C^* algebras and Differential Geometry *

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Abstract

This is the translation to appear in the "SUPERSYMMETRY 2000 - Encyclopaedic Dictionary" of the original Comptes Rendus Note, published in March 1980, in which basic notions of noncommutative geometry were introduced and applied to noncommutative tori. These include connections on finite projective modules, their curvature, and the Chern character. Finite projective modules on the noncommutative two-torus $\mathbb{T}^2_\theta$ were realized as Schwartz spaces of vector valued functions on $\mathbb{R}$. Explicit constant curvature connections were constructed and a basic integrality phenomenon of the total curvature was displayed. The pseudo-differential calculus and the Atiyah-Singer index theorems were extended to Lie group actions on C^* algebras and used to explain the above integrality of the total curvature by an index formula for finite difference-differential operators on the line. Recent interest in the hep-th literature for basic notions of noncommutative geometry in the case of noncommutative tori (cf for instance hep-th/0012145 for an excellent review) prompted us to make the English translation of the original paper available.

Introduction

The theory of C^* algebras is a natural extension of the topology of locally compact spaces. A large portion of the theory has been dealing with the analogue of Radon measures. Among the invariants of algebraic topology it is K theory which is easiest to adapt to the framework of C^* algebras. The study of the analogue of a differentiable structure on the locally compact space X was proposed in [6] as the investigation of derivations of a C^* algebra A. In this note we shall develop the basic notions of differential topology in the special case where the differential structure is obtained through a Lie group G of automorphisms of A. We extend the notions of connection on a bundle, of curvature and Chern classes and construct, given a G-invariant trace τ, a morphism ch_τ from $K_0(A)$ to $H^*_{\mathbb{R}}(G)$, (the cohomology of left invariant differential forms on G). We

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then extend the Atiyah-Singer index theorem. All this study is motivated by a simple example, that of the irrational rotation C^* algebra A_θ of angle θ. This algebra is highly noncommutative, being simple and non type I, and Pimsner and Voiculescu computed $K(A_\theta)$ [3]. We first endow A_θ with the differential structure coming from the obvious action of the compact group $\mathbb{T}^2$, $\mathbb{T} = \{z \in \mathbb{C},\, |z| = 1\}$, we then show that the space of smooth (C^∞) sections of the vector bundle of dimension θ on A_θ is identical with the Schwartz space $\mathcal{S}(\mathbb{R})$, on which A_θ acts by finite difference operators. The following operators

$$D_1\, f = \frac{d}{dt}\, f \qquad \text{and} \qquad D_2\, f = \frac{2\,\pi\, it}{\theta}\, f$$

then define a connection (of constant curvature $1/\theta$) and the index theorem allows to compute the index (integer valued) of polynomials in D_1, D_2 with finite difference operators as coefficients.

Let (A, G, α) be a C^* dynamical system, where G is a Lie group. We shall say that $x \in A$ is of C^∞ class iff the map $g \mapsto \alpha_g(x)$ from G to the normed space A is C^∞. The involutive algebra $A^\infty = \{x \in A,\, x \text{ of class } C^\infty\}$ is norm dense in A.

Let Ξ^∞ be a finite projective module on A^∞, (we shall write it as a right module) ; $\Xi^0 = \Xi^\infty \otimes_{A^\infty} A$ is then a finite projective module on A.

Lemma 1 *For every finite projective module Ξ on A, there exists a finite projective module Ξ^∞ on A^∞, unique up to isomorphism, such that Ξ is isomorphic to $\Xi^\infty \otimes_{A^\infty} A$.*

In the sequel we let Ξ^∞ be a finite projective module on A^∞. An hermitian structure on Ξ^∞ is given by a positive hermitian form $\langle \xi, \eta \rangle \in A^\infty$, $\forall\, \xi, \eta \in \Xi^\infty$ such that

$$\langle \xi \cdot x, \eta \cdot y \rangle = y^* \langle \xi, \eta \rangle\, x\ , \qquad \forall\, \xi, \eta \in \Xi^\infty\ , \quad \forall\, x, y \in A^\infty\,.$$

For $n \in \mathbb{N}$, $\Xi^\infty \otimes \mathbb{C}^n$ is a finite projective module on $M_n(A^\infty) = A^\infty \otimes M_n(\mathbb{C})$, this allows, replacing A by $M_n(A) = A \otimes M_n(\mathbb{C})$ (and the G-action by $\alpha \otimes \mathrm{id}$) to assume the existence of a selfadjoint idempotent $e \in A^\infty$ and of an isomorphism F with the module $e\, A^\infty$ on Ξ^∞. We then endow Ξ^∞ with the following hermitian structure :

$$\langle \xi, \eta \rangle = F^{-1}(\eta)^*\, F^{-1}(\xi) \in A^\infty\,.$$

Let δ be the representation of Lie$G$ in the Lie-algebra of derivations of A^∞ given by

$$\delta_X(x) = \lim_{t \to 0} \frac{1}{t}\, (\alpha_{g_t}(x) - x)\,, \qquad \text{where} \quad \dot{g}_0 = X\,,\ x \in A^\infty\,.$$

Definition 2 *Let Ξ^∞ be a finite projective module on A^∞, a connection (on Ξ^∞) is a linear map ∇ of Ξ^∞ in $\Xi^\infty \otimes (\mathrm{Lie}\ G)^*$ such that, for all $X \in \mathrm{Lie}\ G$ and $\xi \in \Xi^\infty$, $x \in A^\infty$ one has*

$$\nabla_X(\xi \cdot x) = \nabla_X(\xi) \cdot x + \xi \cdot \delta_X(x)\,.$$

We shall say that ∇ is compatible with the hermitian structure iff :

$$\langle \nabla_X\, \xi, \xi' \rangle + \langle \xi, \nabla_X\, \xi' \rangle = \delta_X \langle \xi, \xi' \rangle\,, \qquad \forall\, \xi, \xi' \in \Xi^\infty\,, \quad \forall\, X \in \mathrm{Lie}\ G\,.$$

Every finite projective module, Ξ^∞, on A^∞ admits a connection ; on the module $e\,A^\infty$ the following formula defines the *Grassmannian connection*

$$\nabla_X^0(\xi) = e\,\delta_X(\xi) \in e\,A^\infty\,, \qquad \forall\, \xi \in e\,A^\infty\,, \quad \forall\, X \in \mathrm{Lie}\ G\,.$$

This connection is compatible with the hermitian structure

$$\langle \xi, \eta \rangle = \eta^*\, \xi \in A^\infty\,, \qquad \forall\, \xi, \eta \in e\,A^\infty\,.$$

To the representation δ of Lie G in the Lie-algebra of derivations of A^∞ corresponds the complex $\Omega = A^\infty \otimes \Lambda\ (\mathrm{Lie}\ G)^*$ of left-invariant differential forms on G with coefficients in A^∞. We endow Ω with the algebra structure given by the tensor product of A^∞ by the exterior algebra of $(\mathrm{Lie}\ G)^*_{\mathbb{C}}$ (we use the notation $\omega_1 \wedge \omega_2$ for the product of ω_1 with ω_2, one no longer has, of course, the equality $\omega_2 \wedge \omega_1 = (-1)^{\partial\omega_1\,\partial\omega_2}\, \omega_1 \wedge \omega_2$). The exterior differential d is such that :

1° for $a \in A^\infty$ and $X \in \mathrm{Lie}\ G$ one has $\langle X, da \rangle = \delta_X(a)$;

2° $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p\, \omega_1 \wedge d\omega_2$, $\forall\, \omega_1 \in \Omega^p$, $\forall\, \omega_2 \in \Omega$;

3° $d^2\, \omega = 0$, $\forall\, \omega \in \Omega$.

As $A^\infty \subset \Omega$, Ω is a bimodule on A^∞.

Every connection on $e\,A^\infty$ is of the form $\nabla_X(\xi) = \nabla_X^0(\xi) + \theta_X\, \xi$, $\forall\, \xi \in e\,A^\infty$, $X \in \mathrm{Lie}\ G$, where the form $\theta \in e\,\Omega^1\, e$ is uniquely determined by ∇, one has $\theta_X^* = -\theta_X$, $\forall\, X \in \mathrm{Lie}\ G$ iff ∇ is compatible with the hermitian structure of $e\,A^\infty$.

Definition 3 *Let ∇ be a connection on the finite projective module Ξ^∞ (on A^∞), the curvature of ∇ is the element Θ of $\mathrm{End}_{A^\infty}(\Xi^\infty) \otimes \Lambda^2 (\mathrm{Lie}\ G)^*$ given by*

$$\Theta(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}_{A^\infty}(\Xi^\infty)\,, \qquad \forall\, X, Y \in \mathrm{Lie}\ G\,.$$

We identify $\mathrm{End}(e\,A^\infty)$ with $e\,A^\infty\, e \subset A^\infty$, the curvature Θ_0 of the grassmannian connection is the 2-form $e(de \wedge de) \in \Omega^2$, the curvature of $\nabla = \nabla^0 + \theta\wedge$ is equal to $\Theta_0 + e(d\theta + \theta \wedge \theta)\, e \in \Omega^2$.

Lemma 4 *With the above notations one has*

$$e(d\Theta)\, e = \Theta \wedge \theta - \theta \wedge \Theta\,.$$

Let now τ be a finite G-invariant trace on A. For all $k \in \mathbb{N}$, there exists a unique k-linear map from $\Omega \times \underbrace{\ldots}_{k \text{ times}} \times \Omega$ to $\Lambda(\text{Lie } G)^*_{\mathbb{C}}$ such that,

$$\tau^k(a_1 \otimes \omega_1, ..., a_k \otimes \omega_k) = \tau(a_1 a_2 ... a_k)\, \omega_1 \wedge ... \wedge \omega_k$$

Proposition 5 *With the above notations the following invariant differential form on G, $\tau^k(\Theta, \ldots, \Theta) \in \Lambda^{2k}(\text{Lie } G)^*_{\mathbb{C}}$ is closed and its cohomology class only depends on the finite projective module $\Xi^\infty = e\, A^\infty$ on A^∞.*

We let $H^*_{\mathbb{R}}(G)$ be the cohomology ring of left invariant differential forms on G, for e as above we let

$$\text{Ch}_\tau(e) = \text{class of } \sum \left(\frac{1}{2\,\pi\, i}\right)^k \frac{1}{k!}\, \tau^k(\Theta, \ldots, \Theta) \in H^*_{\mathbb{R}}(G)\,.$$

One thus obtains a morphism ch_τ from the group $K_0(A)$ to $H^*_{\mathbb{R}}(G)$. Replacing (A, G, α) by $(A \otimes C(S^1), G \times \mathbb{R}, \alpha')$, where $\alpha'_{g,s}(x \otimes f) = \alpha_g(x) \otimes f_s$ and $f_s(t) = f(t-s)$, $\forall\, t \in S^1 = \mathbb{R}/\mathbb{Z}$ one extends ch_τ to a morphism from $K(A) = K_0(A) \oplus K_1(A)$ to $H^*_{\mathbb{R}}(G)$.

Special cases

(a) Let $(\alpha_t)_{t \in \mathbb{R}}$ be a one parameter group of automorphisms of A, δ the corresponding derivation, then for $U \in A$, invertible and of class C^∞ one has $\text{ch}_\tau([U]) = (1/2\, i\pi)\, \tau(\delta(U)U^{-1})$. (The non-triviality of this expression was known, cf. [4].)

(b) Let $(\alpha_{t_1,t_2})_{t_1,t_2 \in \mathbb{R}}$ be a two parameter group of automorphisms of A, δ_1, δ_2 the corresponding derivations of A and ε^1, ε^2 the canonical basis of $(\text{Lie } G)^*$, where $G = \mathbb{R}^2$. For every orthogonal projection e in A one has $\text{ch}_\tau(e) = \tau(e) + c_1(e)\, \varepsilon^1 \wedge \varepsilon^2 \in H^*_{\mathbb{R}}(G) = \Lambda(\text{Lie } G)^*$, where $c_1(e)$ is computed for e of class C^∞ by the formula

$$c_1(e) = \frac{1}{2\, i\, \pi}\, \tau(e(\delta_1(e)\, \delta_2(e) - \delta_2(e)\, \delta_1(e)))\,.$$

In particular $c_1(e) = 0$ if e is equivalent to an orthogonal projection fixed by a one parameter subgroup.

Let us now give a concrete example, let $\theta \in [0,1] \backslash \mathbb{Q}$ and A_θ the C^*-algebra generated by two unitaries U_1, U_2 such that $U_1\, U_2 = \lambda\, U_2\, U_1$, $\lambda = \exp(i\, 2\, \pi\, \theta)$. The theorem of Pimsner and Voiculescu [3] determines $K_0(A_\theta) = \mathbb{Z}^2$ and shows that the unique trace τ [with $\tau(1) = 1$] on A_θ defines an isomorphism of $K_0(A_\theta)$ on $\mathbb{Z} + \theta\, \mathbb{Z} \subset \mathbb{R}$. Moreover the construction of Powers and Rieffel [5] exhibits an idempotent $e \in A_\theta$ with $\tau(e) = \theta$; let ρ be the isomorphism of $C(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}$ in A_θ which to the function $t \mapsto \exp(i\, 2\, \pi\, t)$ associates U_1. One has $e_0 = \rho(g)\, U_2 +$

$\rho(f) + (\rho(g)\,U_2)^*$, the conditions on f, $g \in C(S^1)$ are fulfilled if $f(s) = 1$, $\forall\, s \in [1-\theta, \theta]$, $f(s) = 1 - f(s-\theta)$, $\forall\, s \in [\theta, 1]$ and $g(s) = 0$ if $s \in [0, \theta]$, $g(s) = (f(s) - f^2(s))^{1/2}$ if $s \in [\theta, 1]$.

Let us endow A_θ with the action of $\mathbb{T}^2$, where $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ such that $\alpha_{z_1,z_2}(U_j) = z_j\, U_j$, $j = 1, 2$ the corresponding derivations fulfill $\delta_k(U_k) = 2\pi\, i\, U_k$, $\delta_j(U_k) = 0$ if $j \neq k$. The algebra A_θ^∞ is the space of sequences with rapid decay $(s_{n,m})_{n,m \in \mathbb{Z}^2}$ the associated element of A_θ being $\Sigma\, s_{n,m}\, U_1^n\, U_2^m$. For f and g of class C^∞ one has $e_0 \in A_\theta^\infty$ and the computation shows that $c_1(e_0) = -6 \int g^2\, f'\, dt = 6 \int_0^1 (\lambda - \lambda^2)\, d\lambda = 1$. One then concludes that if $e \in A_\theta$ is an idempotent such that $\tau(e) = |p - q\,\theta|$, one has

$$c_1(e) = \pm\, q\,.$$

This example shows the non-triviality of c_1, the integrality property of $c_1 \in \mathbb{Z}$ will be explained later (th. 10), let us now concretely realise the finite projective module $\Xi_{p,q}^\infty$ of dimension $|p - q\,\theta|$ on A_θ^∞.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$, we view $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q$ as the space of sections of the trivial bundle with fiber $\mathbb{C}^q$ on $\mathbb{R}$, so that $\xi(s) \in \mathbb{C}^q$, $\forall\, \xi \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q$, $s \in \mathbb{R}$. Let W_1, W_2 be two unitaries in $\mathbb{C}^q$ such that $W_1\, W_2 = \exp(i\, 2\,\pi/q)\, W_2\, W_1$. We view $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q$ as a right A_θ^∞-module letting :

$$(\xi \cdot U_1)(s) = W_1\, \xi(s - \varepsilon)\,, \qquad \forall\, s \in \mathbb{R}\,, \quad \text{where } \varepsilon = \frac{p}{q} - \theta\,,$$

$$(\xi \cdot U_2)(s) = \exp(i\, 2\,\pi\, s)\, W_2^p\, \xi(s) \qquad \forall\, s \in \mathbb{R}\,.$$

Theorem 6 *The A_θ^∞ module thus defined is projective of finite type, of dimension $|p - q\,\theta|$, the equalities*

$$(\nabla_1\, \xi)(s) = \frac{d}{ds}\, \xi(s) \quad \textit{and} \quad (\nabla_2\, \xi)(s) = \frac{2\,\pi\, i\, s}{\varepsilon}\, \xi(s)$$

define a connection of constant curvature equal to $1/(\theta - p/q)$.

One thus obtains another way to compute c_1, indeed the integral of the curvature is $|p - q\,\theta|/(\theta - (p/q)) = \pm\, q$.

In this example of A_θ the C^*-algebra is far from trivial (it is not of type I), the differential structure coming from the derivations δ_1, δ_2 is however as regular as for a compact smooth manifold, in particular :

1° with $\Delta = \delta_1^2 + \delta_2^2$ the operator $(1 - \Delta)^{-1}$ from A_θ to A_θ is a compact operator (in the usual sense) ;

2° the space A_θ^∞ is a nuclear space (in the sense of Grothendieck).

To elaborate on these facts and tie them with the integrality of the coefficient c_1, let us first remark that (cf. [2]) the crossed product of A_θ by the above action of $\mathbb{T}^2$ is the elementary C^*-algebra k of compact operators in Hilbert space, and let us then go back to the general case of a C^* dynamical system (A, G, α), with as a goal the study of elliptic differential operators of the form (with $G = \mathbb{R}^n$) $D = \sum_{|\alpha| \leq k} a_j\, \delta^j$, where $j = (j_1, \ldots, j_n)$, $a_j \in A^\infty$ and where for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}_n$, $\sigma(\xi) = i^k \sum_{|j|=k} a_j\, \xi^j$ is *invertible* in A.

Pseudo differential calculus and C^* dynamical systems

To simplify let us assume that $G = \mathbb{R}^n$, let B be the crosed product $B = A \times_\alpha \mathbb{R}^n$, $A \subset M(B)$ the canonical isomorphism of A as a subalgebra of the multiplier algebra $M(B)$ of B and $s \rightarrow V_s$ the canonical unitary representation of $\mathbb{R}^n$ on $M(B)$ such that $V_s\, x\, V_s^* = \alpha_s(x)$, $\forall\, x \in A$.

We endow the involutive algebra A^∞ with the family of semi-norms : $p_i(a) = \|\delta_1^{i_1} \ldots \delta_n^{i_n}\, a\|$, where $i_1, \ldots, i_n \in \mathbb{N}$ and let $\mathcal{S}(\mathbb{R}^n, A^\infty)$ be the corresponding Schwartz space : the map $s \rightarrow a(s)$ from $\mathbb{R}^n$ to A^∞ is in $\mathcal{S}$ iff for all multi-indices i, j the function $p_i((\partial/\partial s)^j\, a(s))$ is of rapid decay.

The C^*-algebra $B = A \times_\alpha \mathbb{R}^n$ is the norm closure of the involutive algebra

$$\Sigma = \left\{ \int a(s)\, V_s\, ds\,,\ a \in \mathcal{S}(\mathbb{R}^n, A^\infty) \right\} .$$

We now construct multipliers of B of the form $\int a(s)\, V_s\, ds$, where a is a distribution with values in A^∞ with singular support contained in $\{0\} \subset \mathbb{R}^n$.

Definition 7 *Let $m \in \mathbb{Z}$, $\mathbb{R}_n = (\mathbb{R}^n)^\wedge$ the dual of $\mathbb{R}^n$, and ρ a map of class C^∞ from $\mathbb{R}_n$ to A^∞. We shall say that ρ is a symbol of order m, $\rho \in S^m$ iff :*

1° *for all multi-indices i, j, there exists $C_{ij} < \infty$ such that*

$$p_i \left(\left(\frac{\partial}{\partial\, \xi} \right)^j \rho(\xi) \right) \leq C_{ij} (1 + |\xi|)^{m-|j|}\, ;$$

2° *there exists $\sigma \in C^\infty(\mathbb{R}_n \backslash \{0\}, A^\infty)$ such that when $\lambda \rightarrow +\infty$ one has $\lambda^{-m}\, \rho(\lambda\, \xi) \rightarrow \sigma(\xi)$ [for the topology of $C^\infty(\mathbb{R}_n \backslash \{0\}, A^\infty)$].*

Let $\rho \in S^m$ and $\hat\rho$ its Fourier transform in the sense of distributions [by hypothesis $\rho \in \mathcal{S}'(\mathbb{R}_n, A^\infty)$], it is a distribution with values in A^∞ given by the oscillating integral

$$\hat\rho\,(s) = \int \rho\,(\xi)\, e^{-is \cdot \xi}\, d\xi\,.$$

Its singular support is contained in $\{0\} \subset \mathbb{R}^n$. Moreover $\hat\rho \in \mathcal{S}(\mathbb{R}^n, A^\infty)$ iff ρ is of order $-\infty$.

Proposition 8 (a) *For all* $\rho \in S^m$ *the equality* $P_\rho = \int \hat{\rho}(s)\, V_s\, ds$ *defines a multiplier* P_ρ *of the involutive algebra* Σ.

(b) *Let* $m_1, m_2 \in \mathbb{Z}$, $\rho_j \in S^{m_j}$, $j = 1, 2$. *There exists* $\rho \in S^{m_1+m_2}$ *such that* $P_\rho = P_{\rho_1} P_{\rho_2}$.

(c) *If* $\rho \in S^0$, *then* P_ρ *extends as a multiplier of* $B = A \times_\alpha \mathbb{R}^n$.

(d) *The norm closure* $\mathcal{E}$ *of* $\{P_\rho, \rho \in S^0\}$ *is a sub* C^**-algebra of* $M(B)$.

(e) *Let* S_{n-1} *be the space of half rays* $\mathbb{R}^+ \xi$, $\xi \in \mathbb{R}_n \backslash [0\}$ *and* $P \in \mathcal{E}$, *there exists* $\sigma(P) \in A \otimes C(S^{n-1})$ *such that*

$$\sigma(P)(\mathbb{R}_+ \xi) = \lim_{\tau \to \infty} \hat{\alpha}_{\tau\xi}(P)\,.$$

(f) *The sequence* $0 \longrightarrow A \times_\alpha \mathbb{R}^n \xrightarrow{j} \mathcal{E} \xrightarrow{T} A \otimes C(S^{n-1}) \longrightarrow 0$ *is exact.*

In (e) we use the simple norm convergence of multipliers (viewed as operators in B), and $\hat{\alpha}$ is the dual action. In (f), j is the canonical inclusion of B in $M(B)$ and σ the principal symbol as defined in (e).

To the exact sequence (f) corresponds a six terms exact sequence involving $K(B)$, $K(\mathcal{E})$ and $K(A \otimes C(S^{n-1}))$. In the particular case where $G = \mathbb{R}^n$ as above, one can explicitly compute the index, i.e. the map from $K_1(A \otimes C(S^{n-1}))$ to $K_0(B)$. One has indeed the Thom isomorphism of φ_A, of $K(A)$ on $K(B) = K(A \times_\alpha \mathbb{R}^n)$, defined by induction on n by iterating the isomorphism described in [1].

Let 0 be a point of S^{n-1} and λ the canonical generator of $K(S^{n-1} \backslash \{0\})$, Ψ_A the corresponding isomorphism of $K(A)$ with $K(A \otimes C_0(S^{n-1} \backslash \{0\}))$.

Theorem 9 (index theorem for flows) *One has*

$$\text{Index } P = \varphi_A \circ \Psi_A^{-1}(\tau(P))\,, \qquad \forall\, P \in \mathcal{E}\,,$$

$\sigma(P)$ *invertible.*

On the one hand this theorem is finer than the index theorem for foliations with transverse measures in the case of flows, since it computes the $K_0(C^*(V, \mathcal{F}))$-valued index and not only its composition with the trace. In particular, it shows that the image of $K_0(C^*(V, \mathcal{F}))$ by the trace is equal to the image by the Ruelle-Sullivan cycle $[C]$ of the subgroup of $H^*(V, \mathbb{R})$ range of $K(V)$ by the chern character Ch. Thus, if $H^n(V, \mathbb{R}) = 0$ this range is zero (cf. [1] for an application in the case $n = 1$).

On the other hand, if τ is a finite α-invariant trace on A and $\hat{\tau}$ is the dual trace, theorem 9 allows to compute $\hat{\tau} \circ \text{Ind}$. For D elliptic, one defines $\text{ch}_\tau(\sigma_D)$ as in the classical case as the composition of the map ch_τ defined above with Ψ_A^{-1}, one then has :

Theorem 10 *One has* $\hat{\tau} \circ \text{Index}\, D = \langle \text{ch}_\tau(\sigma_D), v \rangle$, *where* $v = \varepsilon_1 \wedge \ldots \wedge \varepsilon_n$ *is the unique element of the canonical basis of* $\Lambda^n \mathbb{R}^n$.

Theorem 10 remains valid when one replaces $\mathbb{R}^n$ by a Lie group G, in particular let $A = A_\theta$, $G = \mathbb{T}^2$ acting on A_θ as above, then the crossed product $A_\theta \times \mathbb{T}^2$ is isomorphic to the elementary C^*-algebra k, the dual trace $\hat{\tau}$ being the standard trace on k which shows the integrality of $\hat{\tau} \circ \mathrm{Ind}$, as in the case of ordinary compact manifolds.

Let us consider $\mathcal{S}(\mathbb{R})$ as the right A_θ^∞-module $\Xi_{(0,1)}$ described in theorem 6, then $\mathrm{End}(\mathcal{S}(\mathbb{R}))$ is the algebra of finite difference operators of the form $(\Delta F)(s) = \sum \varphi_n(s)\, F(s-n)$, where the φ_n are periodic smooth functions of period θ and where the sequence φ_n is of rapid decay. Thus $\mathrm{End}(\mathcal{S}(\mathbb{R}))$ is isomorphic to $A_{\theta'}^\infty$, where $\theta' = 1/\theta - E(1/\theta)$. Taking for instance f and g periodic with period 1 fulfilling the above conditions relative to θ', $h(s) = g(s\,\theta)$, $k(s) = 1 - 2\, f(s\,\theta)$, the index of the operator

$$P : (PF)(s) = h(s)\, F'(s-1) + h(s+1)\, F'(s+1) + k(s)\, F'(s) + s\, F(s)$$

is equal to $1 + E(1/\theta)$, hence the existence of nonzero solutions of equations $PF = 0$, $F \in \mathcal{S}(\mathbb{R})$, $F \neq 0$.

Remark. Let V be a compact n-dimensional smooth manifold, Ψ a diffeomorphism of V and A the crossed-product C^*-algebra of $C(V)$ by the automorphism Ψ^*, $\Psi^* f = f \circ \Psi$. Let ΛV be the complex of smooth differential forms on V. One can define, essentially as the crossed-product of ΛV by the action of Ψ, a complex $\Omega = \sum_0^{n+1} \Omega^k$. The dense involutive subalgebra Ω^0 of A, has properties similar to the complex $\Omega = A^\infty \times \Lambda(\mathrm{Lie}\ G)$ discussed above. The differentable structure thus defined does not in general correspond to derivations of A.

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