irreducible and by construction each $\varphi_{\beta,\alpha}$ is a type I_∞ factor state. Thus its GNS construction canonically determines the positive operator H , $0 \in \text{Sp } H$, as an unbounded operator affiliated to the weak closure of $C_\mathbb{Q}$. In particular $\varphi_{\beta,\alpha}$ determines canonically the 0-temperature state,

$$\varphi_{\infty,\alpha}(x) = \langle \pi_\alpha(x) \varepsilon_1, \varepsilon_1 \rangle. \quad (6)$$

When we restrict this state $\varphi_{\infty,\alpha}$ to the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of \mathbb{Q}/\mathbb{Z} with rational coefficients, we find the corresponding imbedding of the field \mathbb{Q}^{cycl} of roots of unity in \mathbb{C} :

$$\varphi_{\infty,\alpha}(\sum \lambda_j \gamma_j) = \alpha(\sum \lambda_j \exp(2\pi i \gamma_j)) \quad (7)$$

which of course determines $\alpha \in G$ uniquely.

Next each $\varphi_{\beta,\alpha}$ is a factor state and thus is an extreme point of the weakly compact convex set K_β of KMS_β states. Let $\mathcal{E}(K_\beta)$ be the space of extreme points of K_β . We have shown that the map $\alpha \rightarrow \varphi_{\beta,\alpha}$ is an injection of G in $\mathcal{E}(K_\beta)$. This map is weakly continuous since, as $\beta > 1$, the series $\sum \alpha(\exp 2\pi i k \gamma) k^{-\beta}$ is

uniformly convergent. It remains to show that this map is surjective. Note first that for any element u of W , there exists a corresponding element $\alpha(u)$ of G such that

$$\varphi_{\beta,1} \circ \theta_u = \varphi_{\beta,\alpha(u)} \quad (8)$$

and the map $u \rightarrow \alpha(u)$ is an isomorphism of W with G . Next there exists on $C_{\mathbb{Q}}$ a unique KMS_{β} state which is W -invariant. This holds for any value $\beta \in]0, \infty[$ and follows from Proposition 21 (b) and Proposition 8 (a). Indeed, given such a state φ , one has $\varphi = \varphi \circ E$ where E is the projection $E = \int_W \theta_u du$ of $C_{\mathbb{Q}}$ on $C_{\mathbb{Q}}^W$ and the restriction of φ to $C_{\mathbb{Q}}^W = C^*(\mathbb{N}^*)$ is unique. Thus let ψ be any KMS_{β} state; one has

$$\int_W \psi \circ \theta_u du = \int_W \varphi_{\beta,\alpha(u)} du \quad (9)$$

since both sides are W -invariant KMS_{β} states. Now if $\psi \in \mathcal{E}(K_{\beta})$ is an extreme point, this equality (9) gives two decompositions of the same state as a barycenter of measures over $\mathcal{E}(K_{\beta})$, which is a Choquet simplex (Proposition 2), so that

$$\psi \circ \theta_u \in \{\varphi_{\beta,\alpha(v)} ; v \in W\} \quad \text{for almost all } u.$$

Finally this implies that $\psi = \varphi_{\beta,\alpha(v)} \circ \theta_u^{-1}$ for some $u, v \in W$ and ψ is in the image of the map $\alpha \rightarrow \varphi_{\beta,\alpha}$. Since the map $\alpha \rightarrow \varphi_{\beta,\alpha}$ is continuous and bijective and G is compact, it is homeomorphism with its range $\mathcal{E}(K_{\beta})$ and this proves Theorem 25.

REMARKS 26.

- (1) We shall give in the next section the general formula (for all values of β) for the W -invariant KMS_{β} state φ_{β} on $C_{\mathbb{Q}}$, but we can already find the formula for $\beta > 1$ at this stage using the equality (9). First, using the linear basis $(t_{n,m,\gamma})$ of $\mathcal{H} \subset C_{\mathbb{Q}}$ described in Section 4, one has

$$\varphi(t_{n,m,\gamma}) = 0 \quad \text{if } n/m \neq 1 \quad \forall \varphi \in K_{\beta}. \quad (11)$$

Thus it is enough to determine the restriction of φ_{β} to $C_{\mathbb{Q}}$. For this one can for instance use the right-hand side of (9) and the formula in Theorem 25 (a) for $\varphi_{\beta,\alpha}$. Let $\gamma \in \mathbb{Q}/\mathbb{Z}$, $n \in \mathbb{N}^*$. Then one has

$$\int_G \alpha(\exp 2\pi i n \gamma) d\alpha = \frac{\mu(b/d)}{\varphi(b/d)} \quad (12)$$

where $\gamma = \frac{a}{b}$, $(a, b) = 1$ and $(n, b) = d$ is the g.c.d of n and b . Also μ is the Moebius function and φ the Euler function. Let then ρ_{β} be the multiplicative function such that

$$\sum_{(n,b)=1} n^{-\beta} = \rho_{\beta}(b) \zeta(\beta). \quad (13)$$

One has $\rho_\beta(b) = \prod_{p|b, p \text{ prime}} (1 - p^{-\beta})$.

The unique W -invariant KMS_β state, given by (9) satisfies

$$\varphi_\beta(e(\gamma)) = \sum_{d|b} \frac{\mu(b/d)}{\varphi(b/d)} \rho_\beta(b/d) d^{-\beta} \quad (14)$$

where b is the denominator of the irreducible fraction $\gamma = \frac{a}{b}$. The right-hand side of (14) is a multiplicative function of b and it is also given by

$$\varphi_\beta(e(\gamma)) = b^{-\beta} \prod_{p|b, p \text{ prime}} (1 - p^{\beta-1})(1 - p^{-1})^{-1} \quad (15)$$

as can be seen by computing the right-hand side of (14) when b is a prime power. We shall give another proof of (15) in the next section.

- (2) The statement of Theorem 25 (b) also applies to the 0-temperature states $\varphi_{\infty, \alpha}$. For extreme such states the map $\varphi_{\infty, \alpha}$ restricted to $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \subset C^*(\mathbb{Q}/\mathbb{Z})$ gives the associated embedding of the field \mathbb{Q}^{cycl} of roots of unity in \mathbb{C} , as we have seen above.
- (3) Theorem 25 shows that the partition function of the C^* -dynamical system $(C_{\mathbb{Q}}, \sigma_t)$ is the Riemann zeta function.

§7. Uniqueness of KMS_β states for $\beta \in]0, 1]$

In this section we shall show that for $\beta \in]0, 1]$ there exists a unique KMS_β state on the C^* -algebra $C_{\mathbb{Q}}$. We have seen in Section 5, Proposition 21, that the C^* -subalgebra $C^*(\mathbb{N}^*) \subset C_{\mathbb{Q}}$ generated by the μ_n is the fixed point algebra of the action of W on $C_{\mathbb{Q}}$:

$$C^*(\mathbb{N}^*) = C_{\mathbb{Q}}^W. \quad (1)$$

Since W is a compact abelian group, its action on $C_{\mathbb{Q}}$ has discrete spectrum, and we can consider for each character χ of W the corresponding spectral subspace (cf. [Ped]),

$$C_{\mathbb{Q}, \chi} = \{x \in C_{\mathbb{Q}} ; \theta_u(x) = \chi(u) x \quad \forall u \in W\}. \quad (2)$$

To prove the uniqueness of KMS_β states on $C_{\mathbb{Q}}$ for $0 < \beta \leq 1$, we shall analyze the partial automorphisms of the type III_1 factor associated to $(C^*(\mathbb{N}^*), \varphi_\beta)$ and to a fixed non trivial character χ of W .

We shall show that these partial automorphisms are outer. Given an element V of $C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ (resp. a character χ of W), we shall say that V (resp. χ) is localized in a subset F of \mathcal{P} , the set of finite places of \mathbb{Q} , iff

$$V \in \left(\bigotimes_{p \in F} C(R_p) \right) \otimes 1 \subset C(\mathcal{R}) \quad (3)$$

(resp. if χ , seen as a character of \mathcal{A}^* , factorizes through the projection $W \rightarrow \prod_{p \in F} \mathbb{Q}_p^*$).

Let us now state the main lemma.

LEMMA 27. *Let $\beta \in]0, 1]$ and ψ be a KMS_β state on the C^* -dynamical system $(C_{\mathbb{Q}}, \sigma_t)$. Then:*

- (a) *The restriction of ψ to $C^*(\mathbb{N}^*)$ is equal to φ_β .*
- (b) *Let χ be a non trivial character of W and $V \in C^*(\mathbb{Q}/\mathbb{Z})$ be a partial isometry both localized in a finite set $F \subset \mathcal{P}$, such that*

$$\theta_g(V) = \chi(g) V \quad \forall g \in W.$$

Then $\psi(Vx) = 0 \quad \forall x \in C^(\mathbb{N}^*)$.*

- (c) *The restriction of ψ to the spectral subspaces $C_{\mathbb{Q}, \chi}$, $\chi \neq 1$ is equal to 0.*

Proof. (a) Since the restriction of σ_t to $C^*(\mathbb{N}^*)$ is the one parameter group of automorphisms of Proposition 8, the restriction of ψ to $C^*(\mathbb{N}^*)$ is a KMS_β state and the conclusion follows from Proposition 8 (a).

(b) Let $E = V^*V$. As $C^*(\mathbb{Q}/\mathbb{Z})$ is commutative, one has $E = VV^*$; also E belongs to the algebra $C_{\mathbb{Q}}^W = C^*(\mathbb{N}^*)$. Let α be the automorphism of the reduced algebra $C^*(\mathbb{N}^*)_E$ determined by the equality

$$\alpha(x) = V x V^* \quad \forall x \in C^*(\mathbb{N}^*)_E. \quad (4)$$

Let M be the factor (of type III_1) which is the weak closure of $C^*(\mathbb{N}^*)$ in the G.N.S. representation of φ_β . Let us identify $C^*(\mathbb{N}^*)$ with a weakly dense subalgebra of M and extend the state φ_β to a normal state $\tilde{\varphi}_\beta$ on M . As V belongs to the fixed point algebra of σ_t , it belongs to the centralizer of ψ . It follows that the automorphism α of $C^*(\mathbb{N}^*)_E$ preserves $\tilde{\varphi}_\beta$ and extends to an automorphism of M_E . Let us show that for $\beta \in]0, 1]$ this automorphism is outer. For any $q \in \mathcal{P} \setminus F$ one has

$$E \mu_q \in M_E, \quad \alpha(E \mu_q) = \chi(g_q) E \mu_q \quad (5)$$

where $g_q \in \mathcal{R}^* = \prod_p \mathbb{Z}_p^*$ is given by its components

$$(g_q)_p = q \in \mathbb{Z} \cap \mathbb{Z}_p^* \quad \text{if } q \neq p, \quad (g_q)_q = 1. \quad (6)$$

To prove (5) note that for any $f \in C(\mathcal{R}) = C^*(\mathbb{Q}/\mathbb{Z})$ and $n \in \mathbb{N}^*$ one has (cf. Proposition 18 (e))

$$f \mu_n = \mu_n f_n, \text{ where } f_n(b) = f(nb) \quad \forall b \in \mathcal{R}. \quad (7)$$

Thus if f is localized in $F \subset \mathcal{P}$ and $q \notin F$ one gets $\theta_{g_q}(f) = f_q$,

$$f \mu_q = \mu_q \theta_{g_q}(f). \quad (8)$$

Applying this to $f = V$ one gets $V \mu_q = \chi(g_q) \mu_q V$, i.e. one gets (5). Let us view χ as a character of $(\mathbb{Z}/m\mathbb{Z})^*$ where the prime factors of m all belong to F . Then we can simply write $\chi(q)$ instead of $\chi(g_q)$, for $q \notin F$. It follows from (5) that, modulo inner automorphisms, the automorphism α is the infinite tensor product

$$\alpha = \bigotimes_{q \notin F} \rho_{q, \chi(q)} \quad \text{in} \quad M_{F^c} = \bigotimes_{q \notin F} (M_q, \varphi_{\beta, q}) \quad (9)$$

where for any complex number λ , $|\lambda| = 1$, we let $\rho_{p, \lambda}$ be the automorphism of τ_p such that

$$\rho_{p, \lambda}(\mu_p) = \lambda \mu_p. \quad (10)$$

This automorphism is a special case of $\sigma_{t, p}$ and preserves the state $\varphi_{\beta, p}$ by construction.

For any $\chi \in \widehat{W}$ localized on F we let $\tilde{\theta}_\chi$ be the element of $\text{Out } M = \text{Aut}(M)/\text{Inn}(M)$ determined by the class of

$$\left(\bigotimes_{q \in F} \text{id} \right) \otimes \bigotimes_{q \notin F} \rho_{q, \chi(q)}. \quad (11)$$

LEMMA 28. $\tilde{\theta}_\chi$ is inner relative to φ_β iff the following infinite product converges absolutely in absolute value

$$\prod_{p \in \mathcal{P}} (1 - p^{-\beta}) (1 - \chi(p) p^{-\beta})^{-1}.$$

Proof. One has in the type I_∞ factor M_p associated to $(\tau_p, \varphi_{\beta, p})$ a unitary implementing the automorphism $\rho_{p, \chi(p)}$; it is given by the diagonal operator with eigenvalues $\chi(p)^j$, $j \in \mathbb{N}$. Evaluating the state $\varphi_{\beta, p}$ on this unitary gives

$$(1 - p^{-\beta}) \sum_0^\infty \chi(p)^n p^{-n\beta} = (1 - p^{-\beta})(1 - \chi(p) p^{-\beta})^{-1}$$

and the result follows from general criterions (cf. [Co]). This shows using Dirichlet's theorem [Ser₁] that $\widehat{\theta}_\chi$ is outer (χ non trivial) when $\beta \leq 1$ and is inner (since φ_β is a type I_∞ factor state) for $\beta > 1$.

This lemma shows that, for $\beta \in]0, 1]$, the automorphism α of M_E given by (4) is outer:

$$\{y \in M_E ; \quad y \alpha(x) = x y \quad \forall x \in M_E\} = \{0\}. \quad (12)$$

Now let L be the linear form on M_E given by

$$L(x) = \psi(Vx) \quad \forall x \in C^*(\mathbb{N}^*)_E. \quad (13)$$

The Schwartz inequality $|L(x)|^2 \leq \psi(E) \psi(x^*x)$ shows that it is a normal linear functional on M_E . Let u be the partial isometry $u \in M_E$ of its polar decomposition $L = u |L|$. The KMS_β condition for ψ applied to the pair Vx, y ; $x \in C^*(\mathbb{N}^*), y \in C^*(\mathbb{N}^*)$, shows that L satisfies the α -twisted KMS_β condition, where $L(\sigma_t(y) x)$ is replaced by $L(\sigma_t(y) \alpha(x))$. Now as both V and ψ are σ_t -invariant so is L and hence so are u and $|L|$. It follows that the Radon-Nikodym derivative $(D|L| : D \widetilde{\varphi}_\beta)_t$ belongs to the centralizer of $\widetilde{\varphi}_\beta$ and is of the form h^{it} with

$$|L|(x) = \widetilde{\varphi}_\beta(hx) \quad \forall x \in M_E. \quad (14)$$

From the twisted KMS_β condition we then get

$$z hu = hu \alpha(z) \quad \forall z \in M_E \quad (15)$$

which implies by (12) that $hu = 0$ and that $L = 0$.

This proves Lemma 27 (b). Let us prove 27 (c). It is enough for that purpose, given a character $\chi \in \widehat{W}$ localized in $F \subset \mathcal{P}$, to find a sequence V_n of partial isometries $V_n \in C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$, localized in F and such that

$$\theta_g(V_n) = \chi(g) V_n \quad \forall g \in W ; \quad \varphi_\beta(V_n V_n^*) \xrightarrow{n \rightarrow \infty} 1. \quad (16)$$

Then, given $a \in C_{\mathbb{Q}, \chi}$, one has:

$$\psi(a) = \lim_{n \rightarrow \infty} \psi(V_n V_n^* a) = 0$$

because $V_n^* a \in C_{\mathbb{Q}, 1} = C^*(\mathbb{N}^*)$ and Lemma 27 (b) applies. Finally the construction of the V_n is reduced by Lemma 22 to the construction of continuous functions $V_n \in C(\prod_{p \in F} \mathbb{Z}_p)$ such that

$$V_n(gb) = \chi(g) V_n(b) \quad \forall b \in \prod_F \mathbb{Z}_p, \quad g \in \prod_F \mathbb{Z}_p^* \quad (17)$$

and such that the $|V_n|$ are uniformly bounded and converge pointwise to 1; this is immediate and the existence of the partial isometries V_n follows.

COROLLARY 29. *For any $\beta \in]0, 1]$ there exists at most one KMS_β state on $C_\mathbb{Q}$.*

Proof. The group W is a compact group so that the direct sum of the spectral subspaces $C_{\mathbb{Q}, \chi}$ is dense in $C_\mathbb{Q}$ and this determines ψ uniquely by Lemma 27.

We shall now construct this unique KMS_β state ψ_β on $C_\mathbb{Q}$ in a geometric manner using the action on the product of trees of Section 3. The construction will follow from the following general lemma applied to the C^* -module $\mathcal{E} = C^*(G)e$ over $C^*(\mathbb{N}^*)$ and the time evolution σ_t of $C^*(\mathbb{N}^*)$.

LEMMA 30. *Let C be a unital C^* -algebra, \mathcal{E} a C^* -module over C , $(\sigma_t)_{t \in \mathbb{R}}$ a one-parameter group of automorphisms of C , $\beta \in]0, \infty[$, φ_β a KMS_β state on C , and $\mathcal{H}_{\varphi_\beta}$ the Hilbert space of the GNS construction for φ_β .*

(a) *Let \mathcal{H}_β be the completion of \mathcal{E} for the inner product given by*

$$\langle \xi, \eta \rangle_\beta = \varphi_\beta(\langle \xi, \eta \rangle) \quad \forall \xi, \eta \in \mathcal{E}.$$

Then the action of the endomorphisms, $\text{End}_C(\mathcal{E})$, on \mathcal{E} extends by continuity to \mathcal{H}_β .

(b) *There exists a unique representation ρ of C^0 (the opposite algebra of C) in \mathcal{H}_β such that for any $\xi \in \mathcal{H}_\beta$ and $a \in C$ in the domain of $\sigma_{i\beta/2}$ one has*

$$\rho(a)\xi = \xi\sigma_{i\beta/2}(a).$$

This representation commutes with the left action of $\text{End}_C(\mathcal{E})$.

Proof. The Hilbert space \mathcal{H}_β is the tensor product of C^* modules

$$\mathcal{H}_\beta = \mathcal{E} \otimes_C \mathcal{H}_{\varphi_\beta} \tag{18}$$

so that the first assertion follows. The second assertion also follows, using $\mathcal{H}_{\varphi_\beta}$ as a left Hilbert algebra and the stabilization theorem of Kasparov [Ka].

We apply this lemma with $C = C^*(\mathbb{N}^*)$, $\mathcal{E} = C^*(G)e$, and $\sigma_t \in \text{Aut } C$ given by the time evolution (Proposition 7 (c)) of $C^*(\mathbb{N}^*)$. As \mathcal{E} is a space of functions on Δ , so is each of the \mathcal{H}_β , and for each $\alpha \in \Delta$ we let ε_α be the characteristic function of $\{\alpha\} \subset \Delta$. The vectors ε_α , $\alpha \in \Delta$, are of unit length in each \mathcal{H}_β and always span a dense subspace of \mathcal{H}_β . For $\beta = 1$ they give an orthonormal basis so that $\mathcal{H}_1 = \ell^2(\Delta)$. In order to compute the inner product $\langle \varepsilon_\alpha, \varepsilon_{\alpha'} \rangle_\beta$ in \mathcal{H}_β , we shall first work locally, i.e., we fix the local field $K = \mathbb{Q}_p$ and apply Lemma 30 to

$C = C^*(P_K)_e$ the reduced C^* -algebra of P_K relative to the projection $e = 1_{P_R}$, while the C^* module is $\mathcal{E} = C^*(P_K) e$. We use on C the state $\varphi_{\beta,p}$.

Then Lemma 30 yields an inner product on the space of functions with finite support on the tree $T_p = P_K/P_R$.

Let us now compute explicitly this inner product on the tree T associated to any value β . Thus K is a local field, $K = \mathbb{Q}_p$, and we first note that transporting $\varphi_{\beta,p}$ by the canonical isomorphism of τ_p with the reduced C^* -algebra $C^*(P_K)_e$, its value on a P_R -bi-invariant function $f(s)$, $s \in P_K$ is given by (cf. formula (9) of Section 3),

$$\varphi_{\beta,p}(f) = \left(\sum_{k>0} p^{k(1-\beta)} f \left(\begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right) \right) (1 - p^{\beta-1}) + f \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad (19)$$

Let $g \in P_K$, $g = \begin{bmatrix} 1 & n_0 \\ 0 & h_0 \end{bmatrix}$. The inner product $\langle g \varepsilon_0, \varepsilon_0 \rangle_\beta$, with ε_0 corresponding to the base point, is equal to $\varphi_{\beta,p}(f)$, where the function f is associated by formula (13) of Section 3 to the right cosets P_R and $g P_R$ in P_K/P_R :

$$f(s) = m(g P_R \cap P_R s^{-1}) \quad \forall s \in P_K \quad (20)$$

where m is the left Haar measure on P_K . We just need to evaluate $f(s)$ for $s = \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix}$. One has

$$\begin{aligned} P_R s^{-1} &= \left\{ \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} ; \text{val}(n) \geq 0, \text{val}(h) = 0 \right\} \\ &= \left\{ \begin{bmatrix} 1 & p^{-k} + n \\ 0 & h \end{bmatrix} ; \text{val}(n) \geq 0, \text{val}(h) = 0 \right\}. \end{aligned} \quad (21)$$

All its elements $\begin{bmatrix} 1 & m \\ 0 & a \end{bmatrix}$ satisfy $\text{val}(a) = 0$. This implies that $P_R s^{-1} \cap g P_R \neq \emptyset$ only if $\text{val}(h_0) = 0$. Thus

$$\langle g \varepsilon_0, \varepsilon_0 \rangle = 0 \quad \text{if} \quad \text{val}(h_0) \neq 0 \quad \left(\text{for } g = \begin{bmatrix} 1 & n_0 \\ 0 & h_0 \end{bmatrix} \in P_K \right). \quad (22)$$

Assume now that $\text{val}(h_0) = 0$; replacing g by $g \begin{bmatrix} 1 & 0 \\ 0 & h_0^{-1} \end{bmatrix}$ we can assume that $h_0 = 1$ without changing $g \varepsilon_0$. One has

$$\begin{aligned} g P_R &= \left\{ \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n_1 \\ 0 & h_1 \end{bmatrix} ; \text{val}(n_1) \geq 0, \text{val}(h_1) = 0 \right\} \\ &= \left\{ \begin{bmatrix} 1 & n_1 + h_1 n_0 \\ 0 & h_1 \end{bmatrix} ; \text{val}(n_1) \geq 0, \text{val}(h_1) = 0 \right\}. \end{aligned} \quad (23)$$

One has $g P_R \cap P_R s^{-1} \neq \emptyset$ only if $\text{val}(n_0) = -k$. Let us assume that $\text{val}(n_0) = -k$. Then we need to compute the multiplicative Haar measure of the set of $h_1 \in R^*$ such that $h_1 n_0 = p^{-k} \bmod R$. This holds iff $h_1 \in p^{-k} n_0^{-1} + n_0^{-1} R = p^{-k} n_0^{-1} + p^k R$. The additive and multiplicative Haar measures coincide on R^* up to an overall coefficient $d^* h = \left(1 - \frac{1}{p}\right)^{-1} dh$. Thus we get the equality, with $g = \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} f\left(\begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix}\right) &= (1 - p^{-1})^{-1} p^{-k} & \text{if } k = -\text{val}(n_0) \\ f\left(\begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix}\right) &= 0 & \text{if } k \neq -\text{val}(n_0). \end{aligned} \quad (24)$$

This together with formula (19) gives the equality

$$\varphi_{\beta,p}(f) = p^{-k\beta}(1 - p^{\beta-1})(1 - p^{-1})^{-1}, \quad k = -\text{val}(n_0) \quad (25)$$

i.e.

$$\langle g \varepsilon_0, \varepsilon_0 \rangle_\beta = p^{-k\beta}(1 - p^{\beta-1})(1 - p^{-1})^{-1}, \quad (26)$$

where $g = \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix}$, $k = -\text{val}(n_0)$.

The next step is to understand the geometric meaning of the orbit of $\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} L_0 \right\}$ in the tree T_p and of the value $k = -\text{val}(n)$. Since the action of P_K on the tree T_p fixes a point at ∞ , it preserves the horocycles corresponding to this point. These horocycles are the equivalence classes of the relation $R_\infty : L \sim L'$ iff $\exists q \quad t^q L = t^q L'$ where t is the hyperbolic translation of one unit towards the point at ∞ .

We first check that two lattices L, L' are R_∞ equivalent iff they are on the same orbit of the subgroup $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $n \in K$ of P_K . This subgroup is a normal subgroup and hence defines an equivalence relation which is stable under the left action of P_K . The map t is given by

$$t(g L_0) = g g_p L_0 \quad \forall g \in P_K \quad \text{with } g_p = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}. \quad (27)$$

More generally $t^q(g L_0) = g g_p^q L_0 \quad \forall g \in P_K$. Thus $g_1 L_0 \sim g_2 L_0 (R_q)$ iff $g_1 g_p^q L_0 = g_2 g_p^q L_0$, i.e.

$$g_2^{-1} g_1 \in g_p^q P_R g_p^{-q}. \quad (28)$$

This holds iff $g_2^{-1} g_1$ is of the form $\begin{bmatrix} 1 & m \\ 0 & h \end{bmatrix}$, $\text{val}(h) = 0$, $\text{val}(m) \geq -q$. Thus $g_1 L_0 \sim g_2 L_0 (R_\infty)$ iff $g_2^{-1} g_1 \in K \rtimes R^*$. Two lattices $g L_0$ and $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} g L_0$ obviously

satisfy this relation and conversely, if $g_1 L_0 \sim g_2 L_0(R_\infty)$ we may write g_2 as $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} g_1$ without affecting $g_j L_0$.

Let us then understand the value of $-\text{val}(m)$ as a function of the distance between L_0 and $L = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} L_0$. Let $k = -\text{val}(m)$. We know that $t^k(L_0) = t^k(L)$ and that this fails for $k - 1$. Hence $d(L_0, L) = 2k$. We get

LEMMA 31. *Let $L, L' \in T$ be two lattices.*

- (a) *If they belong to different horocycle equivalence classes, they are orthogonal for $\langle \rangle_\beta$.*
 (b) *If they belong to the same horocycle class with distance $d(L, L') = 2k > 0$ one has*

$$\langle \varepsilon_L, \varepsilon_{L'} \rangle_\beta = p^{-k\beta} (1 - p^{\beta-1})(1 - p^{-1})^{-1}.$$

- (c) *If $L = L'$, then $\langle \varepsilon_L, \varepsilon_{L'} \rangle_\beta = 1$.*

REMARKS.

- (a) For $\beta \rightarrow +\infty$ the above inner product converges to a non zero value only if $L = L'$ or if $L \neq L'$ but $L \sim L'(R_1)$ in which case it goes to $-p^{-1}(1 - p^{-1})^{-1}$.
 (b) For $\beta \rightarrow 0$ the inner product converges to 1 on each horocycle equivalence class, which are then reduced to a single point in \mathcal{H}_β , $\beta = 0$.

Let us now compute the corresponding inner product on $\Delta = \Pi(T_p, L_0) = P_{\mathbb{Q}}^+ / P_{\mathbb{Z}}^+$. We again call L_0 the base point. Given $\begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} = g$, $n \in \mathbb{Q}$, $h \in \mathbb{Q}_+^*$, to get a non zero inner product, $\langle g L_0, L_0 \rangle$, we need that for each place p , $g_p L_0 \sim L_0(R_\infty)$ and hence that $\text{val}(g_p) = 0$. Thus $h = 1$. Let then $N = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} ; n \in \mathbb{Q} \right\}$ and let us understand the inner product on the orbit $N L_0$. We have a basis ε_x parametrized by $x \in \mathbb{Q}/\mathbb{Z}$,

$$\varepsilon_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} L_0.$$

The inner product $\langle \varepsilon_x, \varepsilon_0 \rangle_\beta$ is then given, using Lemma 31, by

$$\langle \varepsilon_x, \varepsilon_0 \rangle_\beta = \prod_{\substack{p \in \mathcal{P} \\ k_p \neq 0}} p^{-k_p \beta} (1 - p^{\beta-1})(1 - p^{-1})^{-1} \quad (29)$$

where $x = a/b$, $(a, b) = 1$ and $b = \prod_p p^{k_p}$ is the decomposition of b as a product of prime powers. More generally, it is invariant by translations, i.e. $\langle \varepsilon_x, \varepsilon_y \rangle_\beta =$

$\langle \varepsilon_{x-y}, \varepsilon_0 \rangle_\beta$; thus the positivity involved is the fact that the function given by (29) is of positive type on the group of roots of unity \mathbb{Q}/\mathbb{Z} . This function is the function ψ_β of Theorem 5.

One has $\langle g_1 L_0, g_2 L_0 \rangle_\beta = 0$ if $g_2^{-1}g_1 \notin N$.

Let us then take an arbitrary orbit $N g L_0$, $g \in P_{\mathbb{Q}}^+$. We need to compute $\langle (1, x) g L_0, (1, y) g L_0 \rangle_\beta$ where $x, y \in \mathbb{Q}$ and $(1, x), (1, y)$ are the corresponding elements of N . One has $\langle (1, x) g L_0, (1, y) g L_0 \rangle = \langle g \varepsilon_{x'}, g \varepsilon_{y'} \rangle = \langle \varepsilon_{x'-y'}, \varepsilon_0 \rangle$ where $(1, x') = g^{-1}(1, x)g$ and $(1, y') = g^{-1}(1, y)g$. Thus we see that the orbits $N \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \varepsilon_0$, $k \in \mathbb{Q}_+^*$, are pairwise orthogonal for $\langle \rangle_\beta$ and the inner product is, up to a relabelling, given by (29) on each of them.

We are now ready to describe the Hilbert spaces \mathcal{H}_β associated by Lemma 30 to the C^* module $\mathcal{E} = Be$ over $C^*(N^*)$ and the KMS state φ_β , and then get the commutant of $P_{\mathbb{Q}}^+$ in \mathcal{H}_β as a unitary representation of the C^* -algebra $C_{\mathbb{Q}}$.

PROPOSITION 32.

- (a) *Let \mathcal{H}_β be the Hilbert space completion of the C^* module $C^*(P_A)e$, over $eC^*(P_A)e = C^*(N^*)$, with state φ_β . Then \mathcal{H}_β has a natural basis indexed by $P_{\mathbb{Q}}^+/P_{\mathbb{Z}}^+$, and its inner product is invariant under left translations by $P_{\mathbb{Q}}^+$ and given by*

$$\begin{aligned} \left\langle \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \varepsilon_e, \varepsilon_e \right\rangle &= 0 && \text{unless } a = 1 \\ \left\langle \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \varepsilon_e, \varepsilon_e \right\rangle &= \psi_\beta(b) \end{aligned}$$

where ψ_β is the function of positive type defined in Theorem 5.

- (b) *The C^* -algebra $C_{\mathbb{Q}}^0$ admits a representation in \mathcal{H}_β given by the right convolution with $\delta^{\beta/2} f$ for any $P_{\mathbb{Z}}^+$ -bi-invariant function f on $P_{\mathbb{Q}}^+$.*
- (c) *The vector $\varepsilon_0 = \text{class of } P_{\mathbb{Z}}^+ \text{ is cyclic for } P_{\mathbb{Q}}^+, \text{ separating for } C_{\mathbb{Q}}, \overline{C_{\mathbb{Q}}} \varepsilon_0 \text{ is the fixed points of } P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+ \text{ and } (C_{\mathbb{Q}})'' \text{ is the commutant of } P_{\mathbb{Q}}^+ \text{ in } \mathcal{H}_\beta.$*
- (d) *The vector ε_0 defines a KMS $_\beta$ state on $C_{\mathbb{Q}}$.*

Proof. The proof of (a) follows from (29). We let $\mathcal{H}_{\beta,1}$ be the subspace of \mathcal{H}_β generated by the orbit $N \varepsilon_0$ of ε_0 under the normal subgroup N of $P_{\mathbb{Q}}^+$. More generally, for $k \in \mathbb{Q}_+^*$ we let

$$\mathcal{H}_{\beta,k} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \mathcal{H}_{\beta,1}. \quad (30)$$

The subspaces $\mathcal{H}_{\beta,k}$ are pairwise orthogonal and \mathcal{H}_β is their direct sum.

Let us prove (b). We know that the action by right convolution of the Hecke algebra \mathcal{H} of $P_{\mathbb{Z}}^+$ -bi-invariant function on $P_{\mathbb{Q}}^+$ yields a representation of \mathcal{H}^0 on the linear span of the natural basis ε_x , $x \in P_{\mathbb{Q}}^+/P_{\mathbb{Z}}^+$ of \mathcal{H}_{β} . This is still true if we twist this action by the (non involutive) automorphism of \mathcal{H} given by multiplication by $\delta^{\beta/2}$. Thus we just need to show that the new representation of \mathcal{H}^0 in \mathcal{H}_{β} is *involutive*. When we restrict this representation to the group ring of \mathbb{Q}/\mathbb{Z} , we obtain (as $\delta = 1$ on double classes in \mathbb{Q}/\mathbb{Z}) that the corresponding representation of \mathbb{Q}/\mathbb{Z} is given by

$$\rho(\gamma) \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \varepsilon_0 = \begin{bmatrix} 1 & b + \gamma \\ 0 & a \end{bmatrix} \varepsilon_0. \quad (31)$$

Thus $\rho(\gamma)$ is diagonal in the decomposition $\mathcal{H}_{\beta} = \oplus \mathcal{H}_{\beta,k}$ and its restriction to $\mathcal{H}_{\beta,k}$ is unitary since the left action of N on $\mathcal{H}_{\beta,k}$ is unitary.

When we restrict the representation ρ of \mathcal{H}^0 to the involutive subalgebra generated by the μ_n , its unitarity follows from Lemma 30 (b). The explicit computation of the isometry $U_p = \rho(\mu_p^*)$ in \mathcal{H}_{β} associated to μ_p^* , p a prime, is the following. We let as above t_p be the hyperbolic translation of one unit of length towards the point at ∞ in the tree T_p . We let it act trivially on the other trees. One then obtains

$$U_p \varepsilon_{\alpha} = p^{\beta/2-1} \sum_{t_p(\alpha')=\alpha} \varepsilon_{\alpha'}. \quad (32)$$

One can check directly using Lemma 31 that U_p is indeed an isometry.

We have thus shown that ρ is an involutive representation of \mathcal{H}^0 in \mathcal{H}_{β} and by Proposition 19, it extends to a representation of $C_{\mathbb{Q}}^0$ in \mathcal{H}_{β} . This proves (b). By construction ε_0 is cyclic for $P_{\mathbb{Q}}^+$. Since the above action of \mathcal{H}^0 (and $C_{\mathbb{Q}}^0$) commutes with $P_{\mathbb{Q}}^+$ the vector ε_0 is separating for $C_{\mathbb{Q}}^0$. The proof of the last statement of (c) is the same as in the case $\beta = 1$ (cf. Lemma 17). The proof of (d) is the same as that of Lemma 16.

Combining Corollary 29 with Proposition 32 (d) we obtain that for $\beta \in]0, 1]$ the state on $C_{\mathbb{Q}}$ given by

$$\varphi(x) = \langle \rho(x) \varepsilon_0, \varepsilon_0 \rangle \quad \forall x \in C_{\mathbb{Q}} \quad (33)$$

is the unique KMS_{β} state. This together with 32 (a) completes the proof of Theorem 5.

REMARKS 33.

- (a) Let on \mathcal{H}_{β} , Δ be the operator of multiplication by k^{β} on $\mathcal{H}_{\beta,k}$. There exists a unique weight ψ_{β} (up to a multiplicative constant) on $(P_{\mathbb{Q}}^+)^{\prime\prime}$ with modular automorphism group given by

$$\sigma_t^{\psi_{\beta}}(\cdot) = \Delta^{it} \cdot \Delta^{-it}. \quad (34)$$

For each $m \in \mathbb{N}^*$, the vector $\varepsilon_m = \begin{bmatrix} 1 & 0 \\ 0 & m^{-1} \end{bmatrix} \varepsilon_e$ is separating for $C''_{\mathbb{Q}}$ but not cyclic. Its cyclic span $\overline{C_{\mathbb{Q}} \varepsilon_m}$ defines a projection $E_m \in (P_{\mathbb{Q}})''$ which belongs to the centralizer of ψ_{β} and on which ψ_{β} is finite. On the subspace E_m the modular operator of the pair $(P''_{\mathbb{Q}}/E_m, C''_{\mathbb{Q}}$ and vector ε_m) is the restriction of Δ . The subspace E_m is the space of fixed points of the subgroup $\begin{bmatrix} 1 & m\mathbb{Z} \\ 0 & 1 \end{bmatrix}$ of $P_{\mathbb{Q}}^+$. These subspaces form a nested family (i.e. $E_m \subset E_{m'}$ if m divides m') which is total in \mathcal{H}_{β} .

- (b) We used throughout this paper the pairs of groups $P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$ instead of the pair $P_{\mathbb{Q}}, P_{\mathbb{Z}} \subset P_{\mathbb{Q}}$. The relation between the corresponding C^* -dynamical systems $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+), \sigma_t$ and $C^*(P_{\mathbb{Q}}, P_{\mathbb{Z}}), \sigma_t$ is quite simple. Indeed the latter is just the fixed point C^* -algebra of the involution α of the former given by the complex conjugation $z \rightarrow \bar{z}$ viewed as an element of $W = \text{Gal}(\mathbb{Q}^{\text{cycl}})$. This is easy to check because the double class X modulo $P_{\mathbb{Z}}$ of an element $g \in P_{\mathbb{Q}}, g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$ is the same as the double class modulo $P_{\mathbb{Z}}$ of $g \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$, $\varepsilon = \text{Sign}(a)$ which allows us to assume $a > 0$. This shows that $P_{\mathbb{Z}}$ -bi-invariant functions on $P_{\mathbb{Q}}$ yield all the $P_{\mathbb{Z}}^+$ invariant functions on $P_{\mathbb{Q}}^+$, which are invariant under the involution

$$g \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad g \in P_{\mathbb{Q}}^+. \quad (35)$$

One thus obtains the σ_t equivariant equality

$$C^*(P_{\mathbb{Q}}, P_{\mathbb{Z}}) = C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)^{\alpha} \quad (36)$$

and one can rewrite the main theorem of this paper in terms of $C^*(P_{\mathbb{Q}}, P_{\mathbb{Z}})$.

- (c) In this paper we have ignored the place at infinity in our treatment of KMS_{β} states or weights and in the construction of $C_{\mathbb{Q}}$ from the action on the product of trees (Section 2). We obtained, for the finite places, the action of P_K on the tree of $SL(2, K)$ as well as the relevant inner product on functions on the tree (Section 5) from the understanding of KMS_{β} weights on $C^*(P_K)$ and of the reduction by the projection $e \in C^*(P_K)$, $e = 1_{P_R}$.

At the infinite place the C^* -dynamical system at hand is $C^*(P_{\mathbb{R}}), \sigma_t$ where $P_{\mathbb{R}}$ is the group of matrices

$$\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \quad ; \quad b \in \mathbb{R}, a \in \mathbb{R}^* \quad (37)$$

and where σ_t is given by the module δ of $P_{\mathbb{R}}$,

$$\sigma_t(f)(g) = \delta(g)^{-it} f(g) \quad \forall f \in L^1(P_{\mathbb{R}}), t \in \mathbb{R}. \quad (38)$$

The C^* -algebra of $P_{\mathbb{R}}$ is, using the identification of \mathbb{R} with its Pontrjagin dual group, given by

$$C^*(P_{\mathbb{R}}) = C_0(\mathbb{R}) \rtimes \mathbb{R}^* \quad (39)$$

where the action of \mathbb{R}^* is by homotheties. This action has two orbits, $\mathbb{R} \setminus \{0\}$ and $\{0\}$ and to the equivariant exact sequence of C^* -algebras

$$0 \rightarrow C_0(\mathbb{R} \setminus \{0\}) \rightarrow C_0(\mathbb{R}) \rightarrow \mathbb{C} \rightarrow 0 \quad (40)$$

corresponds the exact sequence of crossed products:

$$0 \rightarrow \mathcal{K} \rightarrow C^*(P_{\mathbb{R}}) \rightarrow C^*(\mathbb{R}^*) \rightarrow 0 \quad (41)$$

similar to the exact sequence of the Toeplitz C^* -algebra. Here the two-sided ideal \mathcal{K} is the elementary C^* -algebra of compact operators. The representation theory of $C^*(P_{\mathbb{R}})$ immediately follows from (7) and besides the characters of $C^*(\mathbb{R}^*)$ which yield one-dimensional representations of $C^*(P_{\mathbb{R}})$, one has a unique infinite dimensional irreducible representation π . This representation can be described as follows. One lets $\mathcal{H} = L^2(\mathbb{R})$ with

$$(\pi(g)\xi)(t) = |a|^{1/2} \xi(at - b) \quad \forall t \in \mathbb{R}, \xi \in L^2(\mathbb{R}) \quad (42)$$

where $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$ as above, belongs to $P_{\mathbb{R}}$.

For each $\beta \in]0, \infty[$ there exists up to normalization a unique KMS_{β} weight on $C^*(P_{\mathbb{R}})$, given by

$$\varphi_{\beta}(f) = \text{Trace}(\pi(f) \Delta^{-\beta/2}) \quad \forall f \in C^*(P_{\mathbb{R}})^+ \quad (43)$$

where $\Delta = -\frac{d^2}{dt^2}$ is the Laplacian, an unbounded selfadjoint operator in $L^2(\mathbb{R})$.

By construction φ_{β} is a type I_{∞} factor weight which is a dominant weight ([C], [C-T]) on the corresponding type I_{∞} factor. It follows that for any factor M with weight ψ the centralizer of $\psi \otimes \varphi_{\beta}$ is the associated semifinite von Neumann algebra of the continuous decomposition of M , i.e. the crossed product by the modular automorphism group σ_{ψ} ,

$$(M \otimes I_{\infty})_{\psi \otimes \varphi_{\beta}} = M \rtimes_{\sigma_{\psi}} \mathbb{R}. \quad (44)$$

In our case, with $\beta \in]0, 1]$, it is natural to take for M the commutant of $P_{\mathbb{Q}}^+$ acting in the Hilbert space \mathcal{H}_{β} (cf. Proposition 32). To obtain the crossed product (44), it is then natural to use at the infinite place the Hilbert space $\mathcal{H}_{\beta}^{\infty}$ of the GNS construction of the weight φ_{β} on $C^*(P_{\mathbb{R}})$. In the tensor product $\mathcal{H}_{\beta} \otimes \mathcal{H}_{\beta}^{\infty}$ one has a natural product action of the group P_A over the adeles $A = \mathcal{A} \times \mathbb{R}$. The crossed

product (44) is then contained in the commutant of $P_{\mathbb{Q}}^{+}$ which is a *discrete subgroup* of P_A .

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