Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry

Alain Connes[1] **and Henri Moscovici**[2*]

[1] Collège de France, 3, rue Ulm, 75005 Paris
and
I.H.E.S., 35, route de Chartres, 91440 Bures-sur-Yvette
[2] Department of Mathematics, The Ohio State University
231 West 18th Avenue, Columbus, OH 43210-1174, USA

Dedicated to André HAEFLIGER, whose philosophy *of differentiable cohomology has inspired the present paper*

Abstract

We prove a cyclic cohomological analogue of Haefliger's van Est-type theorem for the groupoid of germs of diffeomorphisms of a manifold. The differentiable version of cyclic cohomology is associated to the algebra of transverse differential operators on that groupoid, which is shown to carry an intrinsic Hopf algebraic structure. We establish a canonical isomorphism between the periodic Hopf cyclic cohomology of this extended Hopf algebra and the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields. We then show that this isomorphism can be explicitly implemented at the cochain level, by a cochain map constructed out of a fixed torsion-free linear connection. This allows the direct treatment of the index formula for the hypoelliptic signature operator – representing the diffeomorphism invariant transverse fundamental K-homology class of an oriented manifold – in the general case, when this operator is constructed by means of an arbitrary coupling connection.

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Introduction

The local index formula for hypoelliptic differential operators in a diffeomorphism invariant setting [9] expresses the Chern character of such operators in terms of a particular kind of cocycles of the cyclic bicomplex, satisfying a property analogous to that involved in Haefliger's definition of differentiable cohomology for the groupoid of germs of diffeomorphisms on a manifold [16]. Applying it to the transverse index problem on foliations, we have shown in [10] that, if the complete transversal is chosen to be an affine flat manifold, the "differentiable" cocycle representing the Chern character of such an operator is automatically in the range of a natural characteristic map, originating from the cyclic cohomology of a Hopf algebra $\mathcal{H}_n$ canonically associated to Diff $(\mathbb{R}^n)$. In turn, the periodic Hopf cyclic cohomology of $\mathcal{H}_n$ and its $SO(n)$-relative version were shown to be canonically isomorphic to corresponding Gelfand-Fuchs cohomologies of formal vector fields. The upshot was proving that, in cohomological form, the index formula for transversely hypoelliptic operators on foliations can be expressed in terms of Gelfand-Fuchs classes, transported via the characteristic map. In this paper we incorporate ab initio the curvature into our approach and thus dispense with the geometrically unsatisfactory flatness condition on the transversal.

The new device that allows the direct treatment of the curved case is a "thickened" version $\mathcal{H}_{FM}$ of the Hopf algebra $\mathcal{H}_n$. If one regards $\mathcal{H}_n$ as an algebra of transverse differential operators with constant coefficients, $\mathcal{H}_{FM}$ should be viewed as the algebra of transverse differential operators with variable coefficients on $FM \bar{\rtimes} \Gamma_M$, the étale groupoid of germs of diffeomorphisms of a given n-manifold M lifted to its frame bundle FM. The algebra $\mathcal{H}_{FM}$ is naturally a bimodule over the coefficient ring $\mathcal{R}_{FM} = C^\infty(FM)$ and it affords a Hopf structure only in an extended sense, the coproduct taking values in the tensor product over $\mathcal{R}_{FM}$. Like its precursor $\mathcal{H}_n$, $\mathcal{H}_{FM}$ too gives rise to a natural cyclic module, consisting of $(\mathcal{R}_{FM}, \mathcal{R}_{FM})$-bimodules of multidifferential operators.

The main result of this paper, which can be viewed as a cyclic cohomological analogue of Haefliger's van Est-type theorem for the groupoid of germs of diffeomorphisms of M [16, Theorem IV.4], establishes a canonical isomorphism between the periodic Hopf cyclic cohomology of $\mathcal{H}_{FM}$ and the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$, as well as between their corresponding relative versions. This isomorphism is implemented at the cochain level by a cochain map manufactured out of a given torsion-free connection. In this way we exhibit a purely geometric construction, tracking the displacement of the connection and curvature forms under diffeomorphisms, for the cyclic cohomological counterparts of the classical Chern and secondary classes of foliations ([1], [15]).

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1 The algebra $\mathcal{H}_{FM}$

We begin by introducing $\mathcal{H}_{FM}$, as an algebra of transverse differential operators on the smooth étale groupoid associated to the diffeomorphisms of the base manifold acting on its frame bundle.

Let M be a smooth n-manifold and let FM denote the bundle of frames on M. We denote by Γ_M the pseudogroup of all local diffeomorphisms of M; its elements are partial diffeomorphisms $\psi : \mathrm{Dom}\psi \to \mathrm{Ran}\psi$, with both the domain $\mathrm{Dom}\psi$ and range $\mathrm{Ran}\psi$ open subsets of M. The prolongation to FM of a partial diffeomorphism $\psi \in \Gamma_M$ will be denoted by $\widetilde{\psi}$. The set of all pairs $(u, \widetilde{\varphi})$ with $\varphi \in \Gamma_M$ and $u \in \mathrm{Ran}\widetilde{\varphi}$ will be denoted $FM \rtimes \Gamma_M$. We next form the associated smooth étale groupoid of germs of lifted diffeomorphisms

$$FM \bar{\rtimes} \Gamma_M := \{ [u, \widetilde{\varphi}] ; \quad \varphi \in \Gamma_M, \quad u \in \mathrm{Ran}\widetilde{\varphi} \},$$

where $[u, \widetilde{\varphi}]$ stands for the class of $(u, \widetilde{\varphi}) \in FM \rtimes \Gamma_M$ with respect to the equivalence relation which identifies $(u, \widetilde{\varphi})$ and $(v, \widetilde{\psi})$ if $u = v$ and $\widetilde{\varphi}^{-1}$ coincides with $\widetilde{\psi}^{-1}$ on a neighborhood of u. The corresponding *source* and *target* maps are

$$s[u, \widetilde{\varphi}] = \widetilde{\varphi}^{-1}(u) \in \mathrm{Dom}\widetilde{\varphi}\,, \quad \text{resp.} \quad t[u, \widetilde{\varphi}] = u\,,$$

and the composition rule is

$$[u, \widetilde{\varphi}] \circ [v, \widetilde{\psi}] = [u, \widetilde{\varphi} \circ \widetilde{\psi}]\,, \quad \text{if} \quad v \in \mathrm{Dom}\widetilde{\varphi} \quad \text{and} \quad \widetilde{\varphi}(v) = u\,.$$

We let

$$\mathcal{A} \equiv \mathcal{A}_{FM} := C_c^\infty(FM \bar{\rtimes} \Gamma_M)$$

denote its convolution algebra. From a practical standpoint, it is convenient to regard $\mathcal{A}_{FM}$ as being linearly spanned by monomials of the form

$$f\, U_{\psi}^{*}\,, \quad \text{with} \quad f \in C_c^{\infty}(\mathrm{Dom}\widetilde{\psi})\,,$$

where the asterisk stands for the inverse, with the understanding that

$$f_1\, U_{\psi_1}^{*} \equiv f_2\, U_{\psi_2}^{*} \quad \text{iff} \quad f_1 = f_2 \quad \text{and} \quad \widetilde{\psi}_1|_V = \widetilde{\psi}_2|_V\,,$$

where V is a neighborhood of $\mathrm{Supp}(f_i)$, $i = 1$ or 2. The multiplication rule for such monomials is given by

$$f_1\, U_{\psi_1}^{*} \cdot f_2\, U_{\psi_2}^{*} = f_1(f_2 \circ \widetilde{\psi}_1)\, U_{\psi_2\psi_1}^{*}\,;$$

note that, by hypothesis, the support of $f_1(f_2 \circ \widetilde{\psi}_1)$ is a compact subset of $\mathrm{Dom}\widetilde{\psi_1} \cap \widetilde{\psi}_1^{-1}(\mathrm{Dom}\widetilde{\psi}_2) \subset \mathrm{Dom}(\widetilde{\psi}_2 \circ \widetilde{\psi}_1)$.

The function algebra

$$\mathcal{R} \equiv \mathcal{R}_{FM} := C^{\infty}(FM)$$

acts in two ways on $\mathcal{A}$, by left multiplication operators

$$\alpha(b)(f\, U_{\psi}^{*}) = b \cdot f\, U_{\psi}^{*}\,, \quad b \in \mathcal{R}\,, \tag{1.1}$$

and by right multiplication operators

$$\beta(b)(f\, U_{\psi}^{*}) = \ f\, U_{\psi}^{*} \cdot b = b \circ \widetilde{\psi} \cdot f\, U_{\psi}^{*}\,, \quad b \in \mathcal{R}\,. \tag{1.2}$$

On the other hand, any vector field Z on F can be extended to a linear transformation, although no longer a derivation in general, $Z \in \mathcal{L}(\mathcal{A})$, by setting

$$Z(f\, U_{\psi}^{*}) = Z(f)\, U_{\psi}^{*}\,, \quad f\, U_{\psi}^{*} \in \mathcal{A}. \tag{1.3}$$

The following definition can be easily adapted to cover the case of an arbitrary base manifold. However, for the simplicity of the exposition, we shall *assume from now on that the manifold M admits a finite atlas.*

Definition 1 *A* transverse differential operator *on the groupoid $FM \bar{\rtimes} \Gamma_M$ is an element of the subalgebra of linear operators on $\mathcal{A}$*

$$\mathcal{H} \equiv \mathcal{H}_{FM} \subset \mathcal{L}(\mathcal{A}_{FM})$$

generated by the transformations (1.1), (1.2) and (1.3). A p-differential operator *on* $FM\bar{\rtimes}\Gamma_M$ *is a* p*-linear transformation* H *on* $\mathcal{A}_{FM}$ *with values in* $\mathcal{A}_{FM}$*, of the form*

$$H(a^1, ..., a^p) = \sum_{i=1}^{r} h_i^1(a^1) \cdots h_i^p(a^p) \, , \quad a^1, ..., a^p \in \mathcal{A}_{FM} \, , \tag{1.4}$$

with $h_i^1, \ldots, h_i^p \in \mathcal{H}_{FM}$.

The adjective *transverse* is meant to emphasize the distinction between this notion and that of *longitudinal* or *invariant* (pseudo)differential operator of [4] (cf. also [20] for a systematic treatment of the latter in the context of Lie algebroids.)

There are two built-in algebra homomorphisms $\alpha : \mathcal{R} \to \mathcal{H}$ and $\beta : \mathcal{R} \to \mathcal{H}$, whose images commute

$$\alpha(b_1) \; \beta(b_2) = \beta(b_2) \; \alpha(b_1) \, , \quad \forall \quad b_1, b_2 \in \mathcal{R}; \tag{1.5}$$

we shall view $\mathcal{H}$ as an $(\mathcal{R}, \mathcal{R})$-bimodule with the *left action* of $\mathcal{R}$ given by *left multiplication via* α and the *right action* given by *left multiplication via* β. More generally, for any $p \geq 1$, the space of p-differential operators on $FM\bar{\rtimes}\Gamma_M$

$$\mathcal{H}^{[p]} \equiv \mathcal{H}_{FM}{}^{[p]} \tag{1.6}$$

can be endowed with an $(\mathcal{R}, \mathcal{R})$-bimodule structure in a similar fashion, via left multiplication by means of α, respectively β.

The remainder of this section will be devoted to the proof of two pivotal results, clarifying the structure of these bimodules.

We fix a torsion-free connection on FM, with connection form $\omega = (\omega^i_j)$. We denote by $\theta = (\theta^i)$ the canonical form

$$\theta_u(Z) = u^{-1}(\pi_*(Z)) \, , \quad \forall Z \in T_u FM \, ,$$

where $\pi : FM \to M$ is the projection and the frame $u \in FM$ is viewed as an isomorphism of vector spaces $u : \mathbb{R}^n \xrightarrow{\simeq} T_{\pi(u)}M$.

Let $X_1, ..., X_n$ be the standard horizontal vector fields corresponding to the standard basis of $\mathbb{R}^n$ and $\{Y_i^j\}$ the fundamental vertical vector fields corresponding to the standard basis of $\mathfrak{gl}(n, \mathbb{R})$, such that $\{X_k, Y_i^j\}$ and $\{\theta^k, \omega^i_j\}$

are dual to each other. Since $\{X_k|_u, Y_i^j|_u\}$ form a basis of T_uFM for every $u \in FM$, as an algebra, $\mathcal{H}$ is generated by $\alpha(\mathcal{R})$, $\beta(\mathcal{R})$ and the endomorphisms of $\mathcal{A}$ associated to the vector fields X_k and Y_i^j (cf. (1.3)). The usual commutation relations for the vector fields associated to a torsion-free connection continue to hold in $\mathcal{H}$:

$$
\begin{aligned}
&[Y_i^j, Y_k^\ell] = \delta_k^j Y_i^\ell - \delta_i^\ell Y_k^j \,, \\
&[Y_i^j, X_k] = \delta_k^j X_i \,, \qquad (1.7) \\
&[X_k, X_\ell] = \sum \alpha(R^i_{jk\ell}) Y_i^j \,,
\end{aligned}
$$

where the functions $R^i_{jk\ell} \in \mathcal{R}$ are related to the curvature form Ω of the given connection by

$$\Omega^i_j = \sum_{k<\ell} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell \,, \quad R^i_{jk\ell} = -R^i_{j\ell k} \,.$$

Also, for any $b \in \mathcal{R}$ one has

$$
\begin{aligned}
[Y_i^j, \alpha(b)] &= \alpha(Y_i^j(b)) \,, \\
[X_k,\ \alpha(b)] &= \alpha(X_k(b)) \,.
\end{aligned} \qquad (1.8)
$$

On the other hand, the commutators between the standard horizontal vector fields and $\beta(\mathcal{R})$ introduce new operators acting on $\mathcal{A}$. Indeed, while the fundamental vector fields are Γ_M-invariant and therefore

$$[Y_i^j, \beta(b)] = \beta(Y_i^j(b)) \,, \qquad (1.9)$$

for the standard horizontal vector fields one has, with the usual summation convention,

$$U_\varphi\, X_k\, U_\varphi^* - X_k = \rho^i_{jk}\, Y_i^j \;; \qquad (1.10)$$

the coefficients ρ^i_{jk}, which are functions on $FM \bar{\rtimes} \Gamma_M$ involving the second order jet of the local diffeomorphism, can be alternatively expressed as

$$\rho^i_{jk} = \gamma^i_{jk} \circ \widetilde{\varphi}^{-1} \,, \quad \text{with} \quad \gamma^i_{jk} = \langle \widetilde{\varphi}^* {\omega_j}^i \,, X_k \rangle \,. \qquad (1.11)$$

It then follows that

$$[X_k, \beta(b)] = \beta(X_k(b)) + \beta(Y_i^j(b))\, \delta^i_{jk} \,, \qquad (1.12)$$

where $\delta^i_{jk} \in \mathcal{L}(\mathcal{A})$ are the operators of multiplication by the functions $\gamma^i_{jk} \in C^\infty(FM \bar{\rtimes} \Gamma_M)$,

$$\delta^i_{jk}(f\, U^*_\varphi) := \gamma^i_{jk} \cdot f\, U^*_\varphi = f\, U^*_\varphi \cdot \rho^i_{jk}\,. \tag{1.13}$$

Lemma 2 *The operators δ^i_{jk} are transverse differential operators on the étale groupoid $FM \bar{\rtimes} \Gamma_M$.*

Proof. Due to the existence of a finite atlas on M and using a partition of unity argument, it suffices to show that

$$\beta(\chi \circ \pi)\; \delta^i_{jk} \in \mathcal{H}$$

for any function $\chi \in C^\infty_c(M)$ with support in a coordinate chart $U \subset M$. Equation (1.12) ensures that

$$\beta\left(Y^j_i(b)\right)\; \beta(\chi \circ \pi)\, \delta^i_{jk} \in \mathcal{H}\,, \quad \forall b \in \mathcal{R}\,. \tag{1.14}$$

Choose $b \in M_n(\mathcal{R})$ such that, in local coordinates $u = (x^k,\, y^j_i)$ on $\pi^{-1}(U)$,

$$b(u) = y\,.$$

Then

$$Y^j_i(b^s_\ell)\ \mid_{\pi^{-1}(U)} = \ y^r_i\ \frac{\partial}{\partial y^r_j}\ (y^s_\ell) = y^s_i\, \delta^j_\ell\,. \tag{1.15}$$

We now choose $d \in M_n(\mathcal{R})$ such that its restriction to $\pi^{-1}(U)$ is

$$d(u) = y^{-1}.$$

From (1.15) it follows that

$$\beta(d^r_s)\ \beta\left(\ Y^j_i(b^s_\ell)\right)\ \beta(\chi \circ \pi)\, \delta^i_{jk} \ = \ \beta(\chi \circ \pi)\, \delta^r_{lk}\,,$$

and by (1.14) the left hand side belongs to the algebra $\mathcal{H}$. $\square$

An equivalent form for the equation (1.11) is

$$\widetilde{\varphi}^*\omega \ - \ \omega \ = \ \gamma \cdot \theta\,; \tag{1.16}$$

from this it easily follows that, as a vector valued function on the frame bundle, γ is tensorial with respect to the right action of the structure group

$GL^+(n,\mathbb{R})$. At the infinitesimal level, this translates into the following expressions for the commutators between the Y_j^i and $\delta_{\ell m}^k$:

$$[Y_j^i, \delta_{\ell m}^k] = \delta_\ell^i \, \delta_{jm}^k + \delta_m^i \, \delta_{\ell j}^k - \delta_j^k \, \delta_{\ell m}^i \,, \tag{1.17}$$

The commutators with X_k however yield new operators, involving higher order jets of diffeomorphisms:

$$\delta_{jk,\ell_1 \dots, \ell_r}^i = [X_{\ell_r}, \dots [X_{\ell_1}, \delta_{jk}^i] \dots] \,; \tag{1.18}$$

these operators, acting on $\mathcal{A}$, have the form

$$\delta_{jk,\ell_1 \dots, \ell_r}^i (f \, U_\varphi^*) = \gamma_{jk,\ell_1 \dots \ell_r}^i \cdot f \, U_\varphi^* \,, \qquad \gamma_{jk,\ell_1 \dots \ell_r}^i = X_{\ell_r} \cdots X_{\ell_1}(\gamma_{jk}^i). \tag{1.19}$$

In particular, they form an *abelian subalgebra* and also commute with image of $\mathcal{R}$ through both maps α and β.

We are now ready to prove the main results of this section. Recall that $\mathcal{H}$ is viewed as an $(\mathcal{R},\mathcal{R})$-bimodule under the *left action* of $\mathcal{R}$ via multiplication by the image of α, resp. β.

Proposition 3 *The $(\mathcal{R},\mathcal{R})$-bimodule $\mathcal{H}$ is free over $\mathcal{R} \otimes \mathcal{R}$. The choice of a torsion-free connection on FM gives rise to a Poincaré-Birkhoff-Witt-type basis of $\mathcal{H}$ over $\mathcal{R} \otimes \mathcal{R}$.*

Proof. Once the connection ω is fixed, we use the same notation as above for the associated generators of the algebra $\mathcal{H}$. In addition, we shall need to employ two kinds of multi-indices, whose entries form an increasing sequence with respect to the obvious lexicographic order. The first kind are of the form

$$I = \left\{ i_1 \le \cdots \le i_p \,; \, \begin{pmatrix} j_1 \\ k_1 \end{pmatrix} \le \cdots \le \begin{pmatrix} j_q \\ k_q \end{pmatrix} \right\} ,$$

while the second kind of the form

$$\kappa = \left\{ \begin{pmatrix} i_1 \\ j_1 \, k_1; \ell_1^1 \le \ldots \le \ell_{p_1}^1 \end{pmatrix} \le \cdots \le \begin{pmatrix} i_r \\ j_r \, k_r; \ell_1^r \le \ldots \le \ell_{p_r}^r \end{pmatrix} \right\} .$$

With this notation, we set

$$Z_I = X_{i_1} \ldots X_{i_p} \, Y_{k_1}^{j_1} \ldots Y_{k_q}^{j_q} \quad \text{and} \quad \delta_\kappa = \delta_{j_1 \, k_1; \ell_1^1 \ldots \ell_{p_1}^1}^{i_1} \ldots \delta_{j_r \, k_r; \ell_1^r \ldots \ell_{p_r}^r}^{i_r} \,.$$

Using the relations (1.7)-(1.9), (1.12), Lemma 2, (1.17) and (1.18), it is easy to check that the collection $\{\delta_\kappa \cdot Z_I\}$, where I and κ are multi-indices of the first and second kind respectively, forms a generating set of $\mathcal{H}$ over $\mathcal{R} \otimes \mathcal{R}$. We thus only need to prove that

$$\sum_{I,\kappa} \alpha(\ell_{I,\kappa})\,\beta(r_{I,\kappa})\,\delta_\kappa \cdot Z_I = 0 \quad \Rightarrow \quad \ell_{I,\kappa} \otimes r_{I,\kappa} = 0\,, \quad \forall\,(I,\kappa)\,. \tag{1.20}$$

Evaluating the expression in (1.20) on an arbitrary element $f\,U_\varphi^* \in \mathcal{A}$ one gets for any $u \in FM$

$$\sum_{I,\kappa} \ell_{I,\kappa}(u)\,r_{I,\kappa}(v)\,\gamma_\kappa(u,\widetilde{\varphi})\,(Z_I\,f)(u) = 0\,, \quad v = \widetilde{\varphi}(u)\,. \tag{1.21}$$

Let us fix, for the moment, $u_0, v_0 \in FM$ and set

$$\Gamma^{(1)}\,(u_0,v_0) \;=\; \{\varphi \in \Gamma_M\,;\; \widetilde{\varphi}(u_0) = v_0\}\,.$$

By varying $\varphi \in \Gamma^{(1)}\,(u_0,v_0)$, we shall first prove that (1.21) implies that, for any fixed multi-index of the second kind κ and for any function $f \in C_c{}^\infty(FM)$ supported around u_0, one has

$$\sum_{I} \ell_{I,\kappa}(u_0)\,r_{I,\kappa}(v_0)\,(Z_I\,f)(u_0) = 0\,. \tag{1.22}$$

Let $\Phi_{u_0} : \mathbb{R}^n \xrightarrow{u_0} T_{x_0}\,M \xrightarrow{\exp_{x_0}} M$ be normal coordinates around $x_0 = \pi(u_0)$ and similarly around $y_0 = \pi(v_0)$. Clearly, $\varphi \in \Gamma^{(1)}\,(u_0,v_0)$ iff $\Phi_{v_0}^{-1} \circ \varphi \circ \Phi_{u_0} \in \mathrm{Diff}_0^{(1)}$:= the pseudogroup of all local diffeomorphisms ψ of $\mathbb{R}^n$ such that their first order jet at 0 satisfies $j_0^1(\psi) = j_0^1(\mathrm{Id})\}$.

In normal coordinates and using the customary notation (see [10], also [25], for more detailed local calculations), one has:

$$Y_i^j = y_i^\mu\,\frac{\partial}{\partial\,y_j^\mu} \equiv y_i^\mu\,\partial_\mu^j\,, \tag{1.23}$$

$$X_k = y_k^\mu\,(\partial_\mu - \Gamma_{\alpha\mu}^\nu\,y_j^\alpha\,\partial_\nu^j)\,, \tag{1.24}$$

and

$$\gamma_{ij}^k(x,y,\widetilde{\psi}) = \left(\widetilde{\Gamma}_{\alpha\mu}^\nu(x) - \Gamma_{\alpha\mu}^\nu(x)\right)\,y_j^\alpha\,y_i^\mu\,(y^{-1})_\nu^k \tag{1.25}$$

where
$$\begin{aligned}\widetilde{\Gamma}^{\nu}_{\alpha\mu}(x) &= (\partial\psi(x)^{-1})^{\nu}_{\delta}\,\Gamma^{\delta}_{\varepsilon\zeta}(\psi(x))\,\partial_{\alpha}\,\psi(x)^{\varepsilon}\,\partial_{\mu}\,\psi(x)^{\zeta} \\ &+ (\partial\,\psi(x)^{-1})^{\nu}_{\delta}\,\partial_{\mu}\,\partial_{\alpha}\,\psi(x)^{\delta}\,.\end{aligned} \tag{1.26}$$

Since, by the very choice of local coordinates, $\Gamma^{\nu}_{\alpha\mu}(0) = 0$, it follows from (1.26) that

$$\widetilde{\Gamma}^{\nu}_{\alpha\mu}(0) = \delta^{\gamma}_{\delta}\,\Gamma^{\delta}_{\varepsilon\zeta}(0)\,\delta^{\varepsilon}_{\alpha}\,\delta^{\zeta}_{\mu} + \delta^{\gamma}_{\delta}(\partial_{\mu}\,\partial_{\alpha}\,\psi^{d})(0) \,=\, (\partial_{\mu}\,\partial_{\alpha}\,\psi^{\nu})(0)\,.$$

Therefore, by (1.25), $\forall\,\psi \in \mathrm{Diff}^{(1)}_0$, one has

$$\gamma^{k}_{ij}(0,y,\widetilde{\psi}) = \partial_{\mu}\,\partial_{\alpha}\,\psi^{\nu}(0)\cdot y^{\alpha}_{j}\,y^{\mu}_{i}\,(y^{-1})^{k}_{\nu}\,, \tag{1.27}$$

just like in the affine flat case [10] .

We can now prove (1.22) by induction on the "height" $|\kappa|$ of the multi-index κ, i.e. the number p of indices in $\delta^{i}_{jk;\ell_1\ldots\ell_p}$, counting the number of commutators with X_ℓ.

From (1.24)-(1.26) it follows that the top degree component of the jet of ψ occuring in the expression

$$X_{\ell_p}\ldots X_{\ell_1}\,(\gamma^{i}_{jk})(0,y,\widetilde{\psi})$$

is of the form

$$(\partial_{\ell_1}\ldots\partial_{\ell_p}\,\psi^{\nu}(0))\cdot(y^{\cdot}_{\cdot})\cdots(y^{-1})^{\cdot}_{\cdot}\,.$$

By first choosing ψ with $j^{p-1}_0(\psi) = j^{p-1}_0(\mathrm{Id})$, we can reduce to the situation $|\kappa| = p$, when the equations look just like in the flat case. We can then vary the p-th jet of ψ to get the vanishing of its coefficient and thus drop the height.

We are now reduced to proving that if $\forall\,u,v \in FM$,

$$\sum_I\,\ell_I(u)\,r_I(v)\,(Z_I\,f)(u) = 0\,,\ \forall\,f \in C^{\infty}_c(FM)\,,$$

then, for all indices I of the first kind, one has

$$\ell_I \otimes r_I \,=\, 0\,.$$

This can be done by induction, in the same manner as before, using (1.23) and (1.24). Alternatively, it also follows from the Poincaré-Birkhoff-Witt theorem for Lie algebroids ([23], [20]), applied to TFM. $\square$

Proposition 4 *There is a unique isomorphism of vector spaces*

$$T : \mathcal{H}^{\otimes_{\mathcal{R}} p} \equiv \underbrace{\mathcal{H} \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} \mathcal{H}}_{p\text{-}times} \quad \stackrel{\simeq}{\longrightarrow} \quad \mathcal{H}^{[p]} \,,$$

where $\mathcal{H}^{[p]}$ *(cf. (1.4), (1.6)) is the space of transverse p-differential operators, such that* $\forall\, h_1, \ldots, h_p \in \mathcal{H}$ *and* $\forall\, a^1, \ldots, a^p \in \mathcal{A}$ *,*

$$T(h_1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h_p)\,(a^1, \ldots, a^p) \;=\; h_1(a^1) \cdots h_p(a^p)\,. \qquad (1.28)$$

Proof. The assignment given by (1.28) clearly extends by linearization to a well-defined epimorphism $T : \mathcal{H}^{\otimes_{\mathcal{R}} p} \rightarrow \mathcal{H}^{[p]}$. It remains to prove that $\mathrm{Ker}\; T \;=\; 0$. Assume therefore that

$$\sum_i h_1{}^i(a^1) \cdots h_p{}^i(a^p) \;=\; 0\,, \quad \forall\, a^1, \ldots, a^p \in \mathcal{A}\,. \qquad (1.29)$$

After fixing a Poincaré-Birkhoff-Witt $(\mathcal{R}, \mathcal{R})$-basis $\{B_J = \delta_\kappa \cdot Z_I \,;\, J = I \cup \kappa\}$ of $\mathcal{H}$ as above, we may express each $h_j{}^i$ under the form

$$h_j^i = \sum_J \; \alpha(\ell_j^{i,J})\, \beta(r_j^{i,J})\, B_J\,, \quad \text{with} \quad \ell_j^{i,J},\, r_j^{i,J} \in \mathcal{R}\,.$$

By the hypothesis (1.29), for any $f_1\, U^*_{\varphi_1}, \ldots, f_p\, U^*_{\varphi_p} \in \mathcal{A}$,

$$\sum \; \ell_1^{i,J_1}\; B_{J_1}(f_1\, U^*_{\varphi_1})\, r_1^{i,J_1}\, \ell_2^{i,J_2}\; B_{J_2}(f_2\, U^*_{\varphi_2}) \ldots r_{p-1}^{i,J_{p-1}}\, \ell_p^{i,J_p}\; B_{J_p}(f_p\, U^*_{\varphi_p})\, r_p^{i,J_p} \equiv 0\,.$$

Evaluating the left hand side at $u_0 \in FM$ and setting

$$u_1 = \widetilde{\varphi}_1(u_0)\,,\; u_2 = \widetilde{\varphi}_2(u_1)\,, \ldots, u_p = \widetilde{\varphi}_p(u_{p-1})$$

one gets:

$$\begin{gathered}\sum \ell_1^{i,J_1}(u_0)\, r_1^{i,J_1}(u_1)\, \ell_2^{i,J_2}(u_1) \ldots r_{p-1}^{i,J_{p-1}}(u_{p-1})\, \ell_p^{i,J_p}(u_{p-1})\, r_p^{i,J_p}(u_p) \\ \cdot\gamma_{\kappa_1}(\varphi_1, u_0)\, \gamma_{\kappa_2}(\varphi_2, u_1) \ldots \gamma_{\kappa_p}(\varphi_p, u_{p-1}) \\ \cdot Z_{I_1}(f_1)(u_0)\, Z_{I_2}(f_2)(u_1)\; \ldots\; Z_{I_p}(f_p)(u_{p-1}) \;= 0\,.\end{gathered}$$

Following the same line of arguments as in the preceeding proof, we infer that $\forall\, J_1, \ldots, J_p$ one has

$$\sum_i \ell_1^{i,J_1}(u_0)\, r_1^{i,J_1}(u_1)\, \ell_2^{i,J_2}(u_1) \ldots r_{p-1}^{i,J_{p-1}}(u_{p-1})\, \ell_p^{i,J_p}(u_{p-1})\, r_p^{i,J_p}(u_p) = 0. \quad (1.30)$$

We now choose a basis $\{b_\lambda\}_{\lambda=1}^N$ of the finite-dimensional subspace of $\mathcal{R}$ generated by the functions $\ell_k^{i,J_k}, r_k^{i,J_k}$ and express them as linear combinations, with constant coefficients, of the basis elements:

$$\ell_1^{i,J_1} = \sum_{\lambda_1} c_1^{i,J_1,\lambda_1} b_{\lambda_1}$$

$$r_1^{i,J_1} \ell_2^{i,J_2} = \sum_{\lambda_2} c_2^{i,J_1,J_2,\lambda_2} b_{\lambda_2}$$

$$\dots$$

$$r_{p-1}^{i,J_{p-1}} \ell_p^{i,J_p} = \sum_{\lambda_p} c_p^{i,J_{p-1},J_p,\lambda_p} b_{\lambda_p}$$

$$r_p^{i,J_p} = \sum_{\lambda_{p+1}} c_{p+1}^{i,J_p,\lambda_{p+1}} b_{\lambda_{p+1}} .$$

From (1.30) it then follows that $\forall\, \lambda_1, \dots, \lambda_{p+1}$

$$\sum_i c_1^{i,J_1,\lambda_1} \cdot c_2^{i,J_1,I_2,\lambda_2} \cdot c_{p+1}^{i,J_p,\lambda_{p+1}} = 0 . \tag{1.31}$$

We can conclude that

$$\sum_i h_1^i \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} h_p^i =$$

$$\sum_{i,J} \alpha(\ell_1^{i,J_1}) B_{J_1} \otimes_{\mathcal{R}} \alpha(r_1^{i,J_1} \ell_2^{i,J_2}) B_{J_2} \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \alpha(r_{p-1}^{i,J_{p-1}} \ell_p^{i,J_p}) \beta(r_p^{i,J_p}) B_{J_p}$$

$$= \sum_{J,\lambda} \sum_i c_1^{i,J_1,\lambda_1} \dots c_{p+1}^{i,J_p,\lambda_{p+1}} \alpha(b_{\lambda_1}) B_{J_1} \otimes_{\mathcal{R}} \dots \otimes_{\mathcal{R}} \alpha(b_{\lambda_p}) \beta(b_{\lambda_{p+1}}) B_{J_p}$$

$$= 0 , \quad \text{by } (1.31) . \qquad \square$$

Remark 5 The algebra $\mathcal{H}_{FM}$ acts naturally on the frame bundle $F\mathbb{G}$ of every smooth étale groupoid $\mathbb{G}$ over M. Moreover, if $\mathbb{G}$ is *full* , in the sense that the natural map from $\mathbb{G}$ to the groupoid of jets of diffeomorphisms of M has dense range (i.e. surjects on k-jets for every positive integer k), then $\mathcal{H}_{FM}$ is completely determined by its action on $\mathbb{G}$. Indeed, the proof of Proposition 3 remains unchanged if in Definition 1 one replaces the algebra $C_c^\infty(FM \bar{\rtimes} \Gamma_M)$ by $C_c^\infty(\mathbb{G})$.

2 The Hopf algebraic structure of $\mathcal{H}_{FM}$

We devote this section to the description of the intrinsic Hopf structure carried by the algebra $\mathcal{H} = \mathcal{H}_{FM}$. It fits the pattern of the definition of a *Hopf algebroid*, cf. [22, Appendix 1] and [18], with $\mathcal{H}$ as the total algebra and $\mathcal{R}$ as the base algebra. The source and target homomorphisms $\alpha : \mathcal{R} \to \mathcal{H}$, resp. $\beta : \mathcal{R} \to \mathcal{H}$, obey the commutation property (1.5) and confer to $\mathcal{H}$ its $(\mathcal{R}, \mathcal{R})$-bimodule structure.

In order to define the coproduct, we first note that the generators of $\mathcal{H}$ satisfy product rules when acting on $\mathcal{A}$. Indeed, one easily checks that, for any $a^1, a^2 \in \mathcal{A}$, one has

$$\begin{aligned}
\alpha(\ell)\,(a^1 a^2) &= \alpha(\ell)(a^1)\cdot a^2\,, \qquad \forall\, \ell \in \mathcal{H}\,,\\
\beta(r)\,(a^1 a^2) &= a^1 \cdot \beta(r)(a^2)\,, \qquad \forall\, r \in \mathcal{H}\,,\\
Y_i^j(a^1 a^2) &= Y_i^j(a^1)\, a^2 + a^1\, Y_i^j(a^2)\,,\\
X_k(a^1 a^2) &= X_k(a^1)\, a^2 + a^1\, X_k(a^2) + \delta_{jk}^i(a^1)\, Y_i^j(a^2)\,,\\
\delta_{jk}^i(a^1 a^2) &= \delta_{jk}^i(a^1)\, a^2 + a^1\, \delta_{jk}^i(a^2)\,.
\end{aligned} \tag{2.1}$$

These identities are all of the form

$$h(a^1 a^2) \;=\; \sum h_{(1)}(a^1)\, h_{(2)}(a^2)\,, \quad \text{with } h_{(1)}, h_{(2)} \in \mathcal{H}\,, \tag{2.2}$$

where the sum in the right hand side stands for the customary Sweedler summation convention. By multiplicativity, the rule (2.2) extends to all differential operators $h \in \mathcal{H}$. While the right hand side in (2.2) does not uniquely determine an element $\sum h_{(1)} \otimes h_{(2)} \in \mathcal{H} \otimes \mathcal{H}$, the ambiguity disappears in the tensor product over $\mathcal{R}$, $\sum h_{(1)} \otimes_{\mathcal{R}} h_{(2)} \in \mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$.

Proposition 6 *The formula*

$$\Delta\, h = \sum h_{(1)} \otimes_{\mathcal{R}} h_{(2)}\,, \quad \forall\, h \in \mathcal{H}\,, \tag{2.3}$$

with the right hand side given by the product rule (2.2), defines a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$, *i.e. an* $(\mathcal{R}, \mathcal{R})$*-bimodule map satisfying* *(a)* $\Delta(1) = 1 \otimes 1$;

(b) $(\Delta \otimes_{\mathcal{R}} Id) \circ \Delta = (Id \otimes_{\mathcal{R}} \Delta) \circ \Delta : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{R}} \mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$;

(c) $\Delta(h) \cdot (\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0, \quad \forall b \in \mathcal{R}, h \in \mathcal{H}$,
where we have used the right action of $\mathcal{H} \otimes \mathcal{H}$ on $\mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$ by right multiplication;

(d) $\Delta(h_1 \cdot h_2) = \Delta(h_1) \cdot \Delta(h_2), \qquad \forall h_1, h_2 \in \mathcal{H}$,
where the right hand side makes sense because of (c).

Proof. The fact that the coproduct (2.3) is well-defined is an immediate consequence of Proposition 4, which also ensures that Δ is a bimodule homomorphism satisfying (a). By the same token, the coassociativity property (b) is tantamount to

$$h\,((ab)\,c) = h\,(a\,(bc))\,, \quad \forall h \in \mathcal{H} \quad \text{and} \quad \forall a, b, c \in \mathcal{A}\,,$$

while (c) is equivalent to a special case of the above, corresponding to $b \in \mathcal{R}$. In turn, (c) ensures that the preimage of $\Delta(\mathcal{H})$ in $\mathcal{H} \otimes \mathcal{H}$ is contained in the normalizer

$$\mathcal{N}(\mathcal{J}) = \{\nu \in \mathcal{H} \otimes \mathcal{H}\,;\, \nu \cdot \mathcal{J} \subset \mathcal{J}\}\,, \tag{2.4}$$

where $\mathcal{J}$ is the right ideal generated by $\{\beta(b) \otimes 1 - 1 \otimes \alpha(b)\,;\, b \in \mathcal{R}\}$ in $\mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$, i.e. such that $\mathcal{H} \otimes \mathcal{H} / \mathcal{J} = \mathcal{H} \otimes_{\mathcal{R}} \mathcal{H}$; in particular,

$$\Delta(\mathcal{H}) \subset \mathcal{N}(\mathcal{J}) / \mathcal{J}\,,$$

which is an algebra. $\square$

At this point let us note that the action of $\mathcal{H}$ on $\mathcal{A}$ leaves invariant the subalgebra $\mathcal{R}_0 = C_c^\infty(FM)$, pulled back from the space of units of the groupoid. The restriction of $\mathcal{H}$ to End $(\mathcal{R}_0)$ coincides with the algebra of differential operators on FM, and therefore admits a tautological extension to an action on $\mathcal{R}$. With this clarified, we can proceed to define the counit.

Proposition 7 *The map $\varepsilon : \mathcal{H} \to \mathcal{R}$ defined by*

$$\varepsilon(h) = h(1)\,, \qquad \forall h \in \mathcal{H}\,, \tag{2.5}$$

is a bimodule map such that

(a) $\varepsilon(1) = 1$;

(b) $\mathrm{Ker}\varepsilon$ *is a left ideal of* $\mathcal{H}$;

(c) $(\varepsilon \otimes_{\mathcal{R}} Id) \circ \Delta = (Id \otimes_{\mathcal{R}} \varepsilon) \circ \Delta = Id_{\mathcal{H}}$,
where we have made the identifications $\mathcal{R} \otimes_{\mathcal{R}} \mathcal{H} \simeq \mathcal{H}\,, \mathcal{H} \otimes_{\mathcal{R}} \mathcal{R} \simeq \mathcal{H}$ *, the first via the left action by* α *and the second via the left action by* β*.*

Proof. The fact that $\varepsilon : \mathcal{H} \to \mathcal{R}$ is a bimodule map satisfying (a) and (b) is obvious from the definition. In view of Proposition 4, (c) amounts to

$$h\,(1 \cdot a) \;=\; h\,(a \cdot 1) \;=\; h\,(a)\,, \quad \forall\, h \in \mathcal{H} \quad \text{and} \quad \forall\, a \in \mathcal{A}\,. \qquad \square$$

The last ingredient we need is a twisted version of the antipode. To define it, we recall that $\mathcal{A}$ carries a canonical (up to a scaling factor) *faithful trace* $\tau : \mathcal{A} \to \mathbb{C}$, defined as follows

$$\tau\,(f\,U_\varphi^*) \;=\; \begin{cases} \displaystyle\int_{FM} f\,vol_{FM}\,, & \text{if} \quad \varphi = Id\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{2.6}$$

where vol_{FM} denotes a fixed Γ_M-invariant volume form on the frame bundle, e.g. determined by the choice of a torsion-free. connection

Proposition 8 *The identity*

$$\tau\,(h(a) \cdot b) = \tau\,(a \cdot \widetilde{S}(h)\,(b))\,, \quad \forall\, a, b \in \mathcal{A}\,, \quad \forall\, h \in \mathcal{H}\,, \tag{2.7}$$

is satisfied by all operators $h \in \mathcal{H}$*; it uniquely defines an algebra anti-isomorphism* $\widetilde{S} : \mathcal{H} \to \mathcal{H}$*, such that*

$$\widetilde{S}^2 \;=\; Id_{\mathcal{H}}\,, \tag{2.8}$$

$$\widetilde{S} \circ \beta = \alpha\,, \tag{2.9}$$

$$m_{\mathcal{H}} \circ (\widetilde{S} \otimes_{\mathcal{R}} Id) \circ \Delta = \beta \circ \varepsilon \circ \widetilde{S} : \mathcal{H} \to \mathcal{H}\ , \tag{2.10}$$

where $m_{\mathcal{H}} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ *denotes the multiplication and the left hand side of last identity makes sense because of the preceding identity.*

Proof. Obviously, for any $a^1, a^2 \in \mathcal{A}$, one has

$$\begin{aligned} \tau(\alpha(b)\,(a^1)\cdot a^2) &= \tau(a^1\cdot\beta(b)\,(a^2))\,, \quad \forall\, b\in\mathcal{R}\,, \\ \tau(\beta(b)\,(a^1)\cdot a^2) &= \tau(a^1\cdot\alpha(b)\,(a^2))\,, \quad \forall\, b\in\mathcal{R}\,. \end{aligned} \tag{2.11}$$

Since the fundamental vector fields are Γ_M-invariant, (2.1) and the Stokes formula gives

$$\tau\,(Y_i^j(a^1)\cdot a^2) \,=\, -\,\tau\,(a^1\cdot Y_i^j(a^2)) \,+\, \delta_i^j\,\tau\,(a^1\cdot a^2)\,. \tag{2.12}$$

On the other hand, for the basic horizontal vector fields, again using (2.1) and the Stokes formula, one gets

$$\begin{aligned} \tau\,(X_k(a^1)\cdot a^2) &= -\,\tau\,(a^1\,X_k(a^2)) \,-\, \tau\,(\delta_{jk}^i(a^1)\cdot Y_i^j(a^2)) \\ &= -\,\tau\,(a^1\,X_k(a^2)) \,+\, \tau\,(a^1\cdot\delta_{jk}^i(Y_i^j(a^2))\,. \end{aligned} \tag{2.13}$$

Thus, the generators of $\mathcal{H}$ satisfy (2.7). By multiplicativity, the "integration by part" identity (2.7) extends to all transverse differential operators $h\in\mathcal{H}$ and, since the trace τ is faithful, it uniquely determines the algebra anti-homomorphism $\widetilde{S}:\mathcal{H}\to\mathcal{H}$ satisfying (2.8) and (2.9). Finally, (2.10) follows from the fact that, for any $h\in\mathcal{H}$ and $a^1, a^2\in\mathcal{A}$,

$$\begin{aligned} \tau\,(a^1\cdot a^2\,\widetilde{S}(h)(1)\,) &= \tau\,(h\,(a^1a^2)) = \tau\,(h_{(1)}(a^1)\cdot h_{(2)}(a^2)) \,= \\ &= \tau\,(a^1\cdot\widetilde{S}(h_{(1)})h_{(2)}\,(a^2))\,. \qquad \square \end{aligned} \tag{2.14}$$

Remark 9 $\widetilde{S}:\mathcal{H}\to\mathcal{H}$ is a twisted version of the antipode required by Lu's definition [18]. It already occurs in the flat case [10], for the genuine Hopf algebra $\mathcal{H}_n$, and will play a similar role in the definition of the associated cyclic module. In turn, the bimodule homomorphism $\delta=\varepsilon\circ\widetilde{S}:\mathcal{H}\to\mathcal{R}$ is the analogue of the modular character in [10], [11].

3 Differentiable and Hopf cyclic cohomology

The general framework for cyclic cohomology is that of the category of Λ-modules over the cyclic category Λ (cf. [6]). We recall that the cyclic category Λ is the small category obtained by adjoining cyclic morphisms to the simplicial category Δ. The latter has one object $[q] = \{0 < 1 < \ldots < q\}$ for each integer $q \geq 0$, and is generated by faces $\delta_i : [q-1] \to [q]$, with $\delta_i =$ the injection that misses i, and degeneracies $\sigma_j : [q+1] \to [q]$, with $\sigma_j =$ the surjection which identifies j with $j+1$, satisfying the relations:

$$\delta_j \delta_i = \delta_i \delta_{j-1} \text{ for } i < j\,,\ \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad i \leq j \tag{3.1}$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1}\,, & i < j \\ 1_q\,, & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j\,, & i > j+1. \end{cases}$$

The category Λ is obtained by adjoining for each q an extra morphism $\tau_q : [q] \to [q]$ such that

$$\begin{aligned} &\tau_q \delta_i = \delta_{i-1} \tau_{q-1}\,, \quad 1 \leq i \leq q, \\ &\tau_q \sigma_i = \sigma_{i-1} \tau_{q+1}\,, \quad 1 \leq i \leq q, \\ &\tau_q^{q+1} = 1_q\,. \end{aligned} \tag{3.2}$$

The cyclic cohomology groups of a Λ-module $\mathcal{M} = \{M[q]\}_{q \geq 0}$ in the category of vector spaces (over $\mathbb{C}$) are, by definition, the derived groups

$$HC^q(\mathcal{M}) = Ext^q_\Lambda(\mathbb{C}, \mathcal{M})\,.$$

Using the canonical projective biresolution of the trivial cyclic object [6], these groups can be computed as the cohomology of a bicomplex $CC^{*,*}(\mathcal{M})$ defined as follows

$$\begin{aligned} &CC^{p,r}(\mathcal{M}) = \mathcal{M}[r-p], \quad r \geq p\,, \\ &CC^{p,r}(\mathcal{M}) = 0\,, \quad r < p\,, \end{aligned} \tag{3.3}$$

with vertical boundary operators

$$b = \sum_{i=0}^{q} (-1)^i \delta_i : \mathcal{M}[q-1] \to \mathcal{M}[q] \quad q \geq 0 \tag{3.4}$$

and horizontal boundary operators

$$B = N_q \circ \sigma_{-1} \circ (1_{q+1} - \lambda_{q+1}) : \ \mathcal{M}[q+1] \to \mathcal{M}[q]\,, \tag{3.5}$$

where

$$\begin{aligned}\lambda_q &= (-1)^q \tau_q\,, \quad \sigma_{-1} = \tau_{q+1} \circ \sigma_q : \mathcal{M}[q+1] \to \mathcal{M}[q] \quad \text{and} \\ N_q &= 1_q + \lambda_q + \ldots + \lambda_q{}^q\,.\end{aligned} \tag{3.6}$$

Then $HC^*(\mathcal{M})$ is the cohomology of the first quadrant total complex

$$TC^q(\mathcal{M}) = \sum_{p=0}^{q} CC^{p,q-p}(\mathcal{M})\,, \tag{3.7}$$

while the cohomology of the full direct sum total complex

$$PC^q(\mathcal{M}) = \sum_{p\in\mathbb{Z}} CC^{p,q-p}(\mathcal{M}), \tag{3.8}$$

gives the $\mathbb{Z}/2$–graded *periodic* cyclic cohomology groups $HC^*{}_{\rm per}\,(\mathcal{M})$.

In particular, the cyclic cohomology $HC^*(\mathcal{A})$ of an associative (unital) algebra $\mathcal{A}$, which historically preceded the above definition (cf. [5]), corresponds to the Λ-module $\mathcal{A}^\natural$, with $\mathcal{A}^\natural[q] \equiv C^q(\mathcal{A})$ denoting the linear space of $(q+1)$-linear forms ϕ on $\mathcal{A}$ and with Λ-operators defined as follows: the face operators $\delta_i : C^{q-1}(\mathcal{A}) \to C^q(\mathcal{A})\,, \quad 0 \le i \le q$, are

$$\begin{aligned}&\delta_i\, \phi(a^0, \ldots, a^q) = \phi(a^0, \ldots, a^i a^{i+1}, \ldots, a^q)\,, \quad 0 \le i \le q-1\,, \\ &\delta_q\, \phi(a^0, \ldots, a^q) = \phi(a^q a^0,\, a^1, \ldots, a^{q-1})\,;\end{aligned} \tag{3.9}$$

the degeneracy operators $\sigma_i : C^{q+1}(\mathcal{A}) \to C^q(\mathcal{A})\,, \quad 0 \le i \le q$, are

$$\sigma_i\, \phi(a^0, \ldots, a^q) = \phi(a^0, \ldots, a^i, 1, a^{i+1}, \ldots, a^q) \tag{3.10}$$

and for each $q \ge 0$ the cyclic operator $\tau_q : C^q(\mathcal{A}) \to C^q(\mathcal{A})$ is given by

$$\tau_q\, \phi(a^0, \ldots, a^q) = \phi(a^q,\, a^0, \ldots, a^{q-1})\,. \tag{3.11}$$

We now take $\mathcal{A} = \mathcal{A}_{FM}$ and define its differentiable cyclic cohomology $HC_{\rm d}{}^*(\mathcal{A})$ as follows.

Definition 10 *A cochain $\phi \in C^q(\mathcal{A})$ is called* differentiable *if it is of the form*

$$\phi(a^0, \ldots, a^q) \;=\; \tau\,\left(H(a^0, ..., a^q)\right)\,, \quad a^0, ..., a^p \,\in \mathcal{A}_{FM}\,, \tag{3.12}$$

where

$$H(a^0,...,a^q) = \sum_{i=1}^{r} h_i^0(a^0)\cdots h_i^q(a^q)\,, \qquad {h_i}^j \in \mathcal{H}_{FM}\,,$$

is a $q+1$-differential operator on $FM\bar{\rtimes}\Gamma_M$. The space of such cochains will be denoted ${C_{\rm d}}^q(\mathcal{A})$.

Proposition 11 *The subspace of all differentiable cochains*

$${\mathcal{A}_{\rm d}}^\natural = \{{C_{\rm d}}^q(\mathcal{A})\}_{q\geq 0}$$

forms a Λ-submodule of $\mathcal{A}^\natural$. Furthermore, for any $q \geq 1$, the map

$$\begin{aligned} \chi : C^q(\mathcal{H}) \equiv \underbrace{\mathcal{H}\otimes_{\mathcal{R}}\ldots\otimes_{\mathcal{R}}\mathcal{H}}_{q\text{-}times} \longrightarrow C^q_{\rm d}(\mathcal{A}) \\ \chi(h^1\otimes_{\mathcal{R}}\ldots\otimes_{\mathcal{R}} h^q) = \tau\,(a^0\,h^1(a^1)\cdots h^q(a^q))\,, \quad a^0,...,a^q \in \mathcal{A}\,, \end{aligned} \tag{3.13}$$

is an isomorphism of vector spaces.

Proof. It is easy to check that the Λ-operators (3.9)–(3.11) preserve the property of a cochain of being differentiable, which proves the first claim. Thanks to the integration by parts property (2.7), any differentiable q-cochain can be *normalized*, i.e. put in the form

$$\phi(a^0,\ldots,a^q) \;=\; \sum_{i=1}^{r} \tau\,(a^0\,h_i^1(a^1)\cdots h_i^q(a^q))\,, \quad {h_i}^j \in \mathcal{H}_{FM}\,. \tag{3.14}$$

The q-differential operator

$$H(a^1,\ldots,a^q) \;=\; \sum_{i=1}^{r} h_i^1(a^1)\cdots h_i^q(a^q)$$

is uniquely determined, because of the faithfulness of the canonical trace. Thus, the second assertion follows from Proposition 4. □

Remark 12 By transport of structure, $\mathcal{H}^\natural = \{C^q(\mathcal{H})\}_{q\geq 0}$, with

$$C^0(\mathcal{H}) = \mathcal{R} \quad \text{and} \quad C^q(\mathcal{H}) = \underbrace{\mathcal{H}\otimes_{\mathcal{R}}\ldots\otimes_{\mathcal{R}}\mathcal{H}}_{q\text{-times}}\,, \quad q\geq 1\,, \tag{3.15}$$

acquires the structure of a Λ-module. The expressions of the Λ-operators of $\mathcal{H}^\natural$ are virtually identical to those of the cyclic Λ-module associated to a Hopf algebra in [10], with the obvious modifications required by the replacement of the base ring $\mathbb{C}$ with $\mathcal{R}$.

Thus, the face operators $\delta_i : {C_{\rm d}}^{q-1}(\mathcal{A}) \to {C_{\rm d}}^q(\mathcal{A})\,, \quad 0 \leq i \leq q$, are given by

$$\begin{aligned} \delta_0(h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1}) &= 1 \otimes_{\mathcal{R}} h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1}\,, \\ \delta_j(h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1}) &= h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} \Delta h^j \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1}\,, \\ \forall & \quad 1 \leq j \leq q-1\,, \\ \delta_q(h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1}) &= h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q-1} \otimes_{\mathcal{R}} 1\,, \end{aligned} \tag{3.16}$$

in particular, for $q = 1$

$$\delta_0(b) = 1 \otimes_{\mathcal{R}} b = \beta(b)\,, \quad \delta_1(b) = b \otimes_{\mathcal{R}} 1 = \alpha(b) \quad \forall\, b \in \mathcal{R}\,; \tag{3.17}$$

the degeneracy operators $\sigma_i : C_{\rm d}^{q+1}(\mathcal{A}) \to C_{\rm d}^q(\mathcal{A})\,, \quad 0 \leq i \leq n$, have the expression

$$\begin{aligned} \sigma_i(h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q+1}) \ = \\ h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^i \otimes_{\mathcal{R}} \varepsilon(h^{i+1}) \otimes_{\mathcal{R}} h^{i+2} \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^{q+1} \end{aligned} \tag{3.18}$$

if $1 \leq j \leq q$ and for $q = 0$

$$\sigma_0(h) = \varepsilon(h), \qquad h \in \mathcal{H}\,; \tag{3.19}$$

finally, the cyclic operator $\tau_q : C_{\rm d}^q(\mathcal{A}) \to C_{\rm d}^q(\mathcal{A})$ is

$$\tau_q(h^1 \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} h^q) = (\Delta^{q-1} \widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^q \otimes 1\,, \tag{3.20}$$

where in the right hand side $\mathcal{H} \otimes \ldots \otimes \mathcal{H}$ acts on $\mathcal{H} \otimes_{\mathcal{R}} \ldots \otimes_{\mathcal{R}} \mathcal{H}$ by right multiplication.

The cyclic cohomology of the Λ-module $\mathcal{H}^\natural$ will be denoted $HC^*(\mathcal{H})$. By its very definition, one has a tautological isomorphism

$$T^* : HC^*(\mathcal{H}) \xrightarrow{\simeq} {HC_{\rm d}}^*(\mathcal{A})\,. \tag{3.21}$$

There is a simple way for defining the relative version of this cohomology with respect to any compact subgroup of $K \subset GL(n,\mathbb{R})$, which goes as follows.

For $g \in GL(n,\mathbb{R})$, let $R(g)$ denote its right action on $\mathcal{R} = C^\infty(FM)$. Since the action of $GL(n,\mathbb{R})$ on FM commutes with that of Γ_M, the representation R of $GL(n,\mathbb{R})$ on $\mathcal{R}$ by right translations extends to a natural action by algebra automorphisms of $GL(n,\mathbb{R})$ on $\mathcal{A} = C_c^\infty(FM \bar{\rtimes} \Gamma_M)$. Given a compact subgroup K, we denote by $\mathcal{A}^K$ the subalgebra of K-invariant elements in $\mathcal{A}$ and by $\iota_K : \mathcal{A}^K \to \mathcal{A}$ the corresponding inclusion map. We note that there is an obvious identification

$$\mathcal{A}^K \simeq C_c^\infty(FM/K \bar{\rtimes} \Gamma_M).$$

By definition, the *differentiable cochains relative to* K are those obtained by restricting to $\mathcal{A}^K$ the differentiable cochains on $\mathcal{A}$,

$$C_{\rm d}^q(\mathcal{A}, K) = \iota_K^* \left(C_{\rm d}^q(\mathcal{A})\right).$$

In view of Remark 5, exactly as in the case of $HC^*(\mathcal{H})$, the complex of relative differentiable cochains remains unchanged if Γ_M is replaced by any full subpseudogroup. Because of this, we shall denote the cyclic cohomology of the corresponding Λ-module

$$\mathcal{A}_{\rm d}^{K\natural} = \{{C_{\rm d}}^q(\mathcal{A}, K)\}_{q\geq 0} \tag{3.22}$$

by $HC^*(\mathcal{H}, K)$ and shall refer to it as *the cyclic cohomology of* $\mathcal{H}$ *relative to* K.

As a simple example of an extended Hopf algebra which, without being of the form $\mathcal{H}_{FM}$, gives rise to a cyclic module in a similar fashion, we shall consider the "coarse" extended Hopf algebra over an arbitrary associative unital algebra $\mathcal{K}$ (cf. [18])

$$\mathcal{T} \equiv \mathcal{T}(\mathcal{K}) := \mathcal{K} \otimes \mathcal{K}^{\rm op}.$$

The source and target maps $\alpha, \beta : \mathcal{K} \to \mathcal{T}$ are

$$\alpha(k) = k \otimes 1, \qquad \beta(k) = 1 \otimes k, \qquad k \in \mathcal{K},$$

the coproduct $\Delta : \mathcal{T} \to \mathcal{T} \otimes_{\mathcal{K}} \mathcal{T} \simeq \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}^{\rm op}$ is given by

$$\Delta(\ell \otimes r) = (\ell \otimes 1) \otimes_{\mathcal{K}} (1 \otimes r) \simeq \ell \otimes 1 \otimes r, \quad \forall\, \ell, r \in \mathcal{K}$$

the counit $\varepsilon : \mathcal{T} \to \mathcal{K}$ is

$$\varepsilon(\ell \otimes r) = \ell \cdot r \,, \quad \ell, r \in \mathcal{K}$$

and the antipode $\widetilde{S} \equiv S : \mathcal{T} \to \mathcal{T}$ is

$$S(\ell \otimes r) = r \otimes \ell \,, \quad \forall\, \ell, r \in \mathcal{K} \,.$$

In this case

$$\begin{array}{rcl} C^0(\mathcal{T}) & = & \mathcal{K}\,, \quad \text{and for} \quad q \geq 1\,, \\ C^q(\mathcal{T}) & = & \underbrace{\mathcal{T} \otimes_{\mathcal{K}} \ldots \otimes_{\mathcal{K}} \mathcal{T}}_{q\text{-times}} \simeq \underbrace{\mathcal{K} \otimes \ldots \otimes \mathcal{K}}_{q\text{-times}} \otimes \mathcal{K}^{\rm op} \,. \end{array}$$

With the latter identification, the face operators $\delta_i : C^{q-1}(\mathcal{T}) \to C^q(\mathcal{T})\,$, $0 \leq i \leq q\,$, are given by

$$\delta_0(k^1 \otimes \ldots \otimes k^q) = 1 \otimes k^1 \otimes \ldots \otimes k^q \,, \quad k^1, \ldots, k^q \in \mathcal{K}\,,$$

$$\delta_i(k^1 \otimes \ldots \otimes k^q) = k^1 \otimes \ldots \otimes k^i \otimes 1 \otimes k^{i+1} \otimes \ldots \otimes k^q \,, \quad i = 1, \ldots, q-1$$

$$\delta_q(k^1 \otimes \ldots \otimes k^q) = k^1 \otimes \ldots \otimes k^q \otimes 1$$

for $q \geq 1$ while for $q = 0$,

$$\delta_0(k) = 1 \otimes k \;, \qquad \delta_1(k) = k \otimes 1\,;$$

the degeneracy operators $\sigma_i : C^{q+1}(\mathcal{T}) \to C^q(\mathcal{T})\,, \quad 0 \leq i \leq q\,,$ become

$$\sigma_i(k^1 \otimes \ldots \otimes k^{q+2}) = k^1 \otimes \ldots \otimes k^{i+1}\, k^{i+2} \otimes \ldots \otimes k^{q+2}\,;$$

finally, the cyclic operator $\tau_q : C^q(\mathcal{T}) \to C^q(\mathcal{T})$ is exactly the cyclic permutator

$$\tau_q(k^1 \otimes \ldots \otimes k^{q+1}) = k^2 \otimes \ldots \otimes k^{q+1} \otimes k^1 \,.$$

Lemma 13 *For any associative unital algebra $\mathcal{K}$, one has*

$$HC^*(\mathcal{T}(\mathcal{K})) \simeq HC^*(\mathcal{T}(\mathbb{C})) \equiv HC^*(\mathbb{C})\,,$$

the isomorphism being induced by the unit map $\eta : \mathbb{C} = \mathcal{T}(\mathbb{C}) \to \mathcal{T}(\mathcal{K})$.

Proof. Fix a linear functional $\nu \in \mathcal{A}^*$ with $\nu(1) = 1$ and define the operator $s : C^q(\mathcal{T}) \to C^{q-1}(\mathcal{T})$ by setting

$$s(k^1 \otimes \ldots \otimes k^{q+1}) = \nu(k^1)\, k^2 \otimes \ldots \otimes k^{q+1}\,, \quad k^1, k^2, \ldots, k^{q+1} \in \mathcal{K}\,.$$

It can be easily checked that

$$s\,\delta_i = \delta_{i-1}\, s\,, \qquad i = 1, \ldots, q\,.$$

Thus, the Λ-module map $\eta^\natural : \mathbb{C}^\natural \to \mathcal{T}^\natural$, associated to the unit map, induces an isomorphism for Hochschild and therefore for cyclic cohomology too. □

This lemma will be used to show that, in the case of a flat n-manifold, the cyclic cohomology of the extended Hopf algebra coincides with that of the Hopf algebra $\mathcal{H}_n$ introduced in [10]. We preface the statement with the remark that $\mathcal{H}_n$ can be defined as the subalgebra of $\mathcal{H}_{F\mathbb{R}^n}$ generated over $\mathbb{C}$ by the operators $\{X_k, Y_i^j, \delta_{jk}{}^i\}$ corresponding to the trivial flat connection of $\mathbb{R}^n$.

Proposition 14 *If N^n is a flat affine manifold, then the canonical inclusion $\kappa : \mathcal{H}_n \to \mathcal{H}_{FN}$ associated to the flat connection, induces isomorphism in cyclic cohomology*

$$\kappa_*^K : HC^*(\mathcal{H}_n, K) \stackrel{\simeq}{\longrightarrow} HC^*(\mathcal{H}_{FN}, K)\,,$$

for any compact subgroup $K \subset GL(n, \mathbb{R})$

Proof. As before, we shall use the abbreviated notation

$$\mathcal{H} = \mathcal{H}_{FN}\,, \quad \mathcal{A} = \mathcal{A}_{FN} \quad \text{and} \quad \mathcal{R} = \mathcal{R}_{FN}\,.$$

Also, we shall identify $\mathcal{H}_n$ with its image via the homomorphism of extended Hopf algebras $\kappa : \mathcal{H}_n \to \mathcal{H}$ induced by the Hopf action of $\mathcal{H}_n$ on $\mathcal{A}$.

Applying Proposition 3, one gets a canonical isomorphism of $(\mathcal{R}, \mathcal{R})$-bimodules

$$\mathcal{H} \simeq \alpha(\mathcal{R}) \otimes \beta(\mathcal{R}) \otimes \mathcal{H}_n \simeq \mathcal{T} \otimes \mathcal{H}_n\,, \quad \text{where} \quad \mathcal{T} = \mathcal{R} \otimes \mathcal{R}^{\mathrm{op}}\,. \qquad (3.23)$$

Furthermore, one can easily check that

$$\Delta_{\mathcal{H}} = \Delta_{\mathcal{T}} \otimes \Delta_{\mathcal{H}_n} \quad \text{and} \quad \varepsilon_{\mathcal{H}} = \varepsilon_{\mathcal{T}} \otimes \varepsilon_{\mathcal{H}_n}\,,$$

that is, as coalgebroid, $\mathcal{H}$ is actually isomorphic to the external tensor product between the coarse coalgebroid $\mathcal{T}$ over $\mathcal{R}$ and the coalgebra $\mathcal{H}_n$ over $\mathbb{C}$. This implies that

$$\delta_i{}^{\mathcal{H}} = \delta_i{}^{\mathcal{T}} \otimes \delta_i{}^{\mathcal{H}_n} \quad \text{and} \quad \sigma_j{}^{\mathcal{H}} = \sigma_j{}^{\mathcal{T}} \otimes \sigma_j{}^{\mathcal{H}_n} .$$

Thus, in the category of cosimplicial modules, one has

$$\mathcal{H}^\natural \simeq \mathcal{T}^\natural \times \mathcal{H}_n^\natural .$$

Applying the Eilenberg-Zilber theorem it follows that

$$HH^*(\mathcal{H}) \simeq HH^*(\mathcal{T}) \otimes HH^*(\mathcal{H}_n) \simeq HH^*(\mathcal{H}_n) , \tag{3.24}$$

with the second isomorphism being a consequence of Lemma 13. By the functoriality of Eilenberg-Zilber isomorphism, the composition of the two isomorphisms in (3.24) is induced by the canonical homomorphism $\kappa : \mathcal{H}_n \to \mathcal{H}$, that is

$$\kappa_* : HH^*(\mathcal{H}_n) \xrightarrow{\simeq} HH^*(\mathcal{H}) . \tag{3.25}$$

But $\kappa : \mathcal{H}_n \to \mathcal{H}$ is a homomorphism of extended Hopf algebras and therefore gives rise to a morphism of cyclic modules $\kappa^\natural : \mathcal{H}_n^\natural \to \mathcal{H}^\natural$.

The stated isomorphism for the absolute case now follows from (3.25) and the exact sequence relating Hochschild and cyclic cohomology. The relative case is proved by restriction to K-invariants. □

We are now in a position to prove the first main result of this paper, asserting that $HC^*(\mathcal{H}_{FM})$ and its relative variants depend only on the dimension n of the manifold. In view of with [10, §7, Theorem 11], which identifies $HC^*(\mathcal{H}_n)$ to the Gelfand-Fuchs cohomology, this result provides a cyclic analogue to Haefliger's Theorem IV.4 in [16].

Theorem 15 *For any n-dimensional manifold M and for any compact subgroup $K \subset GL(n, \mathbb{R})$, one has a canonical isomorphism*

$$HC^*(\mathcal{H}_{FM}, K) \simeq HC^*(\mathcal{H}_n, K) .$$

Proof. To construct the stated isomorphism, we shall have to specify certain Morita equivalence data. The resulting isomorphism however will be independent of the choices made.

Concretely, we fix an open cover of M by domains of local charts $\mathcal{U} = \{V_i\}_{1\leq i\leq r}$ together with a partition of unity subordinate to the cover $\mathcal{U}$,

$$\sum \chi_i^2(x) = 1\,, \quad \chi_i \in C_c^\infty(V_i)\,.$$

We begin by forming the smooth étale groupoid

$$G_{\mathcal{U}} = \{(x,i,j);\ x \in V_i \cap V_j\,, \quad 1 \leq i,j \leq r\}\,,$$

whose space of units is the flat manifold $N = \coprod_{i=1}^r V_i \times \{i\}$. One has a natural Morita equivalence between the algebras $C_c^\infty(M)$ and $C_c^\infty(G_{\mathcal{U}})$, and a similar construction continues to function in the presence of the pseudogroup Γ_M, as well as at the level of the frame bundle.

More precisely, there is a *full pseudogroup* $\Gamma_{\mathcal{U}}$ on N such that the corresponding smooth étale groupoid $N \bar{\rtimes} \Gamma_{\mathcal{U}}$ contains $G_{\mathcal{U}}$ and such that the algebra $\mathcal{A} = C_c^\infty(FM \bar{\rtimes} \Gamma_M)$ can be identified with the reduction by a canonical idempotent of the groupoid algebra $\mathcal{B} = C_c^\infty(FN \bar{\rtimes} \Gamma_{\mathcal{U}})$,

$$\mathcal{A} \simeq e\,\mathcal{B}\,e\,, \quad e^2 = e \in \mathcal{B}\,; \tag{3.26}$$

at the same time, one also has a canonical identification

$$C^\infty(FM) \simeq e\,C^\infty(FN)\,e\,.$$

Indeed, the elements of $\mathcal{B} = C_c^\infty(FN \bar{\rtimes} \Gamma_{\mathcal{U}})$ can be represented as finite sums of the form

$$b = \sum_{\varphi\in\Gamma_M} \sum_{i,j} f_{\varphi,ij}\, U_{\varphi_{ij}}^*\,, \quad f_{\varphi,ij} \in C_c^\infty(\mathrm{Dom}\varphi_{ij})\,, \tag{3.27}$$

where, for any $\varphi \in \Gamma_M$,

$$\varphi_{ij} : \varphi^{-1}\,(\varphi\,(\mathrm{Dom}\varphi\,\cap\,V_i)\cap V_j)\times\{i\} \,\to\, \varphi\,(\mathrm{Dom}\varphi\,\cap\,V_i)\cap V_j\times\{j\}$$

stands for the obvious identification given by the restriction of φ. The definition of the multiplication is as follows:

$$\psi_{jk}\circ\varphi_{ij} = (\psi\circ\varphi)_{ik} \quad \text{or equivalently} \quad U_{\varphi_{ij}}^* \cdot U_{\psi_{jk}}^* = U_{(\psi\circ\varphi)_{ik}}^*\,; \tag{3.28}$$

when $\varphi = Id$, instead of $U^*_{\varphi_{ij}}$ we shall simply write U^*_{ij}. The canonical trace $\widetilde{\tau} : \mathcal{B} \to \mathbb{C}$ is related to the trace $\tau : \mathcal{A} \to \mathbb{C}$ by the formula

$$\widetilde{\tau}\left(\sum_{\varphi\in\Gamma_M} \sum_{i,j} f_{\varphi,ij}\, U^*_{\varphi_{ij}}\right) = \tau\left(\sum_{\varphi\in\Gamma_M} \sum_{i} f_{\varphi,ii}\, U^*_{\varphi_{ii}}\right) . \tag{3.29}$$

With the above notation, the idempotent stipulated by (3.26) has the expression

$$e = \sum_{i,j} \widetilde{\chi}_i\, U^*_{ij}\, \widetilde{\chi}_j\,, \quad \text{with} \quad \widetilde{\chi}_i = \chi_i \circ \pi$$

and the claimed identification is given by the map

$$\iota(f\, U^*_\varphi) = \sum_{i,j} \widetilde{\chi}_i\, f\, U^*_{\varphi_{ij}}\, \widetilde{\chi}_j\,, \quad f \in C^\infty_c(FM)\,,\, \varphi \in \Gamma_M\,. \tag{3.30}$$

The corresponding pair of bimodules which implements the Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$ consists of

$$\mathcal{P} = e\mathcal{B} \quad \text{and} \quad \mathcal{Q} = \mathcal{B}e\,, \quad \text{with} \quad \mathcal{P} \otimes_{\mathcal{B}} \mathcal{Q} \simeq \mathcal{A} \quad \text{and} \quad \mathcal{Q} \otimes_{\mathcal{A}} \mathcal{P} \simeq \mathcal{B}\,.$$

Setting for any $i = 1, ..., r$,

$$u_i = \textstyle\sum_k \widetilde{\chi}_k U^*_{ki} \cdot \widetilde{\chi}_i \in \mathcal{P}\,, \quad v_i = \widetilde{\chi}_i \cdot \sum_k U^*_{ik}\widetilde{\chi}_k \in \mathcal{Q}\,, \quad \text{respectively}$$

$$u'_i = \textstyle\sum_k \widetilde{\chi}_k U^*_{ki}\,, \quad e \cdot u'_i = u'_i \quad \text{and} \quad v'_i = \sum_k U^*_{ik}\widetilde{\chi}_k\,, \quad v'_i \cdot e = v'_i\,,$$

one has

$$\sum_i u_i\, v_i = e\,, \quad \text{and} \quad \sum_i v'_i\, u'_i = \sum_i U^*_{ii}\,.$$

The constructive proof of the Morita invariance for the Hochschild and cyclic cohomology of algebras (cf. [19]) associates to these data canonical cochain equivalences $\Psi^* = \{\Psi^q\}$ and $\Theta^* = \{\Theta^q\}$, which are homotopic inverses to each other. We shall only have to check that these cochain maps, as well as the cochain homotopies, preserve the corresponding subcomplexes of differentiable cochains.

The cochain map $\Psi^* : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ is given by

$$\Psi^q(\phi)\left(\textstyle\sum_{i,j} f^0_{ij}\, U^*_{\varphi^0_{ij}}\,, \sum_{i,j} f^1_{ij}\, U^*_{\varphi^1_{ij}}\,, \ldots, \sum_{i,j} f^q_{ij}\, U^*_{\varphi^q_{ij}}\right) =$$

$$\textstyle\sum_{i_0,i_1,\ldots,i_q} \phi\left(f^0_{i_0 i_1}\, U^*_{\varphi^0}\,,\, f^1_{i_1 i_2}\, U^*_{\varphi^1}\,, \ldots,\, f^q_{i_q i_0}\, U^*_{\varphi^q}\right) .$$

Assuming $\phi \in C^q_{\rm d}(\mathcal{A})$, of the form

$$\phi(a^0, \ldots, a^q) \,=\, \tau\left(h^0(a^0)\cdot h^1(a^1)\cdots h^q(a^q)\right)\,, \quad \text{with} \quad h^i \in \mathcal{H}_{FM}, \qquad (3.31)$$

one has

$$\Psi^q(\phi)\,\left(\sum_{i,j} f^0_{ij}\, U^*_{\varphi^0_{ij}}\,, \sum_{i,j} f^1_{ij}\, U^*_{\varphi^1_{ij}}\,, \ldots, \sum_{i,j} f^q_{ij}\, U^*_{\varphi^q_{ij}}\right) \,=$$
$$\sum_{i_0,i_1,\ldots,i_q} \tau\left(h^0(f^0_{i_0\,i_1}\, U^*_{\varphi^0})\, h^1(f^1_{i_1\,i_2}\, U^*_{\varphi^1})\,\ldots\, h^q(\,f^q_{i_q\,i_0}\, U^*_{\varphi^q})\right)\,;$$

using (3.29) it is easily seen that the latter expression is of the form

$$\sum_s \widetilde{\tau}\left(\sum_{i,j}\widetilde{h}^0_s(f^0_{ij}U^*_{\varphi^0_{ij}})\,\cdot\,\sum_{i,j}\widetilde{h}^1_s(f^1_{ij}U^*_{\varphi^1_{ij}})\cdots\sum_{i,j}\widetilde{h}^q_s(f^q_{ij}\,U^*_{\varphi^q_{ij}})\right)\,,$$

with $\widetilde{h}^0_s, \ldots, \widetilde{h}^q_s$ transverse differential operators on $FN\bar{\rtimes}\Gamma_{\mathcal{U}}$.

The cochain map $\Theta^* : C^*(\mathcal{B}) \to C^*(\mathcal{A})$ has the expression

$$\Theta^q(\widetilde{\phi})\;(f^0\,U_{\varphi^0}\,, f^1\,U_{\varphi^1}\,, \ldots, f^q\,U_{\varphi^q}) \,=$$
$$\widetilde{\phi}\;\left(\sum_{i_0,i_1,\ldots,i_q}\widetilde{\chi}_{i_0}f^0U^*_{\varphi^0_{i_0 i_1}}\widetilde{\chi}_{i_1}, \widetilde{\chi}_{i_1}f^1\,U^*_{\varphi^1_{i_1 i_2}}\widetilde{\chi}_{i_2}, \ldots, \widetilde{\chi}_{i_q}f^qU^*_{\varphi^q_{i_q i_0}}\widetilde{\chi}_{i_0}\right)\,.$$

When $\widetilde{\phi} \in C^q_{\rm d}(\mathcal{B})$ is of the form

$$\widetilde{\phi}(b^0, \ldots, b^q) \,=\, \widetilde{\tau}\left(\widetilde{h}^0(b^0)\widetilde{h}^1(b^1)\cdots\widetilde{h}^q(b^q)\right) \quad \text{with} \quad \widetilde{h}^i \in \mathcal{H}_{FN}\,, \qquad (3.32)$$

one gets

$$\Theta^q(\widetilde{\phi})\;(f^0\,U_{\varphi^0}\,, f^1\,U_{\varphi^1}\,, \ldots, f^q\,U_{\varphi^q}) \,=$$
$$\sum_{i_0,i_1,\ldots,i_q}\widetilde{\tau}\left(\widetilde{h}^0(\widetilde{\chi}_{i_0}f^0U^*_{\varphi^0_{i_0\,i_1}}\widetilde{\chi}_{i_1})\cdot\widetilde{h}^1(\widetilde{\chi}_{i_1}f^1U^*_{\varphi^1_{i_1\,i_2}}\widetilde{\chi}_{i_2})\cdots\widetilde{h}^q(\widetilde{\chi}_{i_q}f^qU^*_{\varphi^q_{i_q\,i_0}}\widetilde{\chi}_{i_0})\right);$$

again, one can recognize the last expression to be of the form

$$\sum_s \tau\left(h^0_s(f^0\,U^*_{\varphi^0})\cdot h^1_s(f^1\,U^*_{\varphi^1})\cdots h^q_s(f^q\,U^*_{\varphi^q})\right)\,,$$

with $h^0_s, h^1_s, \ldots, h^q_s \in \mathcal{H}_{FM}$.

Similar arguments apply to the canonical cochain homotopies used in showing that Ψ^* and Θ^* are homotopic inverses to each other in Hochschild cohomology. Since both Ψ^* and Θ^* are actually cochain maps for the cyclic bicomplex, it follows that they induce the stated isomorphism. All these constructions are clearly equivariant with respect to $O(n)$ and therefore apply, by restriction to K-invariants, to the relative cohomology as well. $\square$

We conclude this section with the observation that the ordinary (de Rham) homology of the manifold M can also be construed as differentiable (or Hopf) cyclic cohomology. The extended Hopf algebra responsible for this interpretation is the algebra of the differential operators on FM,

$$\mathcal{D} := \mathcal{D}_{FM} \,,$$

viewed as the algebra of linear transformation of $\mathcal{R}_0 = C_c^\infty(FM)$ generated by the vector fields on FM, acting as derivations, and by the functions in $\mathcal{R} = C^\infty(FM)$, acting as multiplication operators via $\alpha \equiv \beta$. Like its extension $\mathcal{H}$, $\mathcal{D}$ also carries an intrinsic Hopf-algebraic structure in the sense of §2. The coproduct $\Delta : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{R}} \mathcal{D}$ is given by the analogue of (2.3), determined by the Leibniz rule (2.2) corresponding to its action on $\mathcal{R}$. Similarly, the twisted antipode $\widetilde{S} : \mathcal{D} \to \mathcal{D}$ is determined by the analogue of integration by parts formula (2.7), with respect to $\tau|\mathcal{R}_0$ which is given by the integral against the canonical volume form on FM. In other words, $\mathcal{D}$ inherits the quotient Hopf structure corresponding to the restricted action of $\mathcal{H}$ on $\mathcal{R}$.

The differentiable cyclic cochains on $\mathcal{R}_0$ are defined by specializing Definition 10 to the algebra $\mathcal{R}_0$, acted upon by $\mathcal{D}$. By obvious analogues of Proposition 11 and Remark 12, the corresponding Λ-module can also be described purely in terms of the Hopf structure of $\mathcal{D}$. The ensuing cyclic complex, unlike that of $\mathcal{D}$ viewed solely as an algebra (cf. [3], [24], for the cyclic homology of the latter), is quasi-isomorphic to the full cyclic complex of the algebra $\mathcal{R}_0$. In a sense made more precise by the proof below, the differentiable cyclic cochains are in the same relation to the continuous cyclic cochains as the smooth currents are in relation to arbitray currents on FM.

Proposition 16 *The tautological action of $\mathcal{D}_{FM}$ on $C_c^\infty(FM)$ or, equivalently, the inclusion of the differentiable cyclic subcomplex into the usual cyclic complex of $C_c^\infty(FM)$, induces isomorphism in cyclic cohomology, resp.*

relative cyclic cohomology,

$$HC^*_{\rm per}(\mathcal{D}_{FM}) \simeq HC^*_{\rm per}(C_c^\infty(FM)), \quad \textit{resp.}$$

$$HC^*_{\rm per}(\mathcal{D}_{FM}, O(n)) \simeq HC^*_{\rm per}(C_c^\infty(PM)), \qquad PM := FM/O(n).$$

Proof. Since the map that sends $\varpi \in \Omega^{\dim FM - q}(FM)$ to the cyclic cochain on $C_c^\infty(FM)$

$$\gamma_\varpi(f^0, f^1, \ldots, f^q) = \int_{FM} f^0\, df^1 \wedge \ldots \wedge df^q \wedge \varpi$$

induces isomorphism between $\sum_{i \equiv *} H^{n(n+1)-i}(FM)$ and $HC^*_{\rm per}(C_c^\infty(FM))$, it suffices to show that any such cochain γ_ϖ is differentiable. To check this, we fix a torsion-free connection ω on FM and let $\{X_k, Y_i^j\}$ (resp. $\{\theta^k, \omega^i_j\}$) stand for the corresponding basis of the tangent (resp. cotagent) space to FM at any point. Then, for each $\ell = 1, \ldots, q$, one has

$$df_\ell = Y_i^j(f_\ell)\, \omega^i_j + X_k(f_\ell)\, \theta^k.$$

On the other hand, ϖ can be expressed as a linear combination over $\mathcal{R}$ of monomials formed from the basis $\{\theta^k, \omega^i_j\}$. Thus,

$$\gamma_\varpi(f^0, f^1, \ldots, f^q) = \sum_s \int_{FM} f^0\, h^1_s(f^1) \cdots h^q_s(f^q)\, vol_{FM}, \quad \text{with} \quad h^\ell_s \in \mathcal{D}.$$

The relative case can be proved in a similar fashion. □

4 Geometric realization of the cyclic van Est isomorphism

As mentioned before, in conjunction with [10, §7, Theorem 11], Theorem 15 implies that, for any n-dimensional smooth manifold M, $HC^*(\mathcal{H}_{FM})$ is canonically isomorphic to the Gelfand-Fuchs cohomology groups $H^*(\mathfrak{a}_n)$. The purpose of this section is to give a geometric construction of an explicit cochain map, from the Lie algebra cohomology complex to the cyclic bicomplex, which implements this cyclic analogue of the van Est isomorphism. As in the flat case [10, §7], the cochain map will be assembled in two stages. The first consists in mapping the Gelfand-Fuchs cohomology of the Lie algebra

$\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$ to the $\mathcal{G}_M$-equivariant cohomology of FM, where $\mathcal{G}_M := \mathrm{Diff}(M)$. The second involves the canonical map Φ relating the equivariant cohomology to the cyclic cohomology [8, Theorem 14, p.220].

In order to construct the first map, we shall fix a torsion-free linear connection ∇ on M, with connection form $\omega = (\omega^i_j)$. We denote by $F^\infty M$ the bundle of frames of infinite order on M, formed of jets of infinite order $j^\infty_0(\psi)$ of local diffeomorphisms ψ, with source 0, from $\mathbb{R}^n$ to M, and by $\pi_1 : F^\infty M \to FM$ its projection to 1-jets. The connection ∇ determines a cross-section

$$\sigma \equiv \sigma_\nabla : FM \to F^\infty M\,, \quad \pi_1 \circ \sigma = \mathrm{Id}\,,$$

by the formula

$$\sigma(u) = j^\infty_0(\exp^\nabla_x \circ u)\ , \qquad u \in F_x M\,. \tag{4.1}$$

This cross-section is clearly $\mathrm{GL}(n,\mathbb{R})$-equivariant

$$\sigma_\nabla \circ R_a = R_a \circ \sigma_\nabla\,, \qquad a \in \mathrm{GL}(n,\mathbb{R})\,, \tag{4.2}$$

as well as Diff-equivariant,

$$\sigma_{\nabla^\varphi} = \widetilde{\varphi}^{-1} \circ \sigma \circ \widetilde{\varphi}\,, \qquad \forall\, \varphi \in \mathcal{G}_M\,; \tag{4.3}$$

here ∇^φ corresponds to $\widetilde{\varphi}^*\omega$, where by $\widetilde{\varphi}$ we denote the action of φ on both FM^∞ and FM.

For each simplex $(\varphi_0, \ldots, \varphi_p) \times \Delta^p$, we define the map

$$\sigma_\nabla(\varphi_0, \ldots, \varphi_p) : \Delta^p \times FM \to F^\infty M$$

by the formula

$$\sigma_\nabla(\varphi_0, \ldots, \varphi_p)(t, u) = \sigma_{\nabla(\varphi_0,\ldots,\varphi_p;t)}(u) \tag{4.4}$$

where

$$\nabla(\varphi_0, \ldots, \varphi_p; t) = \sum_0^p\ t_i\, \nabla^{\varphi_i}\,.$$

Note that $\sigma_{\nabla(\varphi_0,\ldots,\varphi_p;t)}$ depends only on the class $[\varphi_0, \ldots, \varphi_p; t] \in E\mathcal{G}_M$, where $E\mathcal{G}_M$ is the geometric realization of the simplicial set $N\mathcal{G}_M$, with

$$N\mathcal{G}_M[p]\ =\ \underbrace{\mathcal{G}_M \times\ \cdots\ \times \mathcal{G}_M}_{p+1\text{-times}}\,.$$

Also, in view of (4.3), one has

$$\sigma_\nabla(\varphi_0\,\varphi,\ldots,\varphi_p\,\varphi)(t,u) = \widetilde{\varphi}^{-1}\,\sigma_\nabla(\varphi_0,\ldots,\varphi_p)(t,\widetilde{\varphi}(u))\,. \tag{4.5}$$

At this point, let us recall that the $\mathcal{G}_M$-equivariant cohomology of FM, twisted by the orientation sheaf, can be computed by means of the bicomplex

$$(C^{*,*}(\mathcal{G}_M;FM),\delta,\partial)\ ,$$

defined as follows: $C^{p,m}(\mathcal{G}_M;FM) = 0$ unless $p \geq 0$ and $-\dim FM \leq m \leq 0$, when $C^{p,m}(\mathcal{G}_M;FM) \equiv C^p(\mathcal{G}_M,\Omega_{-m}\,(FM))$ is the space of totally antisymmetric maps $\mu : \mathcal{G}_M^{p+1} \to \Omega_{-m}\,(FM))$, from $\mathcal{G}_M^{p+1}$ to the currents of dimension $-m$ on FM, such that

$$\mu(\varphi_0\,\varphi,\ldots,\varphi_p\,\varphi) = (\widetilde{\varphi}_*)^{-1}\,\mu(\varphi_0,\ldots,\varphi_p)\,, \tag{4.6}$$

with $\widetilde{\varphi}_*$ denoting the transpose of $\widetilde{\varphi}^*$ acting on the forms $\Omega(FM)$; the operator δ is the simplicial coboundary and ∂ is the de Rham boundary for currents.

We denote by $C^*(\mathfrak{a}_n)$ the Lie algebra cohomology complex of the antisymmetric multilinear functionals on the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$, which are continuous with respect to the $\mathcal{I}$-adic topology, i.e. depend only on finite jets at $0 \in \mathbb{R}^n$ of vector fields. The canonical flat connection of $F^\infty M$ determines an isomorphism between $C^*(\mathfrak{a}_n)$ and the space $\Omega^*\left((F^\infty M)\right)^{\Gamma_M}$ of all Γ_M-invariant forms on $F^\infty M$; we shall denote by $\widetilde{\varpi}$ the Γ_M-invariant form corresponding to $\varpi \in C^*(\mathfrak{a}_n)$.

With this notation in place, we now define for any $\varpi \in C^q(\mathfrak{a}_n)$ and for any pair of integers (p,m) such that $p \geq 0$, $-n(n+1) \leq m \leq 0$ and $p+m = q-n(n+1)$, a current of dimension $-m$ on FM by the following formula, where $\eta \in \Omega_c^{\,-m}\,(FM)$,

$$\begin{aligned}\langle C_{p,m}(\varpi)(\varphi_0,\ldots,\varphi_p),\eta\rangle \qquad &= \\ &(-1)^{\frac{m(m+1)}{2}}\int_{\Delta^p\times FM}\sigma_\nabla(\varphi_0,\ldots,\varphi_p)^*(\pi_1^*(\eta)\wedge\widetilde{\varpi}) \\ &= (-1)^{\frac{m(m+1)}{2}}\int_{\Delta^p\times FM}\eta\wedge\sigma_\nabla(\varphi_0,\ldots,\varphi_p)^*(\widetilde{\varpi})\,.\end{aligned} \tag{4.7}$$

Lemma 17 *For any* $\varpi \in C^q(\mathfrak{a}_n)$*, one has* $C_{p,m}(\varpi) \in C^p(\mathcal{G}_M, \Omega_m(FM))$ *and the assignment*

$$C(\varpi) = \sum_{p+m=q-n(n+1)} C_{p,m}(\varpi)\,,$$

defines a map of (total) complexes

$$C : (C^*(\mathfrak{a}_n),\, d) \;\rightarrow\; (TC^*(\mathcal{G}_M; FM),\, \delta + \partial)\;. \tag{4.8}$$

Proof. We first check the identity

$$C_{p,m}(\varpi)(\varphi_0\,\varphi, \ldots, \varphi_p\,\varphi) = \varphi_*^{-1}\, C_{p,m}(\varpi)(\varphi_0, \ldots, \varphi_p)\,. \tag{4.9}$$

Indeed, using (4.5), one has $\forall \eta \in \Omega_c^{-m}(FM))$

$$\int_{\Delta^p \times FM} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^* \, \left(\widetilde{\varphi}^{-1\,*}({\pi_1}^*(\eta)) \wedge \widetilde{\varphi}^{-1\,*}(\widetilde{\varpi})\right)$$
$$= \int_{\Delta^p \times FM} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^* \, \left({\pi_1}^*(\widetilde{\varphi}^{-1\,*}(\eta)) \wedge \widetilde{\varpi}\right)\,.$$

To prove the second claim, we use the Stokes formula:

$$\begin{aligned}
& (-1)^m \int_{\Delta^p \times FM} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^*(\pi_1^*(\eta) \wedge d\varpi) \\
+ \;& \int_{\Delta^p \times FM} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^*(\pi_1^*(d\eta) \wedge \varpi) \\
= \;& \int_{\Delta^p \times FM} d\left(\sigma_\nabla(\varphi_0, \ldots, \varphi_p)^*\, (\pi_1^*(\eta) \wedge \varpi)\right) \\
= \;& \int_{\partial\, \Delta^p \times FM} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^*(\pi_1^*(\eta) \wedge \varpi) \\
= \;& \sum (-1)^i \int_{\Delta^{p-1} \times FM} \sigma_\nabla(\varphi_0, \ldots, \check{\varphi}_i, \ldots, \varphi_p)^*(\pi_1^*(\eta) \wedge \varpi)\,.
\end{aligned}$$

Adjusting for the sign factors, one obtains the stated cochain property

$$C(d\,\varpi) = (\delta + \partial)\, C(\varpi)\,. \qquad \square$$

To simplify the assembling of the second ingredient of our construction, we shall take advantage of the fact that all the cohomological information of the complex $C^*(\mathfrak{a}_n)$ is carried by the image of the truncated Weil complex of $\mathfrak{gl}(n,\mathbb{R})$ via the canonical inclusion (cf. [13], see also [2], [15]). Thus, it suffices to work with the restriction of $C : C^*(\mathfrak{a}_n) \to TC^*(\mathcal{G}_M; FM)$ to the subcomplex $CW^*(\mathfrak{a}_n)$ of $C^*(\mathfrak{a}_n)$, generated as a graded subalgebra by $\{\theta^i_j, R^i_j\}$, where (θ^i_j), resp. (R^i_j), is the image of the "universal connection" matrix, resp. the "universal curvature" matrix, of the Weil complex of $\mathfrak{gl}(n,\mathbb{R})$.

Lemma 18 *For any torsion-free linear connection ∇ on M, with connection form $\omega_\nabla = (\omega^i_j)$ and curvature form $\Omega_\nabla = (\Omega^i_j)$, one has*

$$\sigma^*_\nabla(\widetilde{\theta}^i_j) = \omega^i_j\,, \qquad \sigma^*_\nabla(\widetilde{R}^i_j) = \Omega^i_j\,. \tag{4.10}$$

Proof. Since

$$R^i_j = d\,\theta^i_j + \theta^i_k \wedge \theta^k_j\,, \tag{4.11}$$

the second identity is a consequence of the first. To prove the first, we note that by (4.3) the operator $\omega_\nabla \mapsto \sigma^*_\nabla(\widetilde{\theta})$, acting on the (affine) space of torsion-free connections on FM, is *natural*, i.e. $\mathcal{G}_M$-equivariant. The uniqueness result for natural operators on torsion-free connections [17, 25.3] ensures that the only such operator is the identity. □

Corollary 19 *For any $\varpi \in CW^q(\mathfrak{a}_n)$ and any $\eta \in {\Omega_c}^{-m}(FM)$ the integrand in formula (4.7), defining the group cochain $\langle C_{p,m}(\varpi)(\varphi_0,\ldots,\varphi_p), \eta\rangle$, is a form that depends polynomially on the functions $\gamma^i_{jk}(u,\widetilde{\varphi}_r)$ and $\gamma^i_{jk,\ell}(u,\widetilde{\varphi}_s)$, where $1 \le i,j,k,\ell \le n$, $0 \le r,s \le p$.*

Proof. By the preceding lemma applied to the connection (4.4) together with the identity (1.16), one has

$$\sigma_\nabla(\varphi_0,\ldots,\varphi_p)^*(\widetilde{\theta}^i_j) = \sum_{r=0}^{p} t_r\,\widetilde{\varphi}^*_r\,\omega^i_j \;=\; \omega^i_j + \sum_{r=0}^{p} t_r \sum_k \gamma^i_{jk}(u,\widetilde{\varphi}_r)\,\theta^k\,.$$

Taking the total exterior derivative and using (4.11), one obtains the corresponding expression for $\sigma_\nabla(\varphi_0,\ldots,\varphi_p)^*(\widetilde{R}^i_j)$, which in addition will involve R^i_j as well as the differentials dt_r, $d\,\omega^i_j$ and $d_u\,\gamma^i_{jk}(u,\widetilde{\varphi}_r)$.

Since, by the very definition $CW^*(\mathfrak{a}_n)$, the form $\widetilde{\varpi}$ is a polynomial in $\widetilde{\theta}^i_j$ and $\widetilde{R}^i_j$, the claim follows. $\square$

By composing the restriction to $CW^*(\mathfrak{a}_n)$ of the chain map C of Lemma 17, (4.8) with the canonical map

$$\Phi : (TC^*(\mathcal{G}_M; FM),\, \delta + \partial) \to (PC^*(\mathcal{A}),\, b + B)\ ,$$

where $\mathcal{A} := C_c^\infty(FM \rtimes \mathcal{G}_M)$ is the crossed product algebra, we obtain a new chain map

$$\widetilde{C} \,:=\, \Phi \circ C \,:\, (CW^*(\mathfrak{a}_n),\, d) \;\longrightarrow\; (PC^*(\mathcal{A}),\, b + B)\ . \tag{4.12}$$

Our next task will be to prove that the image of $\widetilde{C}$ actually lands inside the differentiable periodic cyclic complex $PC_{\mathrm{d}}{}^*(\mathcal{A})$.

In preparation for that, let us recall the definition of Φ [8, III.2.δ]. It involves the crossed product algebra $\mathcal{C} = \mathcal{B} \rtimes \mathcal{G}_M$, where

$$\mathcal{B} \,=\, \Omega_c^*(FM)\hat{\otimes} \wedge^*\ \mathbb{C}[\mathcal{G}'_M]\,,$$

is the graded tensor product of the algebra of compactly supported differential forms on FM by the graded algebra over $\mathbb{C}$ generated by the degree 1 anticommuting symbols δ_φ, with $\varphi \in \mathcal{G}_M\,, \varphi \neq 1$ and $\delta_1 = 0$. The crossed product is taken with respect to the tensor product action of $\mathcal{G}_M$, so that the following multiplication rules hold:

$$\begin{aligned} U_\varphi^*\,\eta\, U_\varphi \,&=\, \widetilde{\varphi}^*\eta\,, \qquad \forall\, \eta \in \Omega^*(FM)\,,\\ U_\varphi^*\,\delta_\psi\, U_\varphi \,&=\, \delta_{\psi\circ\varphi} \,-\, \delta_\varphi\,, \quad \forall\, \varphi, \psi \in \mathcal{G}_M\,. \end{aligned} \tag{4.13}$$

The graded algebra $\mathcal{B}$ is endowed with the differential

$$d\,(\eta \otimes \delta_{\varphi_1} \cdots \delta_{\varphi_p}) \,=\, d\eta \otimes \delta_{\varphi_1} \cdots \delta_{\varphi_p}\,, \quad \forall\, \eta \in \Omega_c^*\,(FM)\,,$$

and the crossed product algebra $\mathcal{C}$ acquires the differential

$$d(b\, U_\varphi^*) \,=\, (d\, b)\, U_\varphi^* \,-\, (-1)^{\deg b}\, b\, \delta_\varphi\, U_\varphi^*\,, \quad b \in \mathcal{B},\, \varphi \in \mathcal{G}_M\,.$$

Any cochain $\nu \in C^{p,m}(\mathcal{G}_M; FM))$ gives rise to a linear functional $\widetilde{\nu}$ on $\mathcal{C}$, defined as follows, $\forall\, \eta \in \Omega_c^{-m}\,(FM)$,

$$\widetilde{\nu}\,(\eta \otimes \delta_{\varphi_1} \cdots \delta_{\varphi_p}\, U_\varphi^*) = \begin{cases} \langle\, \nu\,(1, \varphi_1, \ldots, \varphi_p), \eta\,\rangle & \text{if} \quad \varphi = 1\,,\\[1ex] 0 \quad \text{otherwise}\,. \end{cases} \tag{4.14}$$

With this notation in place, the cochain map (4.12) is given by the formula, $\forall \varpi \in CW^q(\mathfrak{a}_n)$,

$$\widetilde{C}_{p,m}(\varpi)(f^0\, U^*_{\varphi^0}, \ldots, f^p\, U^*_{\varphi_p}) :=$$

$$\tfrac{p!}{(q+1)!} \sum_{j=0}^{q=p-m+1} (-1)^{j(q-j)}\, \widetilde{C_{p,m}(\varpi)} \left(d(f^{j+1} U^*_{\varphi^{j+1}}) \ldots f^0\, U^*_{\varphi^0} \ldots d(f^j U^*_{\varphi^j}) \right). \tag{4.15}$$

Lemma 20 *The cochain homomorphism* $\widetilde{C} = \Phi \circ C$ *maps* $CW^*(\mathfrak{a}_n)$ *to* $PC^*(\mathcal{H}_{FM})$ *.*

Proof. We have to show that the cochain (4.15) is differentiable, i.e. of the form (3.12). Since $\{\theta^k, \omega^i_j\}$ forms a basis of the cotangent space at each point of FM, we can express the differential of a function $f \in {C_c}^\infty(FM)$ as

$$df = Y^j_i(f)\, \omega^i_j + X_k(f)\, \theta^k$$

and therefore

$$d(f\, U^*_\varphi) = Y^j_i(f)\, \omega^i_j\, U^*_\varphi \,+\, X_k(f)\, \theta^k\, U^*_\varphi \,-\, f\, \delta_\varphi\, U^*_\varphi\,. \tag{4.16}$$

Furthermore, for the canonical form one has

$$U^*_\varphi\, \theta^k\, U_\varphi \,=\, \theta^k\,, \tag{4.17}$$

while for the connection form, by (1.16),

$$U^*_\varphi\, \omega^i_j\, U_\varphi = \omega^i_j + \gamma^i_{jk}(u, \widetilde{\varphi})\, \theta^k\ . \tag{4.18}$$

Using the identities (4.16), (4.17), (4.18) and Corollary 19, one can now achieve the proof by following the same reasoning as in [10, pp. 232-234], where the similar result was established for the flat case. □

The definition of the above cochain map involves the choice of a torsion-free connection ∇. To emphasize this dependence, we shall use the more suggestive notation

$$\widetilde{C}_\nabla \,=\, \Phi \circ C_\nabla \,:\, CW^*(\mathfrak{a}_n) \,\longrightarrow\, PC^*(\mathcal{H}_{FM})\,.$$

However, the choice of the connection is cohomologically immaterial.

Lemma 21 *Let ∇^0, ∇^1 be torsion-free connections on FM. The corresponding cochain homomorphisms $\widetilde{C}_{\nabla_0}$ and $\widetilde{C}_{\nabla_1}$ are cochain homotopic.*

Proof. With $\sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p) : I \times \Delta^p \times FM \to F^\infty M$ given by

$$\sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)(s, t, u) = \sigma_{(1-s)\nabla^0(\varphi_0,\ldots,\varphi_p;t)+s\nabla^1(\varphi_0,\ldots,\varphi_p;t)}(u)\,,$$

for any $\eta \in \Omega_c^{-m}(FM))$ we define

$$\langle C_{\tilde{\nabla}}(\varpi)(\varphi_0, \ldots, \varphi_p), \eta\rangle = (-1)^{\frac{m(m+1)}{2}} \int_{I\times\Delta^p\times FM} \eta \wedge \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(\widetilde{\varpi})\,.$$

Applying Stokes, one obtains

$$\begin{aligned} & \textstyle\int_{I\times\Delta^p\times FM} \eta \wedge\ \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(d\varpi) \\ + \quad & (-1)^m \int_{I\times\Delta^p\times FM} d\eta \wedge\ \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(\varpi) \\ = \quad & \int_{I\times\Delta^p\times FM} d\left(\eta \wedge \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(\varpi)\right) \\ = \quad & \int_{\partial I\times\Delta^p\times FM} \eta \wedge \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(\varpi) \\ + \quad & \int_{I\times\partial\Delta^p\times FM} \eta \wedge \sigma_{\tilde{\nabla}}(\varphi_0, \ldots, \varphi_p)^*(\varpi)\,. \end{aligned}$$

After adjusting for the sign factors, this gives the desired homotopy formula

$$C_{\nabla^1} - C_{\nabla^0} = C_{\tilde{\nabla}} \circ d - (\partial + \delta) \circ C_{\tilde{\nabla}}\,. \qquad \square$$

Let $K \subset GL(n, \mathbb{R})$ be a compact subgroup. By restricting the map C_∇ to the subcomplex $CW^*(\mathfrak{a}_n, K)$ of K-basic elements in $CW^*(\mathfrak{a}_n)$, one obtains a chain map $C_\nabla^K : C^*(\mathfrak{a}_n, K) \to C^*(\mathcal{G}_M; FM/K)$. Similarly, the map Φ restricts to $\Phi^K : TC^*(\mathcal{G}_M; FM/K) \to PC^*(\mathcal{A}^K)$, where we identify

$$\mathcal{A}^K \simeq \mathcal{A}_{FM/K} := C_c^\infty(FM/K \rtimes \mathcal{G}_M)\,.$$

By composition, one gets the relative chain map

$$\widetilde{C}_\nabla^K := \Phi^K \circ C_\nabla : CW^*(\mathfrak{a}_n, K) \longrightarrow PC^*(\mathcal{H}_{FM}, K)\,. \qquad (4.19)$$

We are now ready to conclude the proof of our main result, which is a refinement of Theorem 15 and provides a geometric realization of the Gelfand-Fuchs characteristic classes in the framework of cyclic cohomology.

Theorem 22 *Let M be a smooth n-dimensional manifold endowed with a torsion-free linear connection ∇ and let $K \subset O(n)$ be a compact subgroup. The cochain homomorphism (4.19) induces isomorphism in cohomology,*

$$\gamma_K^* : \sum_{i \equiv * \bmod 2} H^i(\mathfrak{a}_n, K) \xrightarrow{\simeq} HC^*_{\mathrm{per}}(\mathcal{H}_{FM}, K) . \qquad (4.20)$$

Proof. Returning to the setting of the proof of Theorem 15, we choose an open cover of M by domains of local charts $\mathcal{U} = \{V_i\}_{1 \le i \le r}$, together with a partition of unity $\{\chi_i^2\}_{1 \le i \le r}$ subordinate to the cover $\mathcal{U}$. We then form the flat manifold $N = \coprod_{i=1}^r V_i \times \{i\}$ and the corresponding smooth étale groupoid $\mathcal{G}_{\mathcal{U}}$, such that the algebras $\mathcal{A} = C_c^\infty(FM \rtimes \mathcal{G}_M)$ and $\mathcal{B} = C_c^\infty(FN \rtimes \mathcal{G}_{\mathcal{U}})$ are Morita equivalent.

It suffices to show that by composing $\widetilde{C}_\nabla$ with the cochain equivalence $\Psi^* : C_d^*(\mathcal{A}) \to C_d^*(\mathcal{B})$ one gets a quasi-isomorphism. In turn, the composite map $\Psi^* \circ \widetilde{C}_\nabla$ can be seen to be homotopic to the map

$$\widetilde{C}_{\widetilde{\nabla}} : CW^*(\mathfrak{a}_n) \longrightarrow PC^*(\mathcal{H}_{FN}) ,$$

corresponding to the connection $\widetilde{\nabla}$ on N obtained by restricting ∇ to each V_i, $1 \le i \le r$. We are thus reduced to proving the statement on N, where one can apply Lemma 21 and replace $\widetilde{\nabla}$ by the trivial flat connection ∇_0. In that case the statement follows from [10, §7, Theorem 11], after noticing that the cochain map of [10, Lemma 8] and its variant $\widetilde{C}_{\nabla_0}$ employed here are obviously homotopic.

The relative case can be proven by restriction to K-invariants. □

By composing the isomorphism (4.20) with the natural forgetful homomorphism, one obtains a characteristic homomorphism

$$\chi^*_{O(n)} : H^*(\mathfrak{a}_n, O(n)) \xrightarrow{\gamma^*_{O(n)}} HC^*_{\mathrm{per}}(\mathcal{H}_{FM}, O(n)) \to HC^*_{\mathrm{per}}(\mathcal{A}_{PM})_{(1)} , \qquad (4.21)$$

where $PM = FM/O(n)$; it lands in the cyclic cohomology component $HC^*_{\mathrm{per}}(\mathcal{A}_{PM})_{(1)}$ corresponding to the localization at the identity. One can further compose $\chi^*_{O(n)}$ with the restriction homomorphism

$$\iota_M^* : HC^*_{\mathrm{per}}(\mathcal{A}_{PM})_{(1)} \longrightarrow HC^*_{\mathrm{per}}(C_c^\infty(PM)) ,$$

corresponding to the natural inclusion $\iota_M : C_c^\infty(PM) \to \mathcal{A}_{PM}$. After identifying the target to the twisted cohomology (by the orientation sheaf)

$$H_\tau^*(PM) \simeq H_\tau^*(M)\,,$$

one gets a natural homomorphism

$$\chi_M^* : H^*(\mathfrak{a}_n, O(n)) \longrightarrow H_\tau^*(M)\,.$$

Proposition 23 *The homomorphism* $\chi_M^* \,:\, H^*(\mathfrak{a}_n, O(n)) \longrightarrow H_\tau^*(M)$ *is the classical characteristic homomorphism, expressing the Pontryagin classes* $(p_{i_1}\cdots p_{i_k})\,(M)$ *of* M *as images of the universal Chern classes* $c_{2i_1}\cdots c_{2i_k} \in H^*(\mathfrak{a}_n, O(n))\,, \quad 2i_1 + \ldots + 2i_k \leq n.$

Proof. Let ∇ be a torsion-free connection. For any $\varpi \in CW^q(\mathfrak{a}_n, O(n))$ and any $f_0, f_1, \ldots, f_m \in C_c^\infty(FM/O(n))$, one has (up to sign)

$$\langle C_{0,-m}(\varpi)\,(1)\,,\, f_0\, df_1 \wedge \ldots \wedge df_m\rangle = \int_{PM} f_0\, df_1 \wedge \ldots \wedge df_m \wedge \sigma_\nabla^*(\tilde{\varpi})\,.$$

Thus, the statement is an immediate consequence of Lemma 18. □

5 Application to the transverse index formula

We shall now indicate how the preceding results apply to the cohomological index formula for the transverse fundamental class in K-homology (cf. [9], [10], [12]), in the setting of diffeomorphism invariant geometry.

With the manifold M assumed to be oriented, we first recall the definition of the spectral triple that encodes its diffeomorphism invariant fundamental class. One starts by forming the bundle of local metrics (cf. [7]), $\pi : PM = F^+M/SO(n) \to M$, where F^+M is bundle of oriented frames on M. The vertical subbundle $\mathcal{V} \subset TP$, $\mathcal{V} = \operatorname{Ker}\pi_*$, carries natural inner products on each of its fibers, determined solely by the choice of a $GL^+(n,\mathbb{R})$–invariant Riemannian metric on the symmetric space $GL^+(n,\mathbb{R})/SO(n)$. At the same time, the quotient bundle $\mathcal{N} = (TP)/\mathcal{V}$ comes equipped with its own, tautologically defined, Riemannian structure: every $p \in P$ is an Euclidean structure on $T_{\pi(p)}(M)$ which is identified to $\mathcal{N}_p$ via π_*. By the naturality of the above construction, the pseudogroup of orientation preserving local diffeomorphisms Γ_M^+, acting by prolongation on PM, preserves the "para-Riemannian" structure thus defined. The algebra of "coordinates" of the spectral triple is the convolution algebra of the smooth étale groupoid $PM \bar{\rtimes} \Gamma_M^+$,

$$\mathcal{A}_{PM} = C_c^\infty(PM \bar{\rtimes} \Gamma_M^+) \,.$$

The Hilbert space of the spectral triple is

$$L^2(\wedge^{\cdot} \mathcal{V}^* \otimes \wedge^{\cdot} \mathcal{N}^*, \, vol_P) \,,$$

where vol_P denotes the canonical Γ_M^+-invariant volume form on P. The algebra $\mathcal{A}_{PM}$ acts on this space by multiplication operators

$$((f\, U_\psi^*)\, \xi)(p) = f(p)\, \xi(\widetilde{\psi}(p)) \quad \forall\, p \in PM \,, \; \xi \in L^2(P) \,, \; f\, U_\psi^* \in \mathcal{A}_{PM} \,.$$

To complete the description of the spectral triple, we need to define the operator D, representing the K-homology orientation class of M in a diffeomorphism invariant fashion. It is given by the identity $Q = D|D|$, where the *hypoelliptic signature operator* Q is defined as a graded sum

$$Q = (d_V^*\, d_V - d_V\, d_V^*) \oplus \gamma\, (d_H + d_H^*) \,; \tag{5.1}$$

d_V denotes the vertical exterior derivative and d_H stands for the horizontal exterior differentiation with respect to a fixed torsion-free connection ∇. When $n \equiv 1 \,\text{or}\, 2 \pmod 4$, in order for the vertical component to make sense, one has to replace PM by $PM \times S^1$ so that the dimension of the vertical fiber stays even. As shown in [9], all the commutators $[D, a]$, with $a \in \mathcal{A}_{PM}$, are bounded. Furthermore, for any $f \in C_c{}^\infty(P)$ and any $\lambda \notin \mathbb{R}$, the local resolvent $f(D-\lambda)^{-1}$ is p-summable, for every p that exceeds the Hausdorff dimension $d = \frac{n(n+1)}{2} + 2n$ of PM viewed as a Cartan-Carathéodory metric space.

As a K-homology class, the operator D determines an additive map from the K-theory group $K_*(\mathcal{A}_{PM})$ to $\mathbb{Z}$, via the familiar index pairing:

(0) in the *graded* (or *even*) case,

$$\text{Index}_D([e]) = \text{Index}\,(eD^+e) \,, \quad e^2 = e \in \mathcal{A}_{PM} \,;$$

(1) in the *ungraded* (or *odd*) case,

$$\mathrm{Index}_D([u]) = \mathrm{Index}\,(P^+uP^+)\,, \quad u \in GL_1(\mathcal{A}_{PM})\,,$$

where $P^+ = \frac{1+F}{2}$, with $F = \mathrm{Sign}\ D$.

In cohomological terms, the index pairing can be expressed as as a pairing between cyclic (co)homological classes :

$$\mathrm{Index}_D(\kappa) = \langle ch_*(D),\, \mathrm{ch}^*(\kappa)\rangle \qquad \forall\,\kappa \in K_*(\mathcal{A}_{PM}). \tag{5.2}$$

While $\mathrm{ch}^*(\kappa) \in HC_*^{\mathrm{per}}(\mathcal{A}_{PM})$ is easy to express in terms of the "difference idempotent" determined by the K-theory class κ, the Chern character class in K-homology $ch_*(D) \in HC^*_{\mathrm{per}}(\mathcal{A}_{PM})$ has a more involved definition (cf. [5]). In the odd case it is given by the cyclic cocycle

$$\tau_F(a^0, \ldots, a^p) = \mathrm{Trace}\,(a^0[F, a^1] \ldots [F, a^p])\,, \qquad a^j \in \mathcal{A}_{PM}\,, \tag{5.3}$$

where p is any odd integer exceeding the dimension $d = \frac{n(n+1)}{2} + 2n$ of the spectral triple; in the even case the trace gets replaced by the graded trace Trace_s and p is even. As such, the cocycle (5.3) is inherently difficult to compute, because it involves the operator trace.

In [9, Part I] we used the hypoelliptic calculus on Heisenberg manifolds adapted to the para-Riemannian structure of the manifold PM to prove that the spectral triple constructed above fulfills the hypotheses of the operator theoretic local index theorem of [9, Part II]. In particular, a pseudodifferential operator T in the aforementioned calculus admits a locally computable residue of Wodzicki-Guillemin-Manin-type

$$⨍ T = \frac{1}{(2\pi)^d} \int_{PM} \mathrm{res}_T(u)\, vol_{PM}(u) \tag{5.4}$$

where $\mathrm{res}_T(u)$ is the function obtained by integrating the symbol of T of critical order $-d$, $\sigma^T_{-d}(u, \xi)$, against the Liouville measure of the sphere $||\xi||' = 1$ corresponding to the intrinsic quartic metric of PM. Specializing the local index formula of [9, Part II] to the above spectral triple, assumed for definiteness to be odd-dimensional, one obtains that the Chern character $ch_*(D) \in HC^*_{\mathrm{per}}(\mathcal{A}_{PM})$ can be represented by the cocycle $\Phi_Q = \{\phi_q\}_{q=1,3,\ldots}$ defined as follows:

$$\phi_q(a^0, \ldots, a^q) = \tag{5.5}$$

$$\sum_k (-1)^{|k|}\, \sqrt{2i}(k_1! \cdots k_q!)^{-1}((k_1+1)\ldots(k_1+\ldots+k_q+q))^{-1}\Gamma(|k|+\tfrac{q}{2})\times$$

$$\int\!\!\!\!\!- a^0[Q,a^1]^{(k_1)}\dots[Q,a^q]^{(k_q)}\,|Q|^{-q-2|k|}\,, \quad a^j\in\mathcal{A}_{PM}\,,$$

where we used the abbreviations $|k| = k_1 + \ldots + k_q$ and

$$T^{(i)} = \nabla^i(T) \qquad \text{and} \qquad \nabla(T) = D^2T - TD^2\,.$$

Only finitely many terms in the above sum may not vanish and only finitely many components of Φ_Q are nonzero.

While the expression (5.5) is algorithmically computable in principle, its actual computation is prohibitively difficult to perform in practice. However, as we showed in [10] for the case of the flat connection, from the cohomological standpoint the answer is as reasonable as it could possibly be. (See also [21] for a relevant illustration.)

We are now in a position to remove the flatness assumption from the statement of our transverse index theorem [10, §9, Theorem 5] and formulate it in full generality, for a hypoelliptic signature operator formed with an arbitrary torsion-free coupling connection.

Theorem 24 *Let $Q = D|D|$ be the hypoelliptic signature operator on PM associated to a torsion-free connection ∇. The identity component of its Chern character, $ch_*(D)_{(1)} \in HC^*_{\rm per}(\mathcal{A}_{PM})_{(1)}$, is the image under the characteristic homomorphism (4.21) of a universal class $\mathcal{L}_n \in H^*(\mathfrak{a}_n, SO(n))$,*

$$ch_*(D)_{(1)} \;=\; \chi^*_{SO(n)}(\mathcal{L}_n)\,.$$

In particular, the class $ch_*(D)_{(1)}$ can be represented by a cocycle built out of the connection form $\omega_\nabla = (\omega^i_j)$, its curvature form $\Omega_\nabla = (\Omega^i_j)$ and the corresponding displacement functions on the jet groupoid of $FM\bar{\rtimes}\Gamma_M$, γ^i_{jk} and $\gamma^i_{jk,\ell} \equiv X_\ell\,\gamma^i_{jk}$, $1 \le i,j,k,\ell \le n$.

Proof. In view of the formula (5.5) for $ch_*(D)$, it suffices to show that any cochain on $\mathcal{A}_{PM}$ of the form,

$$\phi(a^0,\ldots,a^q) = \int\!\!\!\!\!- a^0[Q,a^1]^{(k_1)}\dots[Q,a^q]^{(k_q)}\,|Q|^{-(q+|2k|)}\,,$$

with $a^j = f^j\,U^*_{\psi_j} \in \mathcal{A}_{PM}\,, \quad j = 1,\ldots,q$, such that $\psi_q\circ\cdots\circ\psi_1\circ\psi_0 = 1$, belongs to the range of the characteristic map $\chi^*_{SO(n)}$. Using the results in

[10, §9], specifically Lemma 1 and its corollary, one can write such a cochain as a sum of cochains of the form

$$⨍ a^0\, h^1(a^1) \cdots h^q(a^q)\, R\,, \quad h^1, \ldots, h^q \in \mathcal{H}_{FM}\,,$$

with R a pseudodifferential operator in the hypoelliptic calculus. By (5.4), each such expression is indeed of the desired form

$$\tau\left(a^0\, h^1(a^1) \cdots \widetilde{h}^q(a^q)\right)\,,$$

once the local residue function res_R is absorbed into $\widetilde{h}^q = \beta(\mathrm{res}_R)\, h^q$.

Finally, the second assertion follows from Corrolary 19 and Theorem 22. □

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