

# CYCLIC COHOMOLOGY, HOPF ALGEBRAS AND THE MODULAR THEORY

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## Abstract

We associate canonically a cyclic module to any Hopf algebra endowed with a modular pair, consisting of a group-like element and a character, in involution. This provides the key construct allowing to extend cyclic cohomology to Hopf algebras in the non-unimodular case and further to develop a theory of characteristic classes for actions of Hopf algebras compatible not only with traces but also with the modular theory of weights. It applies to ribbon and to coribbon algebras, as well as to quantum groups and their duals.

## 1 Introduction

We solve in this paper the question left open in [1] (and [2]) of extending cyclic cohomology to Hopf algebras in the general non-unimodular case. The setup employed in [1] was still based on a partial unimodularity condition, which was breaking the natural symmetry between a Hopf algebra and its dual.

Our solution relies on the modular theory of weights instead of traces. We use this theory in an algebraic way, by introducing the notion of a  $\sigma$ -trace on an algebra on which our Hopf algebra acts, where  $\sigma$  is a group-like element. This leads to a natural construction of a cyclic module associated

to any Hopf algebra endowed with a *modular pair in involution*, i.e. with a group-like element and a character such that the corresponding doubly twisted antipode has square the identity. The simplicial structure subjacent to this cyclic module involves the coproduct of the Hopf algebra and the group-like element, while the cyclic structure makes use of the product and of the twisted antipode.

The tracial case of this construction, which corresponds to the group-like element being trivial, was initially introduced in [1] under a restrictive assumption, then amended in [2]; cf. also [3], where it was recast in the Cuntz-Quillen formalism – the non-unimodular case can likewise be reformulated. By extending the construction of the cyclic module to the general non-unimodular case, we settle a problem left open in the above mentioned papers, thus laying the groundwork for a theory of characteristic classes for actions of Hopf algebras compatible with the modular theory of weights.

It is important to mention that the non-unimodular case does arise even in the simplest examples and that neglecting its (often hidden) presence would give rise to misleading answers. The property of the existence of a modular pair in involution is intrinsically satisfied by both the ribbon and the coribbon Hopf algebras, as well as by the quantum groups and their duals.

## 2 Characteristic classes for actions of Hopf algebras

In what follows we fix a *modular pair*, consisting of a group-like element  $\sigma$  and a character  $\delta$  of  $\mathcal{H}$  such that

$$\delta(\sigma) = 1.$$

They will play the role of the module of locally compact groups.

We then introduce the twisted antipode,

$$\tilde{S}(y) = \sum \delta(y_{(1)})S(y_{(2)}) , \quad y \in \mathcal{H}, \quad \Delta y = \sum y_{(1)} \otimes y_{(2)}. \quad (2.1)$$

Given an algebra  $A$ , an action of the Hopf algebra  $\mathcal{H}$  on  $A$  is given by a linear map,

$$\mathcal{H} \otimes A \rightarrow A, \quad h \otimes a \rightarrow h(a)$$

satisfying  $h_1(h_2a) = (h_1h_2)(a)$ ,  $\forall h_i \in \mathcal{H}$ ,  $a \in A$  and

$$h(ab) = \sum h_{(1)}(a)h_{(2)}(b) \quad \forall a, b \in A, h \in \mathcal{H}. \quad (2.2)$$

where the coproduct of  $h$  is,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \quad (2.3)$$

**Definition 1** We shall say that a linear form  $\tau$  on  $A$  is a  $\sigma$ -trace under the action of  $\mathcal{H}$  iff one has,

$$\tau(ab) = \tau(b\sigma(a)) \quad \forall a, b \in A.$$

We shall say that a  $\sigma$ -trace  $\tau$  on  $A$  is  $\delta$ -invariant under the action of  $\mathcal{H}$  iff

$$\tau(h(a)b) = \tau(a\tilde{S}(h)(b)) \quad \forall a, b \in A, h \in \mathcal{H}.$$

As in [1] the definition of the cyclic complex for  $HC_{(\delta, \sigma)}^*(\mathcal{H})$  is uniquely dictated in such a way that the following proposition holds,

**Proposition 2** Let  $\tau$  be a  $\delta$ -invariant  $\sigma$ -trace on  $A$ , then the following defines a canonical map from  $HC_{(\delta, \sigma)}^*(\mathcal{H})$  to  $HC^*(A)$ ,

$$\begin{aligned} \gamma(h^1 \otimes \dots \otimes h^n) \in C^n(A), \quad \gamma(h^1 \otimes \dots \otimes h^n)(x^0, \dots, x^n) = \\ \tau(x^0 h^1(x^1) \dots h^n(x^n)). \end{aligned}$$

We shall show below that the required cyclic complex can be implemented whenever the modular pair  $(\delta, \sigma)$  satisfy a natural involutive condition.

### 3 The cyclic module of a Hopf algebra

In this section we shall associate a cyclic complex (in fact a  $\Lambda$ -module, where  $\Lambda$  is the cyclic category), to any Hopf algebra  $\mathcal{H}$  (over  $\mathbb{C}$ ) endowed with a modular pair  $(\delta, \sigma)$  in involution, i.e. satisfying

$$(\sigma^{-1}\tilde{S})^2 = I. \quad (3.1)$$

With the standard notation for unit  $\eta : \mathbb{C} \rightarrow \mathcal{H}$ , counit  $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$  and antipode  $S : \mathcal{H} \rightarrow \mathcal{H}$ , we recall that the  $\delta$ -twisted antipode was defined as

$$\tilde{S}(h) = \sum_{(h)} \delta(h_{(1)}) S(h_{(2)}) \quad , \quad h \in \mathcal{H}. \quad (3.2)$$

The elementary properties of  $S$  imply immediately that  $\tilde{S}$  is an algebra antihomomorphism

$$\begin{aligned} \tilde{S}(h^1 h^2) &= \tilde{S}(h^2) \tilde{S}(h^1) \quad , \quad \forall h^1, h^2 \in \mathcal{H} \\ \tilde{S}(1) &= 1, \end{aligned} \quad (3.3)$$

a coalgebra twisted antimorphism

$$\Delta \tilde{S}(h) = \sum_{(h)} S(h_{(2)}) \otimes \tilde{S}(h_{(1)}) \quad , \quad \forall h \in \mathcal{H}; \quad (3.4)$$

and also that it satisfies

$$\varepsilon \circ \tilde{S} = \delta. \quad (3.5)$$

By transposing and twisting by  $\sigma$  the standard simplicial operators underlying the Hochschild homology complex of an algebra, one associates to  $\mathcal{H}$ , viewed only as a coalgebra, the following cosimplicial module  $\{\mathcal{H}^{\otimes n}\}_{n \geq 1}$ , with face operators  $\delta_i : \mathcal{H}^{\otimes n-1} \rightarrow \mathcal{H}^{\otimes n}$ ,

$$\begin{aligned} \delta_0(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1} \\ \delta_j(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes \Delta h^j \otimes \dots \otimes h^n, \quad \forall 1 \leq j \leq n-1, \\ \delta_n(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma \end{aligned} \quad (3.6)$$

and degeneracy operators  $\sigma_i : \mathcal{H}^{\otimes n+1} \rightarrow \mathcal{H}^{\otimes n}$ ,

$$\sigma_i(h^1 \otimes \dots \otimes h^{n+1}) = h^1 \otimes \dots \otimes \varepsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}, \quad 0 \leq i \leq n. \quad (3.7)$$

The remaining two essential features of a Hopf algebra – *product* and *antipode* – are now brought into play, to define the *cyclic operators*  $\tau_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$ ,

$$\tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^n \otimes \sigma. \quad (3.8)$$

**Theorem 3** Let  $\mathcal{H}$  be a Hopf algebra endowed with a modular pair  $(\delta, \sigma)$  in involution (3.1). Then  $\mathcal{H}_{(\delta, \sigma)}^{\natural} = \{\mathcal{H}^{\otimes n}\}_{n \geq 1}$  equipped with the operators given by (3.6)–(3.8) defines a module over the cyclic category  $\Lambda$ .

*Proof.* The simplicial relations are easy to check and follow from the group-like property of  $\sigma$ . We shall verify, following closely the corresponding computations in [2], the remaining relations of the cyclic category:

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \quad 1 \leq i \leq n, \quad (3.9)$$

$$\tau_n \delta_0 = \delta_n,$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leq i \leq n, \quad (3.10)$$

$$\tau_n \sigma_0 = \sigma_n \tau_{n+1}^2,$$

$$\tau_n^{n+1} = I_n. \quad (3.11)$$

As in [2], we shall only use the basic properties of the product, the coproduct, the antipode and of the twisted antipode (cf. (3.2)–(3.5)), and adhere to the standard notational conventions for the Hopf algebra calculus (cf. [5]).

We first look at the case  $n = 2$ . Thus,

$$\begin{aligned} \tau_2(h^1 \otimes h^2) &= \Delta \tilde{S}(h^1) \cdot h^2 \otimes \sigma = \\ &= \sum \tilde{S}(h^1)_{(1)} h^2 \otimes \tilde{S}(h^1)_{(2)} \sigma \\ &= \sum S(h^1_{(2)}) h^2 \otimes \tilde{S}(h^1_{(1)}) \sigma. \end{aligned}$$

Its square is therefore:

$$\begin{aligned} \tau_2^2(h^1 \otimes h^2) &= \sum S(S(h^1_{(2)})_{(2)} h^2_{(2)}) \tilde{S}(h^1_{(1)}) \sigma \otimes \tilde{S}(S(h^1_{(2)})_{(1)} h^2_{(1)}) \sigma \\ &= \sum S(S(h^1_{(2)(1)}) h^2_{(2)}) \tilde{S}(h^1_{(1)}) \sigma \otimes \tilde{S}(S(h^1_{(2)(2)}) h^2_{(1)}) \sigma \\ &= \sum S(h^2_{(2)}) (S \circ S)(h^1_{(2)(1)}) \tilde{S}(h^1_{(1)}) \sigma \otimes \tilde{S}(h^2_{(1)}) (\tilde{S} \circ S)(h^1_{(2)(2)}) \sigma \\ &= \sum S(h^2_{(2)}) \boxed{S(S(h^1_{(1)(2)})) \tilde{S}(h^1_{(1)(1)})} \sigma \otimes \tilde{S}(h^2_{(1)}) \tilde{S}(S(h^1_{(2)})) \sigma. \end{aligned}$$

The term in the box is computed as follows. With  $k = h_{(1)}^1$ , one has

$$\begin{aligned}
\sum S(S(k_{(2)}))\tilde{S}(k_{(1)}) &= \sum S(S(k_{(2)}))\delta(k_{(1)(1)})S(k_{(1)(2)}) \\
&= \sum S(S(k_{(2)(2)})\delta(k_{(1)})S(k_{(2)(1)}) = \\
&= \sum \delta(k_{(1)})S\left(\sum k_{(2)(1)}S(k_{(2)(2)})\right) \\
&= \sum \delta(k_{(1)})S(\varepsilon(k_{(2)})1) = \\
&= \sum \delta(k_{(1)})\varepsilon(k_{(2)}) = \delta\left(\sum k_{(1)}\varepsilon(k_{(2)})\right) \\
&= \delta(k).
\end{aligned}$$

It follows that

$$\begin{aligned}
\tau_2^2(h^1 \otimes h^2) &= \sum S(h_{(2)}^2)\sigma \underbrace{\delta(h_{(1)}^1) \otimes \tilde{S}(h_{(1)}^2)\tilde{S}(S(h_{(2)}^1))}_{\tilde{S}(h^1)}\sigma \\
&= \sum S(h_{(2)}^2)\sigma \otimes \tilde{S}(h_{(1)}^2)\tilde{S}(h^1)\sigma = \\
&= \sum S(h_{(2)}^2)\sigma \otimes \tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma,
\end{aligned}$$

Thus

$$\begin{aligned}
\tau_2^2(h^1 \otimes h^2) &= \sum S(h_{(2)}^2) \otimes \tilde{S}(h_{(1)}^2) \cdot \sigma \otimes \tilde{S}^2(h^1)\sigma \\
&= \Delta\tilde{S}(h^2) \cdot \sigma \otimes \tilde{S}^2(h^1)\sigma.
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
\tau_2^3(h^1 \otimes h^2) &= \sum S(S(h_{(2)}^2)_{(2)}\sigma)\tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma \otimes \tilde{S}(S(h_{(2)}^2)_{(1)}\sigma)\sigma \\
&= \sum S(S(h_{(2)(1)}^2)\sigma)\tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma \otimes \tilde{S}(S(h_{(2)(2)}^2)\sigma)\sigma \\
&= \sum S(S(h_{(2)(1)}^2)\sigma)\tilde{S}(h_{(1)(1)}^2)\tilde{S}^2(h^1)\sigma \otimes \sigma^{-1}\tilde{S}(S(h_{(2)}^2))\sigma \\
&= \sum \delta(h_{(1)}^2)\sigma^{-1}\tilde{S}^2(h^1)\sigma \otimes \sigma^{-1}\tilde{S}(S(h_{(2)}^2))\sigma = \\
&= \sigma^{-1}\tilde{S}^2(h^1)\sigma \otimes \sigma^{-1}\tilde{S}^2(h^2)\sigma = h^1 \otimes h^2.
\end{aligned}$$

We now pass to the general case. With the standard conventions of notation,

$$\tau_n(h^1 \otimes h^2 \otimes \dots \otimes h^n) = \Delta^{(n-1)}\tilde{S}(h^1) \cdot h^2 \otimes \dots \otimes h^n \otimes \sigma$$

$$= \sum S(h_{(n)}^1)h^2 \otimes S(h_{(n-1)}^1)h^3 \otimes \dots \otimes S(h_{(2)}^1)h^n \otimes \tilde{S}(h_{(1)}^1)\sigma.$$

Upon iterating once

$$\begin{aligned} \tau_n^2(h^1 \otimes \dots \otimes h^n) &= \sum S(S(h_{(n)}^1)_{(n)}h_{(n)}^2))S(h_{(n-1)}^1)h^3 \otimes \\ &\otimes S(S(h_{(n)}^1)_{(n-1)}h_{(n-1)}^2))S(h_{(n-2)}^1)h^4 \otimes \dots \\ &\dots \otimes S(S(h_{(n)}^1)_{(2)}h_{(2)}^2))\tilde{S}(h_{(1)}^1)\sigma \otimes \tilde{S}(S(h_{(n)}^1)_{(1)}h_{(1)}^2))\sigma \\ &= \sum S(h_{(n)}^2)S(S(h_{(n)}^1)_{(1)})S(h_{(n-1)}^1)h^3 \otimes \\ &\otimes S(h_{(n-1)}^2)S(S(h_{(n)}^1)_{(2)})S(h_{(n-2)}^1)h^4 \otimes \dots \\ &\dots \otimes S(h_{(2)}^2)S(S(h_{(n)}^1)_{(n-1)})\tilde{S}(h_{(1)}^1)\sigma \otimes \\ &\otimes \tilde{S}(h_{(1)}^2)\tilde{S}(S(h_{(n)}^1)_{(n)})\sigma = \\ &= \sum S(h_{(n)}^2)S(h_{(n-1)}^1)S(h_{(n)}^1)h^3 \otimes \\ &\otimes S(h_{(n-1)}^2)S(h_{(n-2)}^1)S(h_{(n+1)}^1)h^4 \otimes \dots \\ &\dots \otimes S(h_{(2)}^2)S(S(h_{(2n-2)}^1)) \cdot \tilde{S}(h_{(1)}^1)\sigma \otimes \\ &\otimes \tilde{S}(h_{(1)}^2)\tilde{S}(S(h_{(2n-1)}^1))\sigma. \end{aligned}$$

We pause to note that

$$\sum h_{(n-1)}^1 S(h_{(n)}^1) = \sum h_{(n-1)(1)}^1 S(h_{(n-1)(2)}^1)$$

equals

$$\varepsilon(h_{(n-1)}^1)1,$$

after resetting the indexation. Next

$$\sum \varepsilon(h_{(n-1)}^1)h_{(n-2)}^1$$

gives  $h_{(n-2)}^1$  after another resetting. In turn

$$\sum h_{(n-2)}^1 S(h_{(n-1)}^1)$$

equals

$$\varepsilon(h_{(n-2)}^1)1,$$

and the process continues.

In the last step,

$$\begin{aligned}
& \sum S(h_{(n)}^2)h^3 \otimes S(h_{(n-1)}^2)h^4 \otimes \dots \\
\dots \otimes & S(h_{(2)}^2) \boxed{S(S(h_{(2)}^1))\delta(h_{(1)(1)}^1)S(h_{(1)(2)}^1)} \sigma \otimes \\
& \otimes \tilde{S}(h_{(1)}^2)\tilde{S}(S(h_{(3)}^1))\sigma \\
& = \sum S(h_{(n)}^2)h^3 \otimes S(h_{(n-1)}^2)h^4 \otimes \dots \otimes S(h_{(2)}^2)\sigma \otimes \tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma \\
& = \sum S(h_{(n)}^2) \otimes S(h_{(n-1)}^2) \otimes \dots \otimes \tilde{S}(h_{(1)}^2) \cdot h^3 \otimes h^4 \otimes \dots \otimes \sigma \otimes \tilde{S}^2(h^1)\sigma \\
& = \Delta^{(n-1)}\tilde{S}(h^2) \cdot h^3 \otimes h^4 \otimes \dots \otimes \sigma \otimes \tilde{S}^2(h^1)\sigma,
\end{aligned}$$

with the boxed term simplified as before.

By induction, one obtains for any  $j = 1, \dots, n+1$ ,

$$\tau_n^j(h^1 \otimes \dots \otimes h^n) = \Delta^{n-1}\tilde{S}(h^j) \cdot h^{j+1} \otimes \dots \otimes h^n \otimes \sigma \otimes \dots \otimes \tilde{S}^2(h^{j-1})\sigma,$$

in particular

$$\tau_n^{n+1}(h^1 \otimes \dots \otimes h^n) = \Delta^{n-1}\tilde{S}(\sigma) \cdot \tilde{S}^2(h^1)\sigma \otimes \dots \otimes \tilde{S}^2(h^n)\sigma = h^1 \otimes \dots \otimes h^n.$$

The verification of the compatibility relations (3.9), (3.10) is straightforward. Indeed, starting with the compatibility with the face operators, one has:

$$\begin{aligned}
\tau_n \delta_0(1 \otimes h^1 \otimes \dots \otimes h^{n-1}) &= \tau_n(1 \otimes h^1 \otimes \dots \otimes h^{n-1}) = \\
&= \Delta^{n-1}\tilde{S}(1) \cdot h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma \\
&= h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma \\
&= \delta_n(h^1 \otimes \dots \otimes h^{n-1}),
\end{aligned}$$

then

$$\begin{aligned}
\tau_n \delta_1(h^1 \otimes \dots \otimes h^{n-1}) &= \tau_n(\Delta h^1 \otimes h^2 \otimes \dots \otimes h^{n-1}) \\
&= \sum \tau_n(h_{(1)}^1 \otimes h_{(2)}^1 \otimes h^2 \otimes \dots \otimes h^{n-1}) \\
&= \sum \Delta^{n-1}\tilde{S}(h_{(1)}^1) \cdot h_{(2)}^1 \otimes h^2 \otimes \dots \otimes h^{n-1} \otimes \sigma = \\
&= \sum S(h_{(1)(n)}^1)h_{(2)}^1 \otimes S(h_{(1)(n-1)}^1)h^2 \otimes \dots \\
&\quad \otimes S(h_{(1)(2)}^1)h^{n-1} \otimes \tilde{S}(h_{(1)(1)}^1)\sigma \\
&= \sum \varepsilon(h_{(n)}^1)1 \otimes S(h_{(n-1)}^1)h^2 \otimes \dots
\end{aligned}$$

$$\begin{aligned}
& \otimes S(h_{(1)}^1)h^{n-1} \otimes \tilde{S}(h_{(1)}^1)\sigma \\
&= 1 \otimes S(h_{(n-1)}^1)h^2 \otimes \dots \otimes S(h_{(1)}^1)h^{n-1} \otimes \tilde{S}(h_{(1)}^1)\sigma \\
&= \delta_0 \tau_{n-1}(h^1 \otimes \dots \otimes h^{n-1}),
\end{aligned}$$

and so forth.

Passing now to degeneracies,

$$\begin{aligned}
\tau_n \sigma_0(h^1 \otimes \dots \otimes h^{n+1}) &= \varepsilon(h^1) \tau_n(h^2 \otimes \dots \otimes h^{n+1}) = \\
&= \varepsilon(h^1) \sum S(h_{(n)}^2)h^3 \otimes \dots \otimes S(h_{(2)}^2)h^{n+1} \otimes \tilde{S}(h_{(1)}^2)\sigma,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
& \sigma_n \tau_{n+1}^2(h^1 \otimes \dots \otimes h^{n+1}) = \\
&= \sigma_n \left( \sum S(h_{(n+1)}^2)h^3 \otimes \dots \otimes S(h_{(2)}^2)\sigma \otimes \tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma \right) \\
&= \sum \varepsilon(\tilde{S}(h_{(1)}^2)\tilde{S}^2(h^1)\sigma) S(h_{(n+1)}^2)h^3 \otimes \dots \otimes S(h_{(2)}^2)\sigma \\
&= \varepsilon(\sigma^{-1}\tilde{S}^2(h^1)\sigma) \sum \delta(h_{(1)}^2) S(h_{(n+1)}^2)h^3 \otimes \dots \otimes S(h_{(2)}^2)\sigma \\
&= \varepsilon(h^1) S(h_{(n)}^2)h^3 \otimes \dots \otimes S(h_{(2)}^2)h^{n+1} \otimes \tilde{S}(h_{(1)}^2)\sigma.
\end{aligned}$$

In the next step

$$\begin{aligned}
\tau_n \sigma_1(h^1 \otimes \dots \otimes h^{n+1}) &= \varepsilon(h^2) \tau_n(h^1 \otimes h^3 \otimes \dots \otimes h^{n+1}) \\
&= \varepsilon(h^2) \cdot \Delta^{n-1} \tilde{S}(h^1) \cdot h^3 \otimes \dots \otimes h^{n+1} \otimes \sigma,
\end{aligned}$$

while on the other hand

$$\begin{aligned}
& \sigma_0 \tau_{n+1}(h^1 \otimes \dots \otimes h^{n+1}) = \\
& \sum \sigma_0(S(h_{(n+1)}^1)h^2 \otimes \dots \otimes S(h_{(2)}^1)h^{n+1} \otimes \tilde{S}(h_{(1)}^1))\sigma \\
&= \sum \varepsilon(h^2) \cdot \varepsilon(h_{(n+1)}^1) \cdot S(h_{(n)}^1)h^3 \otimes \dots \otimes S(h_{(2)}^1)h^{n+1} \otimes \tilde{S}(h_{(1)}^1)\sigma \\
&= \sum \varepsilon(h^2) \cdot S(h_{(n-1)}^1)h^3 \otimes \dots \otimes S(h_{(2)}^1)h^{n+1} \otimes \tilde{S}(h_{(1)}^1)\sigma,
\end{aligned}$$

and similarly for  $i = 2, \dots, n$ . ■

The cohomology of the  $(b, B)$ -bicomplex corresponding to the cyclic module  $\mathcal{H}_{(\delta, \sigma)}^1$  is, by definition, the *cyclic cohomology*  $HC_{(\delta, \sigma)}^*(\mathcal{H})$  of  $\mathcal{H}$  relative to the modular pair in involution  $(\delta, \sigma)$ .

## 4 Examples

The main point of the present paper is that thanks to  $\sigma$  we remove the partial unimodularity condition of [1] on a Hopf algebra. As we shall see now, our general condition is fulfilled (modulo the passage to a double cover) by the most popular Hopf algebras, including quantum groups and their duals.

**Proposition 4** *The following Hopf algebras are canonically endowed with modular pairs in involution: ribbon algebras, coribbon algebras and their tensor products, compact quantum groups in the sense of Woronowicz.*

*Proof.* If  $\mathcal{H}$  is quasitriangular with  $R$ -matrix  $R$ , then

$$S^2(h) = u h u^{-1},$$

with

$$u = \sum S(R^{(2)}) R^{(1)}, \quad \varepsilon(u) = 1$$

and

$$\Delta u = (R_{21} R)^{-1} (u \otimes u).$$

By passage to a “double cover” [4], i.e. an embedding in

$$\mathcal{H}(\theta) = \mathcal{H}[\theta]/(\theta^2 - u S(u))$$

one can assume that  $u S(u) = S(u) u$ , which is central, has a central square root  $\theta$ , such that

$$\Delta(\theta) = (R_{21} R)^{-1} (\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta.$$

Taking

$$\sigma = \theta^{-1} u,$$

one gets a canonical group-like element

$$\Delta \sigma = \sigma \otimes \sigma, \quad \varepsilon(\sigma) = 1, \quad S(\sigma) = \sigma^{-1}.$$

It is easy to check that the product of  $\sigma^{-1}$  by the antipode

$$S' = \sigma^{-1} \cdot S,$$

satisfies the required condition

$$S'^2 = 1.$$

Indeed,

$$\begin{aligned} S'^2(h) &= \sigma^{-1} S(\sigma^{-1} S(h)) = \sigma^{-1} S^2(h) \sigma \\ &= \sigma^{-1} u h u^{-1} \sigma = \theta h \theta^{-1} = h. \end{aligned}$$

This shows that  $(\varepsilon, \sigma)$  is a canonical modular pair in involution for  $\mathcal{H}$ .

By dualizing the above definitions one obtains the notion of a *coquasi-triangular*, resp. *coribbon* algebra. Among the most prominent examples of coribbon algebras are the function algebras of the classical quantum groups  $GL_q(N)$ ,  $SL_q(N)$ ,  $SO_q(N)$ ,  $O_q(N)$  and  $Sp_q(N)$ . For a coribbon algebra  $\mathcal{H}$ , the analogue of the above *ribbon group-like element*  $\sigma$  is the *ribbon character*  $\delta \in \mathcal{H}^*$ . The corresponding twisted antipode satisfies again the condition  $\tilde{S}^2 = 1$ , so that  $(\delta, 1)$  is a canonical modular pair in involution for  $\mathcal{H}$ .

Finally, for a compact quantum group in the sense of [6], Theorem 5.6 of [6] describing the modular properties of the Haar measure shows that both the coordinate algebra as well as its dual are provided with a canonical modular pair in involution. ■

## 5 References

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