ON VON NEUMANN ALGEBRAS ARISING FROM BOST-CONNES TYPE SYSTEMS

SERGEY NESHVEYEV

The Bost-Connes system for a number field K, defined for imaginary quadratic fields in [3] and for general fields in [4], can be described as follows [7].

Consider the topological space $\mathcal{G}(K^{ab}/K)\times\mathbb{A}_{K,f}$, where $\mathcal{G}(K^{ab}/K)$ is the Galois group of the maximal abelian extension of K and $\mathbb{A}_{K,f}$ the ring of finite adeles. Define an action of the group $\mathbb{A}^*_{K,f}$ of finite ideles on this space via the Artin map $s\colon I_K\to\mathcal{G}(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma,m)=(\gamma s(j)^{-1},jm)\ \ \text{for}\ \ j\in\mathbb{A}^*_{K,f},\ \ \gamma\in\mathcal{G}(K^{ab}/K),\ \ m\in\mathbb{A}_{K,f}.$$

Consider the quotient space $X=\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathbb{A}_{K,f}$ and the clopen subset $Y=\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\hat{\mathcal{O}}$ of X. The action of $\mathbb{A}^*_{K,f}$ defines an action of $J_K\cong\mathbb{A}^*_{K,f}/\hat{\mathcal{O}}^*$ on X. Put

$$A=(C^*_r(J_K\boxtimes Y),\sigma)=\mathbb{1}_Y(C_0(X)\rtimes J_K)\mathbb{1}_Y.$$

The dynamics on A is defined using the absolute norm $N\colon J_K\to(0,+\infty)$.

It is shown in [7] that for each $\beta\in(0,1]$ there exists a unique KMS$_\beta$-state φ_β on A.

Theorem 1. *The von Neumann algebra M_β generated by A in the GNS-representation corresponding to φ_β is the injective factor of type III$_1$.*

Using the action of $\mathcal{G}(K^{ab}/K)$ and a trick from [8] the proof reduces to a computation of the type of an infinite tensor product of factors of type I. The latter computation is similar to that in [1] and is based on an effective form of Landau's theorem on distribution of prime ideals.

In higher dimensions the situation is more complicated. Consider the GL$_n$-system, a generalization of the GL$_2$-system of Connes and Marcolli [2],

$$A=C^*_r(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{GL}^+_n(\mathbb{Q})\boxtimes_{\mathrm{SL}_n(\mathbb{Z})}(\mathrm{PGL}^+_n(\mathbb{R})\times\mathrm{Mat}_n(\hat{\mathbb{Z}})))$$

with dynamics defined by the determinant homomorphism on $\mathrm{GL}^+_n(\mathbb{Q})$. In [5] we proved that for every $\beta\in(n-1,n]$ there exists a unique KMS$_\beta$-state on A. Denote by M_β the von Neumann algebra generated by A in the corresponding GNS-representation. Consider also the von Neumann subalgebra $N_\beta\subset M_\beta$ generated by the Hecke algebra $\mathcal{H}(\mathrm{Mat}_n(\mathbb{Q})\rtimes\mathrm{GL}^+_n(\mathbb{Q}),\mathrm{Mat}_n(\mathbb{Z})\rtimes\mathrm{SL}_n(\mathbb{Z}))$, see [6].

Theorem 2. *For every $\beta\in(n-1,n]$, the factor M_β is injective. The subalgebra $N_\beta\subset M_\beta$ is an injective subfactor of type III$_1$.*

Presumably M_β is also of type III$_1$.

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Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway.
E-mail address: sergeyn@math.uio.no