ON VON NEUMANN ALGEBRAS ARISING FROM BOST-CONNES TYPE SYSTEMS

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The Bost-Connes system for a number field $K$, defined for imaginary quadratic fields in [3] and for general fields in [4], can be described as follows [7].

Consider the topological space $G(K^{ab}/K) \times \hat{\mathcal{A}}_{K,f}$, where $G(K^{ab}/K)$ is the Galois group of the maximal abelian extension of $K$ and $\mathcal{A}_{K,f}$ the ring of finite ideles. Define an action of the group $\hat{\mathcal{A}}_{K,f}$ of finite ideles on this space via the Artin map $s: I_K \to G(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm) \quad \text{for} \quad j \in \hat{\mathcal{A}}_{K,f}, \quad \gamma \in G(K^{ab}/K), \quad m \in \mathcal{A}_{K,f}.$$

Consider the quotient space $X = G(K^{ab}/K) \times \hat{\mathcal{O}}, \mathcal{A}_{K,f}$ and the clopen subset $Y = G(K^{ab}/K) \times \hat{\mathcal{O}} \circ \hat{\mathcal{O}}$ of $X$. The action of $\hat{\mathcal{A}}_{K,f}$ defines an action of $J_K \cong \hat{\mathcal{A}}_{K,f}/\hat{\mathcal{O}}$ on $X$. Put

$$A = (C^*(J_K \times Y), \sigma) = 1_Y(C_0(X) \rtimes J_K)1_Y.$$

The dynamics on $A$ is defined using the absolute norm $N: J_K \to (0, +\infty)$.

It is shown in [7] that for each $\beta \in (0, 1]$ there exists a unique KMS$_\beta$-state $\varphi_\beta$ on $A$.

**Theorem 1.** The von Neumann algebra $M_\beta$ generated by $A$ in the GNS-representation corresponding to $\varphi_\beta$ is the injective factor of type III$_1$.

Using the action of $G(K^{ab}/K)$ and a trick from [8] the proof reduces to a computation of the type of an infinite tensor product of factors of type I. The latter computation is similar to that in [1] and is based on an effective form of Landau’s theorem on distribution of prime ideals.

In higher dimensions the situation is more complicated. Consider the $GL_n$-system, a generalization of the $GL_2$-system of Connes and Marcolli [2],

$$A = C^*(SL_n(\mathbb{Z}) \ltimes GL_n^{+}(\mathbb{Q}) \boxtimes SL_n(\mathbb{Z}) (PGL_n^{+}(\mathbb{R}) \times Mat_n(\mathbb{Z})))$$

with dynamics defined by the determinant homomorphism on $GL_n^{+}(\mathbb{Q})$. In [5] we proved that for every $\beta \in (n-1, n]$ there exists a unique KMS$_\beta$-state on $A$. Denote by $M_\beta$ the von Neumann algebra generated by $A$ in the corresponding GNS-representation. Consider also the von Neumann subalgebra $N_\beta \subset M_\beta$ generated by the Hecke algebra $\mathcal{H}(Mat_n(\mathbb{Q}) \rtimes GL_n^{+}(\mathbb{Q}), Mat_n(\mathbb{Z}) \rtimes SL_n(\mathbb{Z}))$, see [6].

**Theorem 2.** For every $\beta \in (n-1, n]$, the factor $M_\beta$ is injective. The subalgebra $N_\beta \subset M_\beta$ is an injective subfactor of type III$_1$.

Presumably $M_\beta$ is also of type III$_1$.

**References**


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