

A SHORT SURVEY OF NONCOMMUTATIVE GEOMETRY

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Abstract

We give a survey of selected topics in noncommutative geometry, with some emphasis on those directly related to physics, including our recent work with Dirk Kreimer on renormalization and the Riemann-Hilbert problem. We discuss at length two issues. The first is the relevance of the paradigm of geometric space, based on spectral considerations, which is central in the theory. As a simple illustration of the spectral formulation of geometry in the ordinary commutative case, we give a polynomial equation for geometries on the four dimensional sphere with fixed volume. The equation involves an idempotent e , playing the role of the instanton, and the Dirac operator D . It expresses the gamma five matrix as the pairing between the operator theoretic chern characters of e and D . It is of degree five in the idempotent and four in the Dirac operator which only appears through its commutant with the idempotent. It determines both the sphere and all its metrics with fixed volume form.

We also show using the noncommutative analogue of the Polyakov action, how to obtain the noncommutative metric (in spectral form) on the noncommutative tori from the formal naive metric. We conclude on some questions related to string theory.

I Introduction

The origin of noncommutative geometry is twofold.

On the one hand there is a wealth of examples of spaces whose coordinate algebra is no longer commutative but which have obvious relevance in physics or mathematics. The first examples came from phase space in quantum mechanics but there are many others, such as the leaf spaces of foliations, the duals of nonabelian discrete groups, the space of Penrose tilings, the Brillouin zone in solid state physics, the noncommutative tori which appear naturally in M-theory compactification, and the Adele class space which is a natural geometric space carrying an action of the analogue of the Frobenius for global fields of zero characteristic. Finally various models of space-time itself are interesting examples of noncommutative spaces.

On the other hand the stretching of geometric thinking imposed by passing to noncommutative spaces forces one to rethink about most of our familiar notions. The difficulty is not to add arbitrarily the adjective quantum to our geometric words but to develop far reaching extensions of classical concepts, ranging from the simplest which is measure theory, to the most sophisticated which is geometry itself.

II Measure theory

The extension of the classical concepts has been achieved a long time ago by operator algebraists as far as measure theory is concerned. The theory of nonabelian von Neumann algebras is indeed a far reaching extension of measure theory, whose main surprise is that such an algebra M inherits from its noncommutativity a god-given time evolution.

It is given by the group homomorphism, ([1])

$$\delta : \mathbb{R} \rightarrow \text{Out}(M) = \text{Aut}(M)/\text{Int}(M) \quad (1)$$

from the additive group \mathbb{R} to the group of automorphism classes of M modulo inner automorphisms.

This uniqueness of the, a priori state dependent, modular automorphism group of a state, together with the earlier work of Powers, Araki-Woods and Krieger were the first steps which eventually led to the complete classification of approximately finite dimensional factors (also called hyperfinite).

They are classified by their module,

$$\text{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* , \quad (2)$$

which is a virtual closed subgroup of \mathbb{R}_+^* in the sense of G. Mackey, i.e. an ergodic action of \mathbb{R}_+^* .

The classification involves three independent parts,

- (A) The definition of the invariant $\text{Mod}(M)$ for arbitrary factors.
- (B) The equivalence of all possible notions of approximate finite dimensionality.

(C) The proof that Mod is a complete invariant and that all virtual subgroups are obtained.

The module of a factor M was first defined ([1]) as a closed subgroup of \mathbb{R}_+^* by the equality

$$S(M) = \bigcap_{\varphi} \text{Spec}(\Delta_{\varphi}) \subset \mathbb{R}_+, \quad (3)$$

where φ varies among (faithful, normal) states on M and the operator Δ_{φ} is the *modular operator* of the Tomita-Takesaki theory ([2]).

The virtual subgroup $\text{Mod}(M)$ is the *flow of weights* ([1] [3] [4] [5]) of M . It is obtained from the module δ as the dual action of \mathbb{R}_+^* on the abelian algebra,

$$C = \text{Center of } (M \rtimes_{\delta} \mathbb{R}), \quad (4)$$

where $M \rtimes_{\delta} \mathbb{R}$ is the crossed product of M by the modular automorphism group δ . This takes care of (A), to describe (B) let us simply state the equivalence ([6]) of the following conditions

M is the closure of the union of an increasing sequence of finite dimensional algebras. (5)

M is complemented as a subspace of the normed space of all operators in a Hilbert space. (6)

The condition (5) is obviously what one would expect for an approximately finite dimensional algebra. Condition (6) is similar to *amenability* for discrete groups and the implication (6) \Rightarrow (5) is a very powerful tool.

Besides the reduction from type III to type II ([1] [3]), the proof of (C) involves the uniqueness of the approximately finite dimensional factor of type II_{∞} , [6], the classification of its automorphisms [7] for the III_{λ} case, and the results of Krieger [4] for the III_0 case. The only case which was left open in 1976 was the III_1 case, which was reduced to a problem on the bicentralizer of states [8], this problem was finally settled by U. Haagerup in [9]. Since then, the subject of von-Neumann algebras has undergone two major revolutions, thanks first to the famous work of Vaughan Jones on subfactors and then to the pioneering work of Dan Voiculescu who created and developed the completely new field of free probability theory.

Von Neumann algebras arise very naturally in geometry from foliated manifolds (V, F) . The von Neumann algebra $L^{\infty}(V, F)$ of a foliated manifold is easy to describe, its elements are random operators $T = (T_f)$, i.e. bounded measurable families of operators T_f parametrized by the leaves f of the foliation. For each leaf f the operator T_f acts in the Hilbert space $L^2(f)$ of square integrable densities on the manifold f . Two random operators are identified if they are equal for almost all leaves f (i.e. a set of leaves whose union in V is negligible). The algebraic operations of sum and product are given by,

$$(T_1 + T_2)_f = (T_1)_f + (T_2)_f, \quad (T_1 T_2)_f = (T_1)_f (T_2)_f, \quad (7)$$

i.e. are effected pointwise.

All types of factors occur from this geometric construction and the continuous dimensions of Murray and von-Neumann play an essential role in the longitudinal index theorem.

Finally we refer to [10] for the role of approximately finite dimensional factors in number theory as the missing Brauer theory at Archimedean places.

III Topology

The development of the topological ideas was prompted by the work of Israel Gel'fand, whose C^* algebras give the required framework for noncommutative topology. The two main driving forces were the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as the Atiyah-Singer Index theorem. It has led, through the work of Atiyah, Singer, Brown, Douglas, Fillmore, Miscenko and Kasparov ([11] [12] [13] [14] [15]) to the recognition that not only the Atiyah-Hirzebruch K -theory but more importantly the dual K -homology admit Hilbert space techniques and functional analysis as their natural framework. The cycles in the K -homology group $K_*(X)$ of a compact space X are indeed given by Fredholm representations of the C^* algebra A of continuous functions on X . The central tool is the Kasparov bivariant K -theory. A basic example of C^* algebra to which the theory applies is the group ring of a discrete group and restricting oneself to commutative algebras is an obviously undesirable assumption.

For a C^* algebra A , let $K_0(A)$, $K_1(A)$ be its K theory groups. Thus $K_0(A)$ is the algebraic K_0 theory of the ring A and $K_1(A)$ is the algebraic K_0 theory of the ring $A \otimes C_0(\mathbb{R}) = C_0(\mathbb{R}, A)$. If $A \rightarrow B$ is a morphism of C^* algebras, then there are induced homomorphisms of abelian groups $K_i(A) \rightarrow K_i(B)$. Bott periodicity provides a six term K theory exact sequence for each exact sequence $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ of C^* algebras and excision shows that the K groups involved in the exact sequence only depend on the respective C^* algebras. As an exercise to appreciate the power of this abstract tool one should for instance use the six term K theory exact sequence to give a short proof of the Jordan curve theorem.

Discrete groups, Lie groups, group actions and foliations give rise through their convolution algebra to a canonical C^* algebra, and hence to K theory groups. The analytical meaning of these K theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0 C_r^* G$ where $C_r^* G$ is the reduced C^* algebra of G . Thus an explicit determination of the K theory in this case in particular involves an enumeration of the discrete series.

We introduced with P. Baum ([16]) a geometrically defined K theory which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In the case of semi-simple Lie groups it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [17].

Using elliptic operators we constructed a natural map from our geometrically defined K theory groups to the above analytic (i.e. C^* algebra) K theory groups. Much progress has been made in the past years to determine the range of validity of the isomorphism between the geometrically defined K theory groups and the above analytic (i.e. C^* algebra) K theory groups. We refer to the three Bourbaki seminars ([18], [19], [20]) for an update on this topic.

IV Differential Topology

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles, etc... took place during the eighties thanks to cyclic cohomology which came from two different horizons ([21] [22] [23] [24] [25]). This led for instance to the proof of the Novikov conjecture for hyperbolic groups [26], but got many other applications. Basically, by extending the Chern-Weil characteristic classes to the general framework it allows for many concrete computations of differential geometric nature on noncommutative spaces. It also showed the depth of the relation between the above classification of factors and the geometry of foliations. For instance, using cyclic cohomology together with the following simple fact,

$$\text{“A connected group can only act trivially on a homotopy invariant cohomology theory”}, \quad (1)$$

one proves (cf. [27]) that for any codimension one foliation F of a compact manifold V with non vanishing Godbillon-Vey class one has,

$$\text{Mod}(M) \text{ has finite covolume in } \mathbb{R}_+^*, \quad (2)$$

where $M = L^\infty(V, F)$ and a virtual subgroup of finite covolume is a flow with a finite invariant measure.

In its simplest form, cyclic cohomology is the cohomology theory obtained from the cochain complex of $(n + 1)$ -linear form on \mathcal{A} , n arbitrary, such that

$$\varphi(a^0, a^1, \dots, a^n) = (-1)^n \varphi(a^1, a^2, \dots, a^0) \quad \forall a_j \in \mathcal{A}, \quad (3)$$

with coboundary operator given by

$$\begin{aligned} (b\varphi)(a^0, \dots, a^{n+1}) = \\ \sum_0^n (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, a^1, \dots, a^n) \end{aligned} \quad (4)$$

Its first important role is to provide invariants of K -theory classes as follows. Given an n -dimensional cyclic cocycle on \mathcal{A} , n even, the following scalar is invariant under homotopy for projectors (idempotents) $E \in M_n(\mathcal{A})$,

$$\varphi_n(E, E, \dots, E) \quad (5)$$

where φ has been uniquely extended to $M_n(\mathcal{A})$ using the trace on $M_n(\mathbb{C})$, as in (9) below. This defines a pairing $\langle K(\mathcal{A}), HC(\mathcal{A}) \rangle$ between cyclic cohomology and K -theory.

When we take $\mathcal{A} = C^\infty(M)$ for a manifold M and let

$$\varphi(f^0, f^1, \dots, f^n) = \langle C, f^0 df^1 \wedge df^2 \wedge \dots \wedge df^n \rangle \quad \forall f^j \in \mathcal{A} \quad (6)$$

where C is an n -dimensional closed de Rham current, the above invariant is equal to (up to normalization)

$$\langle C, Ch(E) \rangle \quad (7)$$

where $Ch(E)$ is the Chern character of the vector bundle E on M whose fiber at $x \in M$ is the range of $E(x) \in M_n(\mathbb{C})$. In this example we see that for any permutation of $\{0, 1, \dots, n\}$ one has:

$$\varphi(f^{\sigma(0)}, f^{\sigma(1)}, \dots, f^{\sigma(n)}) = \varepsilon(\sigma) \varphi(f^0, f^1, \dots, f^n) \quad (8)$$

where $\varepsilon(\sigma)$ is the signature of the permutation. However when we extend φ to $M_n(\mathcal{A})$ as $\varphi_n = \varphi \otimes \text{Tr}$,

$$\varphi_n(f^0 \otimes \mu^0, f^1 \otimes \mu^1, \dots, f^n \otimes \mu^n) = \varphi(f^0, f^1, \dots, f^n) \text{Tr}(\mu^0 \mu^1 \dots \mu^n) \quad (9)$$

the property (8) only survives for *cyclic* permutations. This is at the origin of the name, *cyclic cohomology*, given to the corresponding cohomology theory.

Both the Hochschild and Cyclic cohomologies of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a manifold M were computed in [23], [24], thus showing how to extend the familiar differential geometric notions to the general noncommutative case according to the following dictionary:

	Space	Algebra
	Vector bundle	Finite projective module
(A)	Differential form	(Class of) Hochschild cycle
	DeRham current	(Class of) Hochschild cocycle
	DeRham homology	Cyclic cohomology
	Chern Weil theory	Pairing $\langle K(\mathcal{A}), HC(\mathcal{A}) \rangle$

A simple example of cyclic cocycle on a nonabelian group ring is provided by the following formula. Any *group cocycle* $c \in H^*(B\Gamma) = H^*(\Gamma)$ gives rise to a cyclic cocycle φ_c on the algebra $\mathcal{A} = \mathbb{C}\Gamma$

$$\varphi_c(g_0, g_1, \dots, g_n) = \begin{cases} 0 & \text{if } g_0 \dots g_n \neq 1 \\ c(g_1, \dots, g_n) & \text{if } g_0 \dots g_n = 1 \end{cases} \quad (10)$$

where $c \in Z^n(\Gamma, \mathbb{C})$ is suitably normalized, and 10 is extended by linearity to $\mathbb{C}\Gamma$.

Cyclic cohomology has an equivalent description by means of the bicomplex (b, B) which is given by the following operators acting on multi-linear forms on \mathcal{A} ,

$$(b\varphi)(a^0, \dots, a^{n+1}) = \sum_0^n (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, a^1, \dots, a^n) \quad (11)$$

$$B = AB_0, \quad B_0 \varphi(a^0, \dots, a^{n-1}) = \varphi(1, a^0, \dots, a^{n-1}) - (-1)^n \varphi(a^0, \dots, a^{n-1}, 1) \\ (A\psi)(a^0, \dots, a^{n-1}) = \sum_0^{n-1} (-1)^{(n-1)j} \psi(a^j, a^{j+1}, \dots, a^{j-1}). \quad (12)$$

The pairing between cyclic cohomology and K-theory is given in this presentation by the following formula for the Chern character of the class of an idempotent e , up to normalization one has

$$Ch_n(e) = (e - 1/2) \otimes e \otimes e \otimes \dots \otimes e, \quad (13)$$

where e appears $2n$ times in the right hand side of the equation.

At the conceptual level, cyclic cohomology is a way to embed the nonadditive category of algebras and algebra homomorphisms in an additive category of modules. The latter is the additive category of Λ -modules where Λ is the cyclic category. Cyclic cohomology is then obtained as an *Ext* functor ([21]).

The cyclic category is a small category which can be defined by generators and relations. It has the same objects as the small category Δ of totally ordered finite sets and increasing maps which plays a key role in simplicial topology. Let us recall (we shall use it later) that Δ has one object $[n]$ for each integer n , and is generated by faces $\delta_i, [n-1] \rightarrow [n]$ (the injection that misses i), and degeneracies $\sigma_j, [n+1] \rightarrow [n]$ (the surjection which identifies j with $j+1$), with the relations,

$$\delta_j \delta_i = \delta_i \delta_{j-1} \text{ for } i < j, \quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad i \leq j \quad (14)$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j + 1 \\ \delta_{i-1} \sigma_j & i > j + 1. \end{cases}$$

To obtain Λ one adds for each n a new morphism $\tau_n, [n] \rightarrow [n]$ such that,

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \quad 1 \leq i \leq n, \quad \tau_n \delta_0 = \delta_n \\ \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \quad (15) \\ \tau_n^{n+1} = 1_n.$$

The original definition of Λ (cf. [21]) used homotopy classes of non decreasing maps from S^1 to S^1 of degree 1, mapping \mathbb{Z}/n to \mathbb{Z}/m and is trivially equivalent to the above. Given an algebra A one obtains a module over the small category Λ by assigning to each integer $n \geq 0$ the vector space C^n of $n+1$ -linear forms $\varphi(x^0, \dots, x^n)$ on A , while

the basic operations are given by

$$\begin{aligned}
(\delta_i \varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^i x^{i+1}, \dots, x^n), \quad i = 0, 1, \dots, n-1 \\
(\delta_n \varphi)(x^0, \dots, x^n) &= \varphi(x^n x^0, x^1, \dots, x^{n-1}) \\
(\sigma_j \varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^j, 1, x^{j+1}, \dots, x^n), \quad j = 0, 1, \dots, n \\
(\tau_n \varphi)(x^0, \dots, x^n) &= \varphi(x^n, x^0, \dots, x^{n-1}).
\end{aligned} \tag{16}$$

These operations satisfy the relations (14) and (15). This shows that any algebra A gives rise canonically to a Λ -module and allows [21, 28] to interpret the cyclic cohomology groups $HC^n(A)$ as Ext^n functors. All of the general properties of cyclic cohomology such as the long exact sequence relating it to Hochschild cohomology are shared by Ext of general Λ -modules and can be attributed to the equality of the classifying space $B\Lambda$ of the small category Λ with the classifying space BS^1 of the compact one-dimensional Lie group S^1 . One has

$$B\Lambda = BS^1 = P_\infty(\mathbb{C}) \tag{17}$$

For group rings $\mathcal{A} = \mathbb{C}\Gamma$ as above the cyclic cohomology bicomplex corresponds exactly ([29]) to the bicomplex computing the S^1 -equivariant cohomology of the free loop space of the classifying space $B\Gamma$, which is in essence dual to the space of irreducible representations of Γ .

In the recent years J. Cuntz and D. Quillen ([30] [31] [32]) have developed a powerful new approach to cyclic cohomology which allowed them to prove excision in full generality. A great deal of activity has also been generated around the work of Maxim Kontsevich on deformation theory and the Deligne conjecture on the fine structure of the algebra of Hochschild cochains (see [33]).

V Geometry

The basic data of Riemannian geometry [34] consists of a manifold M whose points are locally labeled by a finite number of real coordinates $\{x^\mu\}$ and a *metric*, which is given by the infinitesimal line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

The distance between two points $x, y \in M$ is given by

$$d(x, y) = \text{Inf}\{\text{Length } \gamma \mid \gamma \text{ is a path between } x \text{ and } y\} \tag{2}$$

where

$$\text{Length } \gamma = \int_\gamma ds. \tag{3}$$

One of the main virtues of Riemannian geometry is to be flexible enough to give a good model of space-time in general relativity (up to a sign change) while simple notions of

Euclidean geometry continue to make sense. Homogeneous spaces which are geometries in the sense of the Klein program are too restrictive to achieve that goal. For instance the idea of a straight line gives rise to the notion of geodesic and the geodesic equation

$$\frac{d^2 x^\mu}{dt^2} = -\Gamma_{\nu\rho}^\mu \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \quad (4)$$

where $\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha})$, gives the Newton equation of motion of a particle in the Newtonian potential V provided one uses the metric $dx^2 + dy^2 + dz^2 - (1 + 2V(x, y, z))dt^2$ instead of the Minkowski metric (cf.[35] for the more precise formulation). The next essential point is that the differential and integral calculus is available and allows to go from the local to the global.

The central notion of noncommutative geometry, comes from the identification of the noncommutative analogue of the two basic concepts in Riemann's formulation of Geometry, namely those of manifold and of infinitesimal line element. Both of these noncommutative analogues are of spectral nature and combine to give rise to the notion of spectral triple and spectral manifold, which will be described in detail below. We shall first describe an operator theoretic framework for the calculus of infinitesimals which will provide a natural home for the line element ds .

VI Calculus and Infinitesimals

It was recognized at an early stage of the development of noncommutative geometry that the formalism of quantum mechanics gives a natural home both to infinitesimals (the compact operators in Hilbert space) and to the integral (the logarithmic divergence in an operator trace) thus allowing for the generalization of the differential and integral calculus which is vital for the development of the general theory.

The following is the beginning of a long dictionary which translates classical notions into the language of operators in the Hilbert space \mathcal{H} :

Complex variable	Operator in \mathcal{H}
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order α	Compact operator with characteristic values μ_n satisfying $\mu_n = O(n^{-\alpha})$, $n \rightarrow \infty$
Integral of an infinitesimal of order 1	$\int T =$ Coefficient of logarithmic divergence in the trace of T .

The first two lines of the dictionary are familiar from quantum mechanics. The range of a complex variable corresponds to the *spectrum* of an operator. The holomorphic functional calculus gives a meaning to $f(T)$ for all holomorphic functions f on the

spectrum of T . It is only holomorphic functions which operate in this generality which reflects the difference between complex and real analysis. When $T = T^*$ is selfadjoint then $f(T)$ has a meaning for all Borel functions f .

The size of the infinitesimal $T \in \mathcal{K}$ is governed by the order of decay of the sequence of characteristic values $\mu_n = \mu_n(T)$ as $n \rightarrow \infty$. In particular, for all real positive α the following condition defines infinitesimals of order α :

$$\mu_n(T) = O(n^{-\alpha}) \quad \text{when } n \rightarrow \infty \quad (1)$$

(i.e. there exists $C > 0$ such that $\mu_n(T) \leq Cn^{-\alpha} \quad \forall n \geq 1$). Infinitesimals of order α also form a two-sided ideal and moreover,

$$T_j \text{ of order } \alpha_j \Rightarrow T_1 T_2 \text{ of order } \alpha_1 + \alpha_2. \quad (2)$$

Hence, apart from commutativity, intuitive properties of the infinitesimal calculus are fulfilled.

Since the size of an infinitesimal is measured by the sequence $\mu_n \rightarrow 0$ it might seem that one does not need the operator formalism at all, and that it would be enough to replace the ideal \mathcal{K} in $\mathcal{L}(\mathcal{H})$ by the ideal $c_0(\mathbb{N})$ of sequences converging to zero in the algebra $\ell^\infty(\mathbb{N})$ of bounded sequences. A variable would just be a bounded sequence, and an infinitesimal a sequence $\mu_n, \mu_n \downarrow 0$. However, this commutative version does not allow for the existence of variables with range a continuum since all elements of $\ell^\infty(\mathbb{N})$ have a point spectrum and a discrete spectral measure. Only *noncommutativity* of $\mathcal{L}(\mathcal{H})$ allows for the coexistence of variables with Lebesgue spectrum together with infinitesimal variables. As we shall see shortly, it is precisely this lack of commutativity between the line element and the coordinates on a space that will provide the measurement of distances.

The integral is obtained by the following analysis, mainly due to Dixmier ([36]), of the logarithmic divergence of the partial traces

$$\text{Trace}_N(T) = \sum_0^{N-1} \mu_n(T), \quad T \geq 0. \quad (3)$$

In fact, it is useful to define $\text{Trace}_\Lambda(T)$ for any positive real $\Lambda > 0$ by piecewise affine interpolation for noninteger Λ .

Define for all order 1 operators $T \geq 0$

$$\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_e^\Lambda \frac{\text{Trace}_\mu(T)}{\log \mu} \frac{d\mu}{\mu} \quad (4)$$

which is the Cesaro mean of the function $\frac{\text{Trace}_\mu(T)}{\log \mu}$ over the scaling group \mathbb{R}_+^* . For $T \geq 0$, an infinitesimal of order 1, one has

$$\text{Trace}_\Lambda(T) \leq C \log \Lambda \quad (5)$$

so that $\tau_\Lambda(T)$ is bounded. The essential property is the following *asymptotic additivity* of the coefficient $\tau_\Lambda(T)$ of the logarithmic divergence (5):

$$|\tau_\Lambda(T_1 + T_2) - \tau_\Lambda(T_1) - \tau_\Lambda(T_2)| \leq 3C \frac{\log(\log \Lambda)}{\log \Lambda} \quad (6)$$

for $T_j \geq 0$.

An easy consequence of (6) is that any limit point τ of the nonlinear functionals τ_Λ for $\Lambda \rightarrow \infty$ defines a positive and linear trace on the two-sided ideal of infinitesimals of order 1,

In practice the choice of the limit point τ is irrelevant because in all important examples T is a *measurable* operator, i.e.:

$$\tau_\Lambda(T) \text{ converges when } \Lambda \rightarrow \infty. \quad (7)$$

Thus the value $\tau(T)$ is independent of the choice of the limit point τ and is denoted

$$\int T. \quad (8)$$

The first interesting example is provided by pseudodifferential operators T on a differentiable manifold M . When T is of order 1 in the above sense, it is measurable and $\int T$ is the non-commutative residue of T ([37]). It has a local expression in terms of the distribution kernel $k(x, y)$, $x, y \in M$. For T of order 1 the kernel $k(x, y)$ diverges logarithmically near the diagonal,

$$k(x, y) = -a(x) \log |x - y| + 0(1) \text{ (for } y \rightarrow x) \quad (9)$$

where $a(x)$ is a 1-density independent of the choice of Riemannian distance $|x - y|$. Then one has (up to normalization),

$$\int T = \int_M a(x). \quad (10)$$

The right hand side of this formula makes sense for all pseudodifferential operators (cf. [37]) since one can see that the kernel of such an operator is asymptotically of the form

$$k(x, y) = \sum a_k(x, x - y) - a(x) \log |x - y| + 0(1) \quad (11)$$

where $a_k(x, \xi)$ is homogeneous of degree $-k$ in ξ , and the 1-density $a(x)$ is defined intrinsically.

The same principle of extension of \int to infinitesimals of order < 1 works for hypoelliptic operators and more generally as we shall see below, for spectral triples whose dimension spectrum is simple.

VII Manifolds

As we shall see shortly this framework gives a natural home for the analogue of the infinitesimal line element ds of Riemannian geometry, but we need first to exhibit its compatibility with the notion of manifold.

It was recognized long ago by geometers that the main quality of the homotopy type of a manifold, (besides being defined by a cooking recipe) is to satisfy Poincaré duality not only in ordinary homology but also in K-homology. Poincaré duality in ordinary homology is not sufficient to describe homotopy type of manifolds ([38]) but D. Sullivan ([39]) showed (in the simply connected PL case of dimension ≥ 5 ignoring 2-torsion) that it is sufficient to replace ordinary homology by KO -homology.

The characteristic property of *differentiable manifolds* which is carried over to the noncommutative case is *Poincaré duality* in KO -homology.

Moreover, K -homology admits, as we saw above, a fairly simple definition in terms of Hilbert space Fredholm representations.

In the general framework of Noncommutative Geometry the confluence of the Hilbert space incarnation of the two notions of metric and fundamental class for a manifold led very naturally to define a geometric space as given by a *spectral triple*:

$$(\mathcal{A}, \mathcal{H}, D) \tag{1}$$

where \mathcal{A} is an involutive algebra of operators in a Hilbert space \mathcal{H} and D is a selfadjoint operator on \mathcal{H} . The involutive algebra \mathcal{A} corresponds to a given space M like in the classical duality “Space \leftrightarrow Algebra” in algebraic geometry. The infinitesimal line element in Riemannian geometry is given by the equality

$$ds = 1/D, \tag{2}$$

which expresses the infinitesimal line element ds as the inverse of the Dirac operator D , hence under suitable boundary conditions as a propagator.

The significance of D is two-fold. On the one hand it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration. The exact measurement of distances is performed as follows, instead of measuring distances between points using the formula (5.2) we measure distances between states φ, ψ on $\bar{\mathcal{A}}$ by a dual formula. This dual formula involves *sup* instead of *inf* and does not use paths in the space

$$d(\varphi, \psi) = \text{Sup} \{ |\varphi(a) - \psi(a)| ; a \in \mathcal{A}, \|[D, a]\| \leq 1 \}. \tag{3}$$

A state, is a normalized positive linear form on \mathcal{A} such that $\varphi(1) = 1$,

$$\varphi : \bar{\mathcal{A}} \rightarrow \mathbb{C}, \varphi(a^*a) \geq 0, \quad \forall a \in \bar{\mathcal{A}}, \varphi(1) = 1. \tag{4}$$

In the commutative case the points of the space coincide with the characters of the algebra or equivalently with its pure states (i.e. the extreme points of the convex compact set of states). As it should, this formula gives the geodesic distance in the

Riemannian case. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to a compact Riemannian manifold M , K -oriented by a spin structure, is given by the representation

$$(f\xi)(x) = f(x)\xi(x) \quad \forall x \in M, f \in \mathcal{A}, \xi \in \mathcal{H} \quad (5)$$

of the algebra \mathcal{A} of functions on M in the Hilbert space

$$\mathcal{H} = L^2(M, S) \quad (6)$$

of square integrable sections of the spinor bundle. The operator D is the Dirac operator (cf. [40]). The commutator $[D, f]$, for $f \in \mathcal{A} = C^\infty(M)$ is the Clifford multiplication by the gradient ∇f and its operator norm is:

$$\|[D, f]\| = \text{Sup}_{x \in M} \|\nabla f(x)\| = \text{Lipschitz norm } f. \quad (7)$$

Let $x, y \in M$ and φ, ψ be the corresponding characters: $\varphi(f) = f(x)$, $\psi(f) = f(y)$ for all $f \in \mathcal{A}$. Then formula (3) gives the same result as formula (5.2), i.e. it gives the geodesic distance between x and y .

Unlike the formula (5.2) the dual formula (3) makes sense in general, namely, for example for discrete spaces and even for totally disconnected spaces.

The second role of the operator D is to define the fundamental class of the space X in K -homology, according to the following table,

Space X	Algebra \mathcal{A}
$K_1(X)$	Stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$
$K_0(X)$	Stable homotopy class of $\mathbb{Z}/2$ graded spectral triple

(i.e. for K_0 we suppose that \mathcal{H} is $\mathbb{Z}/2$ -graded by γ , where $\gamma = \gamma^*$, $\gamma^2 = 1$ and $\gamma a = a\gamma \quad \forall a \in \mathcal{A}$, $\gamma D = -D\gamma$).

This description works for the complex K -homology which is 2-periodic. We shall come back later to its refinement to KO -homology.

VIII Operator theoretic Index Formula

Before entering in the detailed discussion of the spectral notion of manifold let us mention the local index formula. This result allows, using the infinitesimal calculus, to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$.

The Fredholm index of the operator D determines (in the odd case) an additive map $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$ given by the equality

$$\varphi([u]) = \text{Index}(PuP), \quad u \in GL_1(\mathcal{A}) \quad (1)$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}(D)$.

This map is computed by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, \dots, a^n) = \text{Trace}(a^0[F, a^1] \dots [F, a^n]) \quad \forall a^j \in \mathcal{A} \quad (2)$$

where $F = \text{Sign } D$ and we assume that the dimension p of our space is finite, which means that $(D + i)^{-1}$ is of order $1/p$, also $n \geq p$ is an odd integer. There are similar formulas involving the grading γ in the even case, and it is quite satisfactory ([41] [42]) that both cyclic cohomology and the chern Character formula adapt to the infinite dimensional case in which the only hypothesis is that $\exp(-D^2)$ is a trace class operator.

It is difficult to compute the cocycle τ in general because the formula (2) involves the ordinary trace instead of the local trace \oint and it is crucial to obtain a local form of the above cocycle.

This problem is solved by a general formula [43] which we now describe
Let us make the following regularity hypothesis on $(\mathcal{A}, \mathcal{H}, D)$

$$a \text{ and } [D, a] \in \cap \text{Dom } \delta^k, \quad \forall a \in \mathcal{A} \quad (3)$$

where δ is the derivation $\delta(T) = [[D], T]$ for any operator T .

We let \mathcal{B} denote the algebra generated by $\delta^k(a)$, $\delta^k([D, a])$. The usual notion of *dimension* of a space is replaced by the *dimension spectrum* which is a subset of \mathbb{C} . The precise definition of the dimension spectrum is the subset $\Sigma \subset \mathbb{C}$ of singularities of the analytic functions

$$\zeta_b(z) = \text{Trace}(b|D|^{-z}) \quad \text{Re } z > p, \quad b \in \mathcal{B}. \quad (4)$$

The dimension spectrum of a manifold M is the set $\{0, 1, \dots, n\}$, $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z \notin \mathbb{R}$ in the dimension spectrum.

We assume that Σ is discrete and simple, i.e. that ζ_b can be extended to \mathbb{C}/Σ with simple poles in Σ .

We refer to [43] for the case of a spectrum with multiplicities. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple satisfying the hypothesis (3) and (4). The local index theorem is the following, [43]:

1. The equality

$$\oint P = \text{Res}_{z=0} \text{Trace}(P|D|^{-z})$$

defines a trace on the algebra generated by \mathcal{A} , $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.

2. There is only a finite number of non-zero terms in the following formula which defines the odd components $(\varphi_n)_{n=1,3,\dots}$ of a cocycle in the bicomplex (b, B) of \mathcal{A} ,

$$\varphi_n(a^0, \dots, a^n) = \sum_k c_{n,k} \oint a^0 [D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} |D|^{-n-2|k|} \quad \forall a^j \in \mathcal{A}$$

where the following notations are used: $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2T - TD^2$, k is a multi-index, $|k| = k_1 + \dots + k_n$,

$$c_{n,k} = (-1)^{|k|} \sqrt{2i} (k_1! \dots k_n!)^{-1} ((k_1+1) \dots (k_1+k_2+\dots+k_n+n))^{-1} \Gamma\left(|k| + \frac{n}{2}\right).$$

3. The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.

For the normalization of the pairing between HC^* and $K(\mathcal{A})$ see [44]. In the even case, i.e. when \mathcal{H} is $\mathbb{Z}/2$ graded by γ ,

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma a = a\gamma \quad \forall a \in \mathcal{A}, \quad \gamma D = -D\gamma,$$

there is an analogous formula for a cocycle (φ_n) , n even, which gives the Fredholm index of D with coefficients in K_0 . However, φ_0 is not expressed in terms of the residue \oint because it is not local for a finite dimensional \mathcal{H} .

IX Diffeomorphism invariant Geometry

The power of the above operator theoretic local trace formula lies in its generality. We showed in [45] how to use it to compute the index of transversally hypoelliptic operators for foliations ([46]). This allows to give a precise meaning to diffeomorphism invariant geometry on a manifold M , by the construction of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where the algebra \mathcal{A} is the crossed product of the algebra of smooth functions on the finite dimensional bundle P of metrics on M by the natural action of the diffeomorphism group of M . The operator D is an hypoelliptic operator which is directly associated to the reduction of the structure group of the manifold P to a group of triangular matrices whose diagonal blocks are orthogonal. By construction the fiber of $P \xrightarrow{\pi} M$ is the quotient $F/O(n)$ of the $GL(n)$ -principal bundle F of frames on M by the action of the orthogonal group $O(n) \subset GL(n)$. The space P admits a canonical foliation: the vertical foliation $V \subset TP$, $V = \text{Ker } \pi_*$ and on the fibers V and on $N = (TP)/V$ the following Euclidean structures. A choice of $GL(n)$ -invariant Riemannian metric on $GL(n)/O(n)$ determines a metric on V . The metric on N is defined tautologically: for every $p \in P$ one has a metric on $T_{\pi(p)}(M)$ which is isomorphic to N_p by π_* .

The computation of the local index formula for diffeomorphism invariant geometry [45] was quite complicated even in the case of codimension 1 foliations: there were innumerable terms to be computed; this could be done by hand, by 3 weeks of eight hours per day tedious computations, but it was of course hopeless to proceed by direct computations in the general case. Henri and I finally found how to get the answer for the general case after discovering that the computation generated a Hopf algebra $\mathcal{H}(n)$ which only depends on $n = \text{codimension of the foliation}$, and which allows to organize the computation provided cyclic cohomology is suitably adapted to Hopf algebras.

Hopf algebras arise very naturally from their actions on noncommutative algebras [47]. Given an algebra A , an action of the Hopf algebra \mathcal{H} on A is given by a linear map,

$$\mathcal{H} \otimes A \rightarrow A, \quad h \otimes a \rightarrow h(a)$$

satisfying $h_1(h_2a) = (h_1h_2)(a)$, $\forall h_i \in \mathcal{H}$, $a \in A$ and

$$h(ab) = \sum h_{(1)}(a)h_{(2)}(b) \quad \forall a, b \in A, h \in \mathcal{H}. \quad (1)$$

where the coproduct of h is,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \quad (2)$$

In concrete examples, the algebra A appears first, together with linear maps $A \rightarrow A$ satisfying a relation of the form (1) which dictates the Hopf algebra structure. This is exactly what occurred in the above example (see [45] for the description of $\mathcal{H}(n)$ and its relation with $\text{Diff}(\mathbb{R}^n)$).

The theory of characteristic classes for actions of \mathcal{H} extends the construction [48] of cyclic cocycles from a Lie algebra of derivations of a C^* algebra A , together with an *invariant trace* τ on A .

This theory was developed in [45] in order to solve the above computational problem for diffeomorphism invariant geometry but it was shown in [49] that the correct framework for the cyclic cohomology of Hopf algebras is that of modular pairs in involution. It is quite satisfactory that exactly the same structure emerged from the analysis of locally compact quantum groups. The resulting cyclic cohomology appears to be the natural candidate for the analogue of Lie algebra cohomology in the context of Hopf algebras. We fix a group-like element σ and a character δ of \mathcal{H} with $\delta(\sigma) = 1$. They will play the role of the module of locally compact groups.

We then introduce the twisted antipode,

$$\tilde{S}(y) = \sum \delta(y_{(1)})S(y_{(2)}), \quad y \in \mathcal{H}, \quad \Delta y = \sum y_{(1)} \otimes y_{(2)}. \quad (3)$$

We associate a cyclic complex (in fact a Λ -module, where Λ is the cyclic category), to any Hopf algebra together with a modular pair in involution. By this we mean a pair (σ, δ) as above, such that the (σ, δ) -twisted antipode is an involution,

$$(\sigma^{-1}\tilde{S})^2 = I. \quad (4)$$

Then $\mathcal{H}_{(\delta, \sigma)}^{\natural} = \{\mathcal{H}^{\otimes n}\}_{n \geq 1}$ equipped with the operators given by the following formulas (5)–(7) defines a module over the cyclic category Λ . By transposing the standard simplicial operators underlying the Hochschild homology complex of an algebra, one associates to \mathcal{H} , viewed only as a coalgebra, the natural cosimplicial module $\{\mathcal{H}^{\otimes n}\}_{n \geq 1}$, with face operators $\delta_i : \mathcal{H}^{\otimes n-1} \rightarrow \mathcal{H}^{\otimes n}$,

$$\begin{aligned} \delta_0(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1} \\ \delta_j(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes \Delta h^j \otimes \dots \otimes h^n, \quad \forall 1 \leq j \leq n-1, \\ \delta_n(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma \end{aligned} \quad (5)$$

and degeneracy operators $\sigma_i : \mathcal{H}^{\otimes n+1} \rightarrow \mathcal{H}^{\otimes n}$,

$$\sigma_i(h^1 \otimes \dots \otimes h^{n+1}) = h^1 \otimes \dots \otimes \varepsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}, \quad 0 \leq i \leq n. \quad (6)$$

The remaining two essential features of a Hopf algebra – *product* and *antipode* – are brought into play, to define the *cyclic operators* $\tau_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$,

$$\tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^n \otimes \sigma. \quad (7)$$

The theory of characteristic classes applies to actions of the Hopf algebra on an algebra endowed with a δ -invariant σ -trace. A linear form τ on A is a σ -trace under the action of \mathcal{H} iff one has,

$$\tau(ab) = \tau(b\sigma(a)) \quad \forall a, b \in A.$$

A σ -trace τ on A is δ -invariant under the action of \mathcal{H} iff

$$\tau(h(a)b) = \tau(a\tilde{S}(h)(b)) \quad \forall a, b \in A, \quad h \in \mathcal{H}.$$

The definition of the cyclic complex $HC_{(\delta, \sigma)}^*(\mathcal{H})$ is uniquely dictated in such a way that the following defines a canonical map from $HC_{(\delta, \sigma)}^*(\mathcal{H})$ to $HC^*(A)$,

$$\begin{aligned} \gamma(h^1 \otimes \dots \otimes h^n) \in C^n(A), \quad \gamma(h^1 \otimes \dots \otimes h^n)(x^0, \dots, x^n) = \\ \tau(x^0 h^1(x^1) \dots h^n(x^n)). \end{aligned}$$

X Hopf algebras, Renormalization and the Riemann-Hilbert problem

I have been for many years fascinated by those topics in theoretical physics which combine mathematical sophistication together with validation by experiments. A prominent example is Quantum Field Theory, not in its abstract formulation but in its computational power, as a mysterious new calculus, known as perturbative renormalization. It is heartening that some hard workers ([50] [51]) continue to dig in the bottom of that mine and actually find gold. I had the luck to meet one of them, Dirk Kreimer, and to join him in trying to unveil the secret beauty of these computations.

Dirk Kreimer showed ([52] [53] [54]) that for any quantum field theory, the combinatorics of Feynman graphs is governed by a Hopf algebra \mathcal{H} whose antipode involves the same algebraic operations as in the Bogoliubov-Parasiuk-Hepp recursion and the Zimmermann forest formula.

His Hopf algebra is commutative as an algebra and we showed in [55] that it is the dual Hopf algebra of the envelopping algebra of a Lie algebra \underline{G} whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of \mathcal{H} .

We also showed that, using dimensional regularization, the bare (unrenormalized) theory gives rise to a loop

$$\gamma(z) \in G, \quad z \in C \quad (1)$$

where C is a small circle of complex dimensions around the integer dimension D of space-time. Our main result ([56] [57]) which relies on all the previous work of Dirk is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of γ .

The Birkhoff decomposition is the factorization

$$\gamma(z) = \gamma_-(z)^{-1} \gamma_+(z) \quad z \in C \quad (2)$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and γ_{\pm} are loops with values in G ,

$$\gamma(z) \in G \quad \forall z \in \mathbb{C}$$

and γ_{\pm} are boundary values of holomorphic maps (still denoted by the same symbol)

$$\gamma_{\pm} : C_{\pm} \rightarrow G. \quad (3)$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (2) is unique (under suitable regularity conditions). It is intimately tied up to the classification of holomorphic G -bundles on the Riemann sphere $P_1(\mathbb{C})$ and for $G = \text{GL}_n(\mathbb{C})$ to the Riemann-Hilbert problem. The Riemann-Hilbert problem comes from Hilbert's 21st problem which he formulated as follows:

“Prove that there always exists a Fuchsian linear differential equation with given singularities and given monodromy.”

In this form it admits a positive answer due to Plemelj and Birkhoff. When formulated in terms of linear systems of the form,

$$y'(z) = A(z) y(z), \quad A(z) = \sum_{\alpha \in S} \frac{A_{\alpha}}{z - \alpha}, \quad (4)$$

where S is the given finite set of singularities, $\infty \notin S$, the A_{α} are complex matrices such that

$$\sum A_{\alpha} = 0 \quad (5)$$

to avoid singularities at ∞ , the answer is not always positive ([58], [59]), but the solution exists when the monodromy matrices M_{α} are sufficiently close to 1. It can then be explicitly written as a series of polylogarithms.

For $G = \text{GL}_n(\mathbb{C})$ the existence of the Birkhoff decomposition (2) is equivalent to the vanishing,

$$c_1(L_j) = 0 \quad (6)$$

of the Chern numbers $n_j = c_1(L_j)$ of the holomorphic line bundles of the Birkhoff-Grothendieck decomposition,

$$E = \oplus L_j \quad (7)$$

where E is the holomorphic vector bundle on $P_1(\mathbb{C})$ associated to γ , i.e. with total space:

$$(C_+ \times \mathbb{C}^n) \cup_{\gamma} (C_- \times \mathbb{C}^n). \quad (8)$$

When G is a simply connected nilpotent complex Lie group the existence (and uniqueness) of the Birkhoff decomposition (2) is valid for any γ . When the loop $\gamma : C \rightarrow G$ extends to a holomorphic loop: $C_+ \rightarrow G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z \in C_+$ the evaluation,

$$\gamma \rightarrow \gamma_+(z) \in G \quad (9)$$

is a natural principle to extract a finite value from the singular expression $\gamma(z)$. This extraction of finite values coincides with the removal of the pole part when G is the additive group \mathbb{C} of complex numbers and the loop γ is meromorphic inside C_+ with z as its only singularity.

As I mentioned earlier our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of the loop given by the unrenormalized theory γ .

We showed that the group G is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms, thanks to the relation that we had uncovered in [55] between the Hopf algebra of rooted trees and the Hopf algebra \mathcal{H} of section 9 involved in the computation of the index formula. The analysis of the relation between these two groups is intimately connected with the renormalization group and anomalous dimensions, this will be the content of our coming paper [60].

XI Spectral Manifolds

Let us now turn to manifolds and explain by giving concrete examples the content of our characterization ([61]) of spectral triples associated to ordinary Riemannian manifolds. It will be crucial that it applies to any Riemannian metric with fixed volume form. What we shall show in particular is that even in that classical case there is a definite advantage in dealing with the slightly noncommutative algebra of matrices of functions. The pair given by the algebra and the Dirac operator is then the solution of a remarkably simple polynomial equation. We shall also give a very natural "quantization" of the volume form of the manifolds which will appear most naturally in our examples, namely the spheres S^n for $n=1, 2$ and 4 .

Let us start with the simplest example, namely, let us show that the geometry of the circle S^1 of length 2π is completely specified by the presentation:

$$U^{-1}[D, U] = 1, \text{ where } UU^* = U^*U = 1. \quad (1)$$

Of course D is as above an unbounded selfadjoint operator. We let \mathcal{A} be the algebra of smooth functions of the single element U . One has $S^1 = \text{Spectrum}(\mathcal{A})$ as one easily

checks using the invariance of the spectrum of U by rotations implied by the above equation. Any element a of \mathcal{A} is of the form $a = f(U)$ and one has

$$[D, a] = U \left(\frac{\partial}{\partial U} f \right) (U) = g(U) \quad (2)$$

and thus,

$$\|[D, a]\| = \sup_x |g(x)|, \quad g = U \frac{\partial}{\partial U} f. \quad (3)$$

This shows that the metric on $S^1 = \text{Spectrum}(\mathcal{A})$ given by (6.3) is the standard Riemannian metric of length 2π . Let us now assume that $ds = D^{-1}$ is an infinitesimal of order 1. It is easy to see that this holds iff the commutant of the algebra generated by U and D is finite dimensional. We then claim that

$$\int f |ds| = n\pi^{-1} \int f(x) \sqrt{g} dx \quad \forall f \in \mathcal{A} \quad (4)$$

where the metric g on S^1 is the above Riemannian metric and where the integer n is the index

$$n = -\text{Index}(PUP), \quad (5)$$

where P is the projector $P = \frac{1+F}{2}$, $F = \text{Sign}(D)$. This formula is simple to prove directly since it is enough to check it for irreducible pairs U, D in which case the spectrum of D is of the form,

$$\text{Spec}(D) = \mathbb{Z} + \lambda \quad (6)$$

for some λ , while U is the shift.

It is important for our later purpose to understand that it is a special case of the general index formula. Indeed both sides of 4 are translation invariant and the equality for $f = 1$ follows from

$$\text{Index}(PUP) = -1/2 \int U^{-1} [D, U] |ds| \quad (7)$$

which follows from the following expression [44] for the n -dimensional Hochschild class of the Chern character of a spectral triple of dimension n ,

$$\tau_n(a^0, \dots, a^n) = \int a^0 [D, a^1] \dots [D, a^n] |D|^{-n} \quad \forall a^j \in \mathcal{A}$$

where we insert a γ in the even case. This formula is weaker than the local index formula of section 8 since it only gives the n -dimensional Hochschild class of the character, but it has the superiority to hold in full generality, with no assumption on the dimension spectrum. It is easy to use it to compute the index pairing with K -theory classes which come from the algebraic K -theory group K_n since the Chern character of such classes is an n -dimensional Hochschild cycle. In the above toy example, U defines an element in $K_1(\mathcal{A})$ and its Chern character is the 1-dimensional Hochschild cycle $U^{-1} \otimes U$ so that 7 follows.

Of course this toy example is a bit too simple, but the above K-theory discussion tells us how to proceed to higher dimension by relying on the formula (13) of section 4 for the Chern character and requiring the vanishing of the lower components.

It is crucial that we do not restrict ourselves to the homogeneous case.

We shall now show that all geometries with fixed total area on the 2-sphere S^2 are indeed described by the following even analogues of equation (1),

$$\left\langle e - \frac{1}{2} \right\rangle = 0, \quad \left\langle \left(e - \frac{1}{2} \right) [D, e] [D, e] \right\rangle = \gamma \quad (8)$$

where, as above, $D = D^*$ is an unbounded selfadjoint operator and $e, e^* = e, e^2 = e$ is a selfadjoint idempotent.

The right hand side of (8) namely γ , is the $\mathbb{Z}/2$ grading of the Hilbert space \mathcal{H} which is a characteristic feature of even dimensions, as we saw above. One has,

$$\gamma^2 = 1, \quad \gamma = \gamma^*, \quad \gamma e = e \gamma, \quad D \gamma = -\gamma D. \quad (9)$$

We still need to explain the symbol $\langle T \rangle$ for any operator T in \mathcal{H} . We fix a subalgebra $M \subset \mathcal{L}(\mathcal{H})$ isomorphic to $M_2(\mathbb{C})$ and let,

$$\langle T \rangle = E_M(T) \quad (10)$$

where E_M is the conditional expectation onto its commutant M' , given for instance as the integral over its unitary group of the conjugates $u T u^*$ of T . We assume that D and γ commute with M ,

$$D \in M', \quad \gamma \in M'. \quad (11)$$

One has the factorization $\mathcal{L}(\mathcal{H}) = M_2(\mathbb{C}) \otimes M'$, and any $T \in \mathcal{L}(\mathcal{H})$ can be uniquely written as,

$$T = \sum \varepsilon_{ij} T^{ij} \quad T^{ij} \in M' \quad (12)$$

where ε_{ij} are the usual matrix units in $M_2(\mathbb{C})$. We can apply (12) to $T = e$ and we let \mathcal{A} be the algebra of operators generated by the components e^{ij} of e . Let us show that \mathcal{A} is abelian and is the algebra of functions on the 2-sphere S^2 .

We let $t = e^{11}, z = e^{12}$ so that

$$e^{22} = 1 - t, \quad e^{21} = z^* \quad (13)$$

using $\langle e - \frac{1}{2} \rangle = 0$ and $e = e^*$. Also $t = t^*$ and $0 \leq t \leq 1$ follow from $e = e^*$ and $e^2 = e$. Thus $e = \begin{bmatrix} t & z \\ z^* & 1-t \end{bmatrix}$ and the equation $e^2 = e$ means that $t^2 + z z^* = t$, $t z + z(1-t) = z$, $z^* t + (1-t) z^* = z^*$, $z^* z + (1-t)^2 = (1-t)$. This shows that $z z^* = z^* z$ and that $t z = z t$ so that \mathcal{A} is abelian.

It also shows that the joint spectrum X of t and z in $\mathbb{C} \times \mathbb{C}$ is a compact subset of

$$\{(t, z) \in [0, 1] \times \mathbb{C}; (t^2 - t) + |z|^2 = 0\} = P_1(\mathbb{C}). \quad (14)$$

Let us now compute the left hand side of (8) using $e = \begin{bmatrix} t & z \\ z^* & (1-t) \end{bmatrix}$ and the notation

$$dx = [D, x]. \quad (15)$$

We just expand the product of matrices,

$$\begin{bmatrix} (t - \frac{1}{2}) & z \\ z^* & (\frac{1}{2} - t) \end{bmatrix} \begin{bmatrix} dt & dz \\ dz^* & -dt \end{bmatrix} \begin{bmatrix} dt & dz \\ dz^* & -dt \end{bmatrix} \quad (16)$$

and take the sum of the diagonal elements. We get the terms,

$$\begin{aligned} & \left(t - \frac{1}{2}\right) (dt dt + dz dz^*) + z (dz^* dt - dt dz^*) \\ & + z^* (dt dz - dz dt) + \left(\frac{1}{2} - t\right) (dz^* dz + dt dt) \\ & = \left(t - \frac{1}{2}\right) (dz dz^* - dz^* dz) + z (dz^* dt - dt dz^*) \\ & + z^* (dt dz - dz dt). \end{aligned}$$

Thus the second equation (8) is equivalent to,

$$\begin{aligned} & \left(t - \frac{1}{2}\right) ([D, z] [D, z^*] - [D, z^*] [D, z]) + \\ & z ([D, z^*] [D, t] - [D, t] [D, z^*]) + \\ & z^* ([D, t] [D, z] - [D, z] [D, t]) = \gamma. \end{aligned} \quad (17)$$

Equivalently we can write it as,

$$\pi(c) = \gamma \quad (18)$$

where c is the Hochschild 2-cycle,

$$c \in Z_2(\mathcal{A}, \mathcal{A}) \quad (19)$$

given by the formula,

$$c = \left(t - \frac{1}{2}\right) \otimes (z \otimes z^* - z^* \otimes z) + z \otimes (z^* \otimes t - t \otimes z^*) + z^* \otimes (t \otimes z - z \otimes t) \quad (20)$$

and where π is the canonical map from Hochschild chains to operators ([44]),

$$\pi(a^0 \otimes a^1 \otimes \cdots \otimes a^n) = a^0 [D, a^1] \cdots [D, a^n] \quad \forall a^j \in \mathcal{A}. \quad (21)$$

We let $v \in C^\infty(S^2, \wedge^2 T^*)$ be the 2-form on $S^2 = P_1(\mathbb{C})$ associated to the Hochschild class of c ([24]). It is given up to normalization by,

$$v = \frac{1}{1-2t} dz \wedge d\bar{z} \quad (22)$$

and vanishes nowhere on S^2 .

We shall now show that *any* Riemannian metric g on S^2 whose associated volume form is equal to v ,

$$\sqrt{g} d^2x = v \quad (23)$$

gives canonically a solution to our equations (8)–(11).

It is very important for our later considerations on gravity that not only the round metric but all possible metrics fulfilling (23) actually appear as solutions.

The solution associated to a given metric g fulfilling (23) is constructed as follows, one lets

$$\mathcal{H} = L^2(S^2, S) \otimes \mathbb{C}^2 \quad (24)$$

be the direct sum of two copies of the space of L^2 spinors on S^2 . The algebra M is just,

$$M = \mathbb{C} \otimes M_2(\mathbb{C}). \quad (25)$$

The operator D is given by,

$$D = \not{\partial} \otimes 1 \quad (26)$$

where $\not{\partial}$ is the Dirac operator (of the metric g). Finally the $\mathbb{Z}/2$ grading is

$$\gamma = \gamma_5 \otimes 1 \quad (27)$$

where γ_5 is the chirality operator on spinors. We identify S^2 with $P_1(\mathbb{C})$ which is the space

$$P_1(\mathbb{C}) = \{x \in M_2(\mathbb{C}), x^2 = x = x^*, \text{ trace } x = 1\} \quad (28)$$

and we let

$$e \in C^\infty(S^2) \otimes M_2(\mathbb{C}) \quad (29)$$

be the corresponding selfadjoint idempotent in \mathcal{H} where $C^\infty(S^2)$ is acting by multiplication operators in $L^2(S^2, S)$.

One has (9)–(11) by construction as well as $\langle e - \frac{1}{2} \rangle = 0$ using (28).

Let us check the second equality of (8), or rather the equivalent form (17). For any $f \in \mathcal{A} = C^\infty(S^2)$ one has,

$$[D, f] = df \otimes 1 \quad (30)$$

where $df = [\not{\partial}, f]$ is the Clifford multiplication by the differential of the function f .

For any $f^0, f^1, f^2 \in \mathcal{A}$ one has,

$$f^0 ([D, f^1] [D, f^2] - [D, f^2] [D, f^1]) = \rho \gamma \quad (31)$$

where ρ is the smooth function such that

$$f^0 df^1 \wedge df^2 = \rho \sqrt{g} d^2x \quad (32)$$

where $\sqrt{g} d^2x$ is the volume form of the metric g . By (23) we have $\sqrt{g} d^2x = v$ and by construction of v as the 2-form associated to the class of C we get from (31), (32) that

$$\pi(c) = \rho \gamma, \quad v = \rho v \quad (33)$$

i.e. $\rho = 1$.

This is enough to check that any Riemannian metric g on S^2 with volume form equal to v does give a solution of equations (8)–(11).

To establish the converse one still needs technical assumptions in order to use the theorem of ([61]), the main additional hypothesis being the order one condition which requires,

$$[[D, e^{ij}], e^{k\ell}] = 0 \quad \forall i, j, k, \ell. \quad (34)$$

Let us show now that the index formula (4) admits a perfect analogue in the general framework of solutions of (8)–(11), assuming the following control of the dimension,

$$ds = D^{-1} \text{ is of order } \frac{1}{2}, \quad (35)$$

i.e. the n^{th} characteristic value $\mu_n(D^{-1})$ is of order of $n^{-1/2}$ as $n \rightarrow \infty$.

One has $e \in M_2(\mathcal{A})$ and the chern character of e in the cyclic homology bicomplex (b, B) is given by its components,

$$\left\langle \left(e - \frac{1}{2} \right) \underbrace{e \otimes \cdots \otimes e}_{2n} \right\rangle = \text{ch}_{2n}(e) \quad (36)$$

where the $\langle \rangle$ means that we take the $M_2(\mathbb{C})$ trace of the corresponding elements.

Let us recall the index formula,

$$\text{Index } D_e^+ = \langle \text{ch}(e), \text{ch}(D) \rangle \quad (37)$$

which computes the index of the compression $e D^+ e$ of $D^+ : \frac{1+\gamma}{2} \mathcal{H} \rightarrow \frac{1-\gamma}{2} \mathcal{H}$, in terms of the pairing between cyclic homology and cyclic cohomology. In general this requires the full knowledge of the chern character $\text{ch}(D)$ in cyclic cohomology.

However in our case (8) shows that $\text{ch}_0(e) = 0$, so that $\text{ch}_2(e)$ is a Hochschild cycle. Moreover by (35) all the higher components of $\text{ch}(D)$ vanish and ([44]) its component of degree 2, $\text{ch}_2(D)$ has a Hochschild class given by,

$$\tau_2(a^0, a^1, a^2) = \int \gamma a^0 [D, a^1] [D, a^2] D^{-2}. \quad (38)$$

The integral \int is a trace and when specializing (38) to $a^j = e$ we can replace the integrand by its average $\langle (e - \frac{1}{2}) [D, e] [D, e] D^{-2} \rangle = \gamma D^{-2}$.

Since $\gamma^2 = 1$ we thus obtain,

$$\int ds^2 = \text{Index } D_e^+. \quad (39)$$

In particular the area, taken in suitable units, is “quantized” by this equation since the index is always an integer.

This simple fact will take more meaning in the 4-dimensional case where the Einstein-Hilbert action will appear.

To close the discussion of this 2-dimensional example we note that the natural algebra here is not \mathcal{A} but rather $M_2(\mathcal{A})$ which admits an amazingly simple presentation. It is generated by $M_2(\mathbb{C})$ and e with the only relations

$$e = e^*, \quad e^2 = e, \quad \left\langle e - \frac{1}{2} \right\rangle = 0 \quad (40)$$

where $\langle \cdot \rangle$ is the conditional expectation on the commutant of the subalgebra $M_2(\mathbb{C})$. Indeed the above computations show that the C^* algebra generated by $M_2(\mathbb{C})$ and e with the relations (40) is,

$$C(S^2, M_2(\mathbb{C})) = C(S^2) \otimes M_2(\mathbb{C}). \quad (41)$$

Let us now move on to the 4-dimensional case.

We first determine the C^* algebra generated by $M_4(\mathbb{C})$ and a projection $e = e^*$ such that $\langle e - \frac{1}{2} \rangle = 0$ as above and whose matrix expression (12) is of the form,

$$[e^{ij}] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \quad (42)$$

where each q_{ij} is a 2×2 matrix of the form,

$$q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}. \quad (43)$$

Since $e = e^*$, both q_{11} and q_{22} are selfadjoint, moreover since $\langle e - \frac{1}{2} \rangle = 0$, we can find $t = t^*$ such that,

$$q_{11} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \quad q_{22} = \begin{bmatrix} (1-t) & 0 \\ 0 & (1-t) \end{bmatrix}. \quad (44)$$

We let $q_{12} = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}$, we then get from $e = e^*$,

$$q_{21} = \begin{bmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{bmatrix}. \quad (45)$$

We thus see that the commutant \mathcal{A} of $M_4(\mathbb{C})$ is generated by t, α, β and we need to find the relations imposed by the equality $e^2 = e$.

In terms of $e = \begin{bmatrix} t & q \\ q^* & 1-t \end{bmatrix}$, the equation $e^2 = e$ means that $t^2 - t + qq^* = 0$, $t^2 - t + q^*q = 0$ and $[t, q] = 0$. This shows that t commutes with α, β, α^* and β^* and since $qq^* = q^*q$ is a diagonal matrix

$$\alpha\alpha^* = \alpha^*\alpha, \quad \alpha\beta = \beta\alpha, \quad \alpha^*\beta = \beta\alpha^*, \quad \beta\beta^* = \beta^*\beta \quad (46)$$

so that the C^* algebra \mathcal{A} is abelian, with the only further relation, (besides $t = t^*$),

$$\alpha\alpha^* + \beta\beta^* + t^2 - t = 0. \quad (47)$$

This is enough to check that,

$$\mathcal{A} = C(S^4) \quad (48)$$

where S^4 appears naturally as quaternionic projective space,

$$S^4 = P_1(\mathbb{H}). \quad (49)$$

The original C^* algebra is thus,

$$B = C(S^4) \otimes M_4(\mathbb{C}). \quad (50)$$

The analogue of (8) is,

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^{2n} \right\rangle = 0, \quad n = 0, 1 \text{ and } = \gamma \text{ for } n = 2. \quad (51)$$

As above we assume,

$$D \in M', \quad \gamma \in M' \quad (52)$$

where $M = M_4(\mathbb{C})$ is the algebra of 4×4 matrices.

We shall first check by a direct computation that the equality $\langle (e - \frac{1}{2}) [D, e]^2 \rangle = 0$ is automatic with our choice of e (42). We use (15) for notational convenience and first compute exactly as in (16), with z replaced by $q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}$. We thus get,

$$\begin{aligned} \langle (e - 1/2) [D, e]^2 \rangle &= \left\langle \left(t - \frac{1}{2} \right) (dq dq^* - dq^* dq) \right. \\ &\quad \left. + q (dq^* dt - dt dq^*) + q^* (dt dq - dq dt) \right\rangle \end{aligned} \quad (53)$$

where the expectation in the right hand side is relative to $M_2(\mathbb{C})$.

The diagonal elements of $\omega = dq dq^*$ are

$$\omega_{11} = d\alpha d\alpha^* + d\beta d\beta^*, \quad \omega_{22} = d\beta^* d\beta + d\alpha^* d\alpha$$

while for $\omega' = dq^* dq$ we get,

$$\omega'_{11} = d\alpha^* d\alpha + d\beta d\beta^*, \quad \omega'_{22} = d\beta^* d\beta + d\alpha d\alpha^*.$$

It follows that, since t is diagonal,

$$\left\langle \left(t - \frac{1}{2} \right) (dq dq^* - dq^* dq) \right\rangle = 0. \quad (54)$$

The diagonal elements of $\rho = q dq^* dt = \rho$ are

$$\rho_{11} = \alpha d\alpha^* dt + \beta d\beta^* dt, \quad \rho_{22} = \beta^* d\beta dt + \alpha^* d\alpha dt$$

while for $\rho' = q^* dq dt$ they are

$$\rho'_{11} = \alpha^* d\alpha dt + \beta d\beta^* dt, \quad \rho'_{22} = \beta^* d\beta dt + \alpha d\alpha^* dt.$$

Similarly for $\sigma = q dt dq^*$ and $\sigma' = q^* dt dq$ one gets the required cancellations so that,

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^2 \right\rangle = 0, \quad (55)$$

holds irrespective of the operator D fulfilling (52).

As in (22) we let v be the natural volume form on S^4 given by,

$$v = \frac{1}{1-2t} d\alpha \wedge d\bar{\alpha} \wedge d\beta \wedge d\bar{\beta}. \quad (56)$$

We shall now show that any Riemannian metric g on S^4 whose associated volume form is v gives a solution to (51), (52), thus,

$$\sqrt{g} d^4x = v. \quad (57)$$

For this we proceed exactly as in (24)–(33) above and we need to check that the Hochschild cycle c obtained in the computation of

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^4 \right\rangle = \pi(c) \quad (58)$$

is totally antisymmetric, i.e. of the form,

$$c = \sum_{i, \sigma} \varepsilon(\sigma) a_0^i \otimes a_{\sigma(1)}^i \otimes \cdots \otimes a_{\sigma(4)}^i \quad (59)$$

where σ ranges through all 24 permutations of $\{1, \dots, 4\}$. With the above notations one has,

$$\left(e - \frac{1}{2} \right) [D, e]^4 = \begin{bmatrix} t - \frac{1}{2} & q \\ q^* & \frac{1}{2} - t \end{bmatrix} \begin{bmatrix} dt & dq \\ dq^* & -dt \end{bmatrix}^4 \quad (60)$$

and the sum of the diagonal elements is,

$$\begin{aligned} & \left(t - \frac{1}{2} \right) ((dt^2 + dq dq^*)^2 + (dt dq - dq dt) (dq^* dt - dt dq^*)) \\ & - \left(t - \frac{1}{2} \right) ((dt^2 + dq^* dq)^2 + (dq^* dt - dt dq^*) (dt dq - dq dt)) \\ & + q ((dq^* dt - dt dq^*) (dt^2 + dq dq^*) + (dq^* dq + dt^2) (dq^* dt - dt dq^*)) \\ & + q^* ((dt^2 + dq dq^*) (dt dq - dq dt) + (dt dq - dq dt) (dq^* dq + dt^2)). \end{aligned}$$

Since t and dt are diagonal 2×2 matrices of operators and the same diagonal terms appear in $dq dq^*$ and $dq^* dq$ as we saw in the proof of (54), the first two lines only contribute by,

$$\left\langle \left(t - \frac{1}{2} \right) (dq dq^* dq dq^* - dq^* dq dq^* dq) \right\rangle. \quad (61)$$

Similarly the two last lines only contribute by,

$$\begin{aligned} & \langle q^* (dt dq dq^* dq - dq dt dq^* dq + dq dq^* dt dq - dq dq^* dq dt) \\ & - q (dt dq^* dq dq^* - dq^* dt dq dq^* + dq^* dq dt dq^* - dq^* dq dq^* dt) \rangle. \end{aligned} \quad (62)$$

The direct computation of (61) then gives

$$\sum \varepsilon(\sigma) \left(t - \frac{1}{2} \right) da_{\sigma(1)}^0 da_{\sigma(2)}^0 da_{\sigma(3)}^0 da_{\sigma(4)}^0 \quad (63)$$

where $a_1^0 = \alpha$, $a_2^0 = \bar{\alpha}$, $a_3^0 = \beta$, $a_4^0 = \bar{\beta}$.

The direct computation of (78) gives

$$\sum_{i,\sigma} \varepsilon(\sigma) a_0^i da_{\sigma(1)}^i da_{\sigma(2)}^i da_{\sigma(3)}^i da_{\sigma(4)}^i \quad (64)$$

where $i \in \{1, 2, 3, 4\}$ and,

$$\begin{aligned} a_0^1 &= \alpha, & a_1^1 &= t, & a_2^1 &= \bar{\alpha}, & a_3^1 &= \bar{\beta}, & a_4^1 &= \beta \\ a_0^2 &= \bar{\alpha}, & a_1^2 &= t, & a_2^2 &= \alpha, & a_3^2 &= \beta, & a_4^2 &= \bar{\beta} \\ a_0^3 &= \beta, & a_1^3 &= t, & a_2^3 &= \bar{\beta}, & a_3^3 &= \bar{\alpha}, & a_4^3 &= \alpha \\ a_0^4 &= \bar{\beta}, & a_1^4 &= t, & a_2^4 &= \beta, & a_3^4 &= \alpha, & a_4^4 &= \bar{\alpha}. \end{aligned}$$

We thus obtain the required formula for the cycle c . When $dx = [D, x]$ with D the Dirac operator associated to a Riemannian metric g on S^4 we get as above, using the Clifford algebra, that

$$\pi(c) = \rho \gamma \quad (65)$$

where ρ is the smooth function such that

$$\omega = \rho \sqrt{g} d^4 x \quad (66)$$

where ω is the differential form associated to c . Now, up to normalization one has,

$$\begin{aligned} \omega &= \left(t - \frac{1}{2} \right) d\alpha \wedge d\bar{\alpha} \wedge d\beta \wedge d\bar{\beta} - \alpha dt \wedge d\bar{\alpha} \wedge d\beta \wedge d\bar{\beta} \\ &\quad + \bar{\alpha} dt \wedge d\alpha \wedge d\beta \wedge d\bar{\beta} - \beta dt \wedge d\bar{\beta} \wedge d\alpha \wedge d\bar{\alpha} \\ &\quad + \bar{\beta} dt \wedge d\beta \wedge d\alpha \wedge d\bar{\alpha}, \end{aligned}$$

which using $t^2 - t + \alpha \bar{\alpha} + \beta \bar{\beta} = 0$ gives up to a factor 2,

$$\omega = \frac{1}{2t-1} d\alpha \wedge d\bar{\alpha} \wedge d\beta \wedge d\bar{\beta}. \quad (67)$$

Thus by hypothesis on g we get $\rho = 1$ and $\pi(c) = \gamma$ which by the above computation means,

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^4 \right\rangle = \gamma. \quad (68)$$

This shows that any Riemannian structure, with the given volume form on $M = S^4$, does give us a solution to our basic equation. Conversely exactly as in the two dimensional case we get, provided that $ds = D^{-1}$ is of order $\frac{1}{4}$,

$$\int ds^4 = -\text{Index } D_e^+. \quad (69)$$

In particular the 4-dimensional volume, taken in suitable units, is “quantized” by this equation since the index is always an integer.

Let $\pi = (e, D, \gamma)$ be a solution of equations (42) (51) (52) above and let us assume (34), together with harmless regularity conditions, [61]. Then there exists a unique Riemannian structure g on M such that the geodesic distance is given by

$$d(x, y) = \text{Sup} \{|a(x) - a(y)| ; a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

The metric $g = g(\pi)$ depends only on the unitary equivalence class of π . The fiber of the map $\{\text{unitary equivalence classes}\} \rightarrow g(\pi)$ is an affine space \mathcal{A} on which the functional $\int ds^2$ is a positive quadratic form with a unique real minimum π_0 which is the representation described above in $L^2(S^4, S)$ given by multiplication operators and the Dirac operator associated to the Levi–Civita connection of the metric g .

The value of $\int ds^2$ on π_0 is the Hilbert–Einstein action of the metric g ,

$$\int ds^2 = -(48\pi^2)^{-1} \int r \sqrt{g} d^4x, .$$

We use the convention that the scalar curvature r is positive for the round sphere S^4 , in particular, the sign of the action $\int ds^2$ is the correct one for the Euclidean formulation of gravity. We refer to [61], [62], [63] for detailed computations.

XII Noncommutative Spectral Manifolds

The main nuance in passing to the noncommutative case is that, since the diagonal in the square no longer corresponds to an algebra homomorphism (the map $x \otimes y \rightarrow xy$ is no longer an algebra homomorphism), the algebra $\mathcal{A} \otimes \mathcal{A}^0$ now plays a central role. The *fundamental class* of a noncommutative space (cf [64]), is a class μ in the KR -homology of the algebra $\mathcal{A} \otimes \mathcal{A}^0$ equipped with the involution

$$\tau(x \otimes y^0) = y^* \otimes (x^*)^0 \quad \forall x, y \in \mathcal{A} \quad (1)$$

where \mathcal{A}^0 denotes the algebra opposite to \mathcal{A} . The KR -homology cycle representing μ is given by a spectral triple, as above, equipped with an anti-linear isometry J on \mathcal{H} which implements the involution τ ,

$$JwJ^{-1} = \tau(w) \quad \forall w \in \mathcal{A} \otimes \mathcal{A}^0, \quad (2)$$

Instead of giving the action of the algebra $\mathcal{A} \otimes \mathcal{A}^0$ in \mathcal{H} one can equivalently give an action of \mathcal{A} satisfying the commutation rule, $[a, b^0] = 0 \quad \forall a, b \in \mathcal{A}$ where

$$b^0 = Jb^*J^{-1} \quad \forall b \in \mathcal{A} \quad (3)$$

KR -homology ([15] [65]) is periodic with period 8 and the dimension modulo 8 is specified by the following commutation rules. One has $J^2 = \varepsilon$, $JD = \varepsilon'DJ$, $J\gamma = \varepsilon''\gamma J$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ and with n the dimension modulo 8,

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

The anti-linear isometry J is given in Riemannian geometry by the charge conjugation operator and in the noncommutative case by the Tomita-Takesaki antilinear conjugation operator [2]. Given an involutive algebra of operators \mathcal{A} on the Hilbert space \mathcal{H} , the Tomita-Takesaki theory associates to all vectors $\xi \in \mathcal{H}$, cyclic for \mathcal{A} and for its commutant \mathcal{A}'

$$\overline{\mathcal{A}\xi} = \mathcal{H}, \quad \overline{\mathcal{A}'\xi} = \mathcal{H} \quad (4)$$

an anti-linear isometric involution $J : \mathcal{H} \rightarrow \mathcal{H}$ obtained from the polar decomposition of the operator

$$S a \xi = a^* \xi \quad \forall a \in \mathcal{A}. \quad (5)$$

It satisfies the following commutation relation:

$$J \mathcal{A}'' J^{-1} = \mathcal{A}'. \quad (6)$$

In particular $[a, b^0] = 0 \quad \forall a, b \in \mathcal{A}$ where

$$b^0 = J b^* J^{-1} \quad \forall b \in \mathcal{A} \quad (7)$$

so \mathcal{H} becomes an \mathcal{A} -bimodule using the representation of the opposite algebra. The class μ specifies only the stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ equipped with the isometry J (and $\mathbb{Z}/2$ -grading γ if n is even). The non-triviality of this homotopy class shows up in the intersection form

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \rightarrow \mathbb{Z}$$

which is obtained from the Fredholm index of D with coefficients in $K_*(\mathcal{A} \otimes \mathcal{A}^0)$. Note that it is defined without using the diagonal map $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, which is not a homomorphism in the noncommutative case. This form is quadratic or symplectic according to the value of n modulo 8.

The Kasparov intersection product [15] allows to formulate the Poincaré duality in terms of the invertibility of μ ,

$$\exists \beta \in KR_n(\mathcal{A}^0 \otimes \mathcal{A}), \quad \beta \otimes_{\mathcal{A}} \mu = \text{id}_{\mathcal{A}^0}, \quad \mu \otimes_{\mathcal{A}^0} \beta = \text{id}_{\mathcal{A}}.$$

It implies the isomorphism $K_*(\mathcal{A}) \xrightarrow{\cap \mu} K^*(\mathcal{A})$.

The condition that D is an operator of order one becomes

$$[[D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A}.$$

(Notice that since a and b^0 commute this condition is equivalent to $[[D, a^0], b] = 0 \quad \forall a, b \in \mathcal{A}$.)

One can show that the von Neumann algebra \mathcal{A}'' generated by \mathcal{A} in \mathcal{H} is automatically finite and hyperfinite and there is a complete list of such algebras up to isomorphism as we saw in section 2. The algebra \mathcal{A} is stable under smooth functional calculus in its norm closure $A = \bar{\mathcal{A}}$ so that $K_j(\mathcal{A}) \simeq K_j(A)$, i.e. $K_j(\mathcal{A})$ depends only on the underlying topology (defined by the C^* algebra A). The integer $\chi = \langle \mu, \beta \rangle \in \mathbb{Z}$ gives the Euler characteristic in the form

$$\chi = \text{Rang } K_0(\mathcal{A}) - \text{Rang } K_1(\mathcal{A})$$

and the general operator theoretic index formula of section 8 gives a local formula for χ .

The group $\text{Aut}^+(\mathcal{A})$ of automorphisms α of the involutive algebra \mathcal{A} , which are implemented by a unitary operator U in \mathcal{H} commuting with J ,

$$\alpha(x) = U x U^{-1} \quad \forall x \in \mathcal{A} ,$$

plays the role of the group $\text{Diff}^+(M)$ of diffeomorphisms preserving the K-homology fundamental class for a manifold M .

In the general noncommutative case, parallel to the normal subgroup $\text{Int } \mathcal{A} \subset \text{Aut } \mathcal{A}$ of inner automorphisms of \mathcal{A} ,

$$\alpha(f) = u f u^* \quad \forall f \in \mathcal{A} \quad (8)$$

where u is a unitary element of \mathcal{A} (i.e. $uu^* = u^*u = 1$), there exists a natural foliation of the space of spectral geometries on \mathcal{A} by equivalence classes of *inner deformations* of a given geometry. To understand how they arise we need to understand how to transfer a given spectral geometry to a Morita equivalent algebra. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and the Morita equivalence [66] between \mathcal{A} and an algebra \mathcal{B} where

$$\mathcal{B} = \text{End}_{\mathcal{A}}(\mathcal{E}) \quad (9)$$

where \mathcal{E} is a finite, projective, hermitian right \mathcal{A} -module, one gets a spectral triple on \mathcal{B} by the choice of a *hermitian connection* on \mathcal{E} . Such a connection ∇ is a linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ satisfying the rules ([44])

$$\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da \quad \forall \xi \in \mathcal{E} , a \in \mathcal{A} \quad (10)$$

$$(\xi, \nabla \eta) - (\nabla \xi, \eta) = d(\xi, \eta) \quad \forall \xi, \eta \in \mathcal{E} \quad (11)$$

where $da = [D, a]$ and where $\Omega_D^1 \subset \mathcal{L}(\mathcal{H})$ is the \mathcal{A} -bimodule of operators of the form

$$A = \sum a_i [D, b_i] , \quad a_i, b_i \in \mathcal{A} . \quad (12)$$

Any algebra \mathcal{A} is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the above construction in the above context one gets the inner deformations of the spectral geometry.

Such a deformation is obtained by the following formula (with suitable signs depending on the dimension mod 8) without modifying neither the representation of \mathcal{A} in \mathcal{H} nor the anti-linear isometry J

$$D \rightarrow D + A + JAJ^{-1} \quad (13)$$

where $A = A^*$ is an arbitrary selfadjoint operator of the form 12. The action of the group $\text{Int}(\mathcal{A})$ on the spectral geometries is simply the following gauge transformation of A

$$\gamma_u(A) = u[D, u^*] + uAu^*. \quad (14)$$

The required unitary equivalence is implemented by the following representation of the unitary group of \mathcal{A} in \mathcal{H} ,

$$u \rightarrow uJuJ^{-1} = u(u^*)^0. \quad (15)$$

The transformation (13) is the identity in the usual Riemannian case. To get a nontrivial example it suffices to consider as we did in section 11, the product of a Riemannian triple by the unique spectral geometry on the finite-dimensional algebra $\mathcal{A}_F = M_N(\mathbb{C})$ of $N \times N$ matrices on \mathbb{C} , $N \geq 2$. One then has $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\text{Int}(\mathcal{A}) = C^\infty(M, PSU(N))$ and inner deformations of the geometry are parameterized by the gauge potentials for the gauge theory of the group $SU(N)$. The space of pure states of the algebra \mathcal{A} , $P(\mathcal{A})$, is the product $P = M \times P_{N-1}(\mathbb{C})$ and the metric on $P(\mathcal{A})$ determined by the formula (6.3) depends on the gauge potential A . It coincide with the Carnot metric [67] on P defined by the horizontal distribution given by the connection associated to A . The group $\text{Aut}(\mathcal{A})$ of automorphisms of \mathcal{A} is the following semi-direct product

$$\text{Aut}(\mathcal{A}) = \mathcal{U} \rtimes \text{Diff}^+(M) \quad (16)$$

of the local gauge transformation group $\text{Int}(\mathcal{A})$ by the group of diffeomorphisms. In dimension $n = 4$, the Hilbert–Einstein action functional for the Riemannian metric and the Yang–Mills action for the vector potential A appear with the correct signs in the asymptotic expansion for large Λ of the number $N(\Lambda)$ of eigenvalues of D which are $\leq \Lambda$ (cf. [68]),

$$N(\Lambda) = \# \text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]. \quad (17)$$

This step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda) \rangle + N_{\text{osc}}(\Lambda).$$

The oscillatory part $N_{\text{osc}}(\Lambda)$ is the same as for a random matrix, governed by the statistics dictated by the symmetries of the system and does not concern us here. The average part $\langle N(\Lambda) \rangle$ is computed by a semiclassical approximation from local expressions involving the familiar heat equation expansion. Other nonzero terms in the asymptotic expansion are cosmological, Weyl gravity and topological terms. As we saw above in our characterization of section 11 we are only dealing with metrics with a fixed volume form so that the bothering cosmological term does not enter in the variational equations associated to the spectral action $\langle N(\Lambda) \rangle$. It is tempting to speculate that the phenomenological Lagrangian of physics, combining matter and gravity appears from the solution of an extremely simple operator theoretic equation along the lines described above. As a starting point for such investigations see [69].

XIII Noncommutative Tori

A more sophisticated example of a spectral manifold is provided by the noncommutative torus \mathbb{T}_θ^2 . The parameter $\theta \in \mathbb{R}/\mathbb{Z}$ defines the following deformation of the algebra of smooth functions on the torus \mathbb{T}^2 , with generators U, V . The relations

$$VU = \exp 2\pi i \theta UV \quad \text{and} \quad UU^* = U^*U = 1, \quad VV^* = V^*V = 1 \quad (1)$$

define the presentation [48] of the involutive algebra $\mathcal{A}_\theta = \{\sum a_{n,m} U^n V^m; a = (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}$ where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences with rapid decay. We shall first describe a completely canonical procedure for constructing the K -cycle $(\mathfrak{K}, D, \gamma)$ over \mathcal{A}_θ from the fundamental class in cyclic cohomology, i.e., the choice of orientation, and the formal positive element

$$G = dU(dU)^* + dV(dV)^* \in \Omega_+^2(\mathcal{A}_\theta), \quad (2)$$

which specifies the metric in the naive classical sense.

This transition from the $g_{\mu\nu}$ to the spectral triple extends in principle to arbitrary formal metrics $G \in \Omega_+^2(\mathcal{A}_\theta)$ but we stick to this specific flat example for simplicity. The construction will be possible thanks to the noncommutative analogue of the Polyakov action of string theory.

We need first to explain briefly how this works in the commutative case. The basic data is the fundamental class in cyclic cohomology, and the formal positive element

$$G = \sum_{\mu, \nu=1}^d g_{\mu\nu} dx^\mu (dx^\nu)^* \in \Omega_+^2(\mathcal{A}), \quad (3)$$

The first key notion is that of positivity in Hochschild cohomology. By definition (cf. [70]) a Hochschild cocycle ψ on a $*$ -algebra \mathcal{A} is *positive* if it has *even* dimension $n = 2m$ and the following equality defines a positive sesquilinear form on the vector space $\mathcal{A}^{\otimes(m+1)}$:

$$\langle a^0 \otimes a^1 \otimes \cdots \otimes a^m, b^0 \otimes b^1 \otimes \cdots \otimes b^m \rangle = \psi(b^{0*} a^0, a^1, \dots, a^m, b^{m*}, \dots, b^{1*}) \quad (4)$$

for any $a^j, b^j \in \mathcal{A}$.

In general the positive Hochschild cocycles form a convex cone

$$Z_+^n(\mathcal{A}, \mathcal{A}^*) \subset Z^n(\mathcal{A}, \mathcal{A}^*) \quad (5)$$

in the vector space Z^n of Hochschild cocycles on \mathcal{A} .

Let M be a 2-dimensional oriented compact manifold, \mathcal{A} be the $*$ -algebra of smooth functions on M and take for the class C the fundamental class, i.e. the class of the de Rham current C

$$\langle C, f^0 df^1 \wedge df^2 \rangle = \frac{-1}{2\pi i} \int_M f^0 df^1 \wedge df^2 \quad \forall f^j \in C^\infty(M). \quad (6)$$

There is a natural correspondence between conformal structures on M and *extreme points* of $Z_+^2 \cap C$. Thus, let g be a conformal structure on M or equivalently, since M is oriented, a *complex* structure. Then, to the Lelong notion of positive current corresponds the positivity in the above sense of the following Hochschild 2-cocycle:

$$\varphi_g(f^0, f^1, f^2) = \frac{i}{\pi} \int_M f^0 \partial f^1 \wedge \bar{\partial} f^2, \quad (7)$$

where ∂ and $\bar{\partial}$ are inherited from the complex structure. The mapping $g \mapsto \varphi_g$ is an injection, since one can read off from φ_g what it means for a function f to be holomorphic in a given small open set $U \subset M$. Each φ_g is an extreme point of the convex set $Z_+^2 \cap C$, and, conversely, the exposed points of this convex set can be determined as follows: for any element of the *dual cone* $(Z_+^2)^\wedge$ of Z_+^2 , of the form

$$G = \sum_{\mu, \nu=1}^d g_{\mu\nu} dx^\mu (dx^\nu)^* \in \Omega^2(\mathcal{A}), \quad (8)$$

where $g_{\mu\nu}$ is a *positive* element of $M_d(\mathcal{A})$, one can show, assuming a suitable condition of nondegeneracy, that the linear form

$$\langle G, \varphi \rangle = \sum \varphi(g_{\mu\nu}, x^\mu, (x^\nu)^*) \quad (9)$$

attains its minimum at a unique point in $Z_+^2 \cap C$, and that this point is equal to φ_g , where g is the conformal structure on M associated with the classical Riemannian metric

$$g = \sum g_{\mu\nu} dx^\mu (dx^\nu)^*. \quad (10)$$

This allows us to understand the complex structures on M as the solutions of a variational problem involving the fundamental class of M and *positivity* in Hochschild cohomology. This problem is by no means restricted in its formulation to the *commutative* case, but it requires the notion of fundamental class in cyclic cohomology. It can be taken as a starting point for developing complex geometry in the noncommutative case.

Let us now show that the previous considerations extend without change to the noncommutative case and treat the noncommutative torus from a metric point of view.

The cyclic cohomology group $HC^0(\mathcal{A}_\theta)$ is 1-dimensional and is generated by the unique trace τ_0 of \mathcal{A}_θ ,

$$\tau_0 \left(\sum a_{n,m} U^n V^m \right) = a_{0,0} \in \mathbb{C}, \quad (11)$$

whereas the cyclic cohomology $HC^2(\mathcal{A}_\theta)$ is two dimensional and besides $S\tau_0 \in HC^2$ (where S is the periodicity operator in cyclic cohomology), is generated by the class of the cyclic 2-cocycle

$$\tau_2(a^0, a^1, a^2) = 2\pi i \sum_{\substack{n_0+n_1+n_2=0 \\ m_0+m_1+m_2=0}} (n_1 m_2 - n_2 m_1) a_{n_0, m_0}^0 a_{n_1, m_1}^1 a_{n_2, m_2}^2. \quad (12)$$

Note that only the *class* of this cocycle matters, not the above specific representative. This nuance is very important since the above class only involves the smooth algebra \mathcal{A}_θ ; we shall now fix the metric.

$$G = dU(dU)^* + dV(dV)^* \in \Omega_+^2(\mathcal{A}_\theta). \quad (13)$$

On the intersection of the cyclic cohomology class $\tau_2 + b(\text{Ker}B)$ with the positive cone Z_+^2 in Hochschild cohomology, the functional G defined by

$$\varphi \in Z^2 \mapsto \langle G, \varphi \rangle = \varphi(1, U, U^*) + \varphi(1, V, V^*) \quad (14)$$

reaches its minimum at a unique point φ_2 given by

$$\varphi_2(a^0, a^1, a^2) = 2\pi \sum_{\substack{n_0+n_1+n_2=0 \\ m_0+m_1+m_2=0}} (n_1 - im_1)(-n_2 - im_2) a_{n_0, m_0}^0 a_{n_1, m_1}^1 a_{n_2, m_2}^2. \quad (15)$$

We then use the noncommutative analogue of a conformal structure, i.e., the positive cocycle φ_2 together with the trace τ_0 , to construct the analogue of the Dirac operator for \mathcal{A}_θ , that is, we shall obtain a $(2, \infty)$ -summable K -cycle (\mathfrak{H}, D) on \mathcal{A}_θ . The Hilbert space \mathfrak{H} is the direct sum $\mathfrak{H} = \mathfrak{H}^+ \oplus \mathfrak{H}^-$ of the Hilbert space $\mathfrak{H}^- = L^2(\mathcal{A}_\theta, \tau_0)$ of the G.N.S. construction of τ_0 , and a Hilbert space \mathfrak{H}^+ of forms of type $(1, 0)$ on the noncommutative torus which is obtained canonically from φ_2 as follows: Let \mathcal{A} be a $*$ -algebra and let $\varphi_2 \in Z_+^2(\mathcal{A}, \mathcal{A}^*)$ be a positive Hochschild 2-cocycle on \mathcal{A} . Let \mathfrak{H}^+ be the Hilbert space completion of $\Omega^1(\mathcal{A})$ equipped with the inner product

$$\langle a^0 da^1, b^0 db^1 \rangle = \varphi_2(b^{0*} a^0, a^1, b^{1*}). \quad (16)$$

Then the actions of \mathcal{A} on \mathfrak{H}^+ by left and right multiplications are unitary. They are automatically bounded if \mathcal{A} is a pre- C^* -algebra.

Thus, \mathfrak{H}^+ is a bimodule over \mathcal{A} and the differential $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$ gives a derivation which, for reasons that will become clear, we shall denote by $\partial : \mathcal{A} \rightarrow \mathfrak{H}^+$.

In our specific example, the computation is straightforward and gives $\mathfrak{H}^+ = L^2(\mathcal{A}_\theta, \tau_0)$ as an \mathcal{A}_θ -bimodule and $\partial : \mathcal{A} \rightarrow \mathfrak{H}^+$ given by $\partial = \frac{1}{\sqrt{2\pi}}(\delta_1 - i\delta_2)$, where δ_1, δ_2 are the standard derivations of \mathcal{A}_θ .

$$\delta_1 = 2\pi i U \frac{\partial}{\partial U}, \delta_2 = 2\pi i V \frac{\partial}{\partial V} \quad (17)$$

so that $\delta_1(\sum b_{nm} U^n V^m) = 2\pi i \sum n b_{nm} U^n V^m$ and similarly for δ_2 . One has of course

$$\delta_1 \delta_2 = \delta_2 \delta_1 \quad (18)$$

and the δ_j are derivations of the algebra \mathcal{A}_θ ,

$$\delta_j(bb') = \delta_j(b)b' + b\delta_j(b') \quad \forall b, b' \in \mathcal{A}_\theta. \quad (19)$$

One can immediately check the following: Let $\mathcal{A} = \mathcal{A}_\theta$ act on the left on both $\mathfrak{H}^- = L^2(\mathcal{A}_\theta, \tau_0)$ and \mathfrak{H}^+ . Then, the operator

$$D = \begin{bmatrix} 0 & \partial \\ \partial^* & 0 \end{bmatrix} \quad (20)$$

in $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ defines a $(2, \infty)$ -summable K -cycle over \mathcal{A}_θ . The $\mathbb{Z}/2$ -grading γ is given by the matrix $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the real structure J is given simply in terms of the Tomita-Takesaki antilinear isometry (cf. [61]).

Translation invariant geometries on \mathbb{T}_θ^2 are parameterized by complex numbers τ with positive imaginary part like in the case of elliptic curves. Up to isometry the geometry depends only on the orbit of τ under the action of $PSL(2, \mathbb{Z})$. However, a new phenomenon appears in the noncommutative case, namely, the *Morita equivalence* which relates the algebras \mathcal{A}_{θ_1} and \mathcal{A}_{θ_2} if θ_1 and θ_2 are in the same orbit of the $PSL(2, \mathbb{Z})$ action on \mathbb{R} [66, 71]. We first need to give a concrete description of the finite projective modules over \mathcal{A}_θ , it is obtained by combining the results of [48] [72] [73]. The finite projective modules are classified up to isomorphism by a pair of integers (p, q) such that $p + q\theta \geq 0$. Let us describe the simplest example of the modules $\mathcal{H}_{p,q}^\theta$. The underlying linear space is the usual Schwartz space,

$$\mathcal{S}(\mathbb{R}) = \{\xi, \xi(s) \in \mathbb{C} \quad \forall s \in \mathbb{R}\} \quad (21)$$

of smooth function on the real line whose all derivatives are of rapid decay. The right module structure is given by the action of the generators U, V

$$(\xi U)(s) = \xi(s + \theta), \quad (\xi V)(s) = e^{2\pi i s} \xi(s) \quad \forall s \in \mathbb{R}. \quad (22)$$

One of course checks the relation (1), and it is a beautiful fact that as a right module over \mathcal{A}_θ the space \mathcal{S} is *finitely generated* and *projective* (i.e. complements to a free module). It follows that it has the correct algebraic attributes to deserve the name of “noncommutative vector bundle” over \mathbb{T}_θ^2 according to the first line of the dictionary of section 4,

$$\text{Space } \mathbb{T}_\theta^2 \qquad \text{Algebra } \mathcal{A}_\theta$$

Vector bundle Finite projective module.

The algebraic counterpart of a vector bundle E on a space X is its space of smooth sections $C^\infty(X, E)$ and one can in particular compute its dimension by computing the trace of the identity endomorphism of E . If one applies this method in the above noncommutative example, one finds

$$\dim_{\mathcal{A}_\theta}(\mathcal{S}) = \theta. \quad (23)$$

The appearance of non integral dimension displays a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when one looks from the measure theoretic point of view (Section 2). The von Neumann algebra which describes the noncommutative torus \mathbb{T}_θ^2 from the measure theoretic point of view is the well known hyperfinite factor R of type II_1 . In particular the classification of finite projective modules over R is given by a positive real number, the Murray and von Neumann *dimension*,

$$\dim_R(\mathcal{E}) \in \mathbb{R}_+. \quad (24)$$

The next point ([48]) is that even though the *dimension* of the above module is irrational, when we compute the analogue of the first Chern class, i.e. of the integral of the curvature of the vector bundle, we obtain an integer. We first need to determine the connections (in the sense of Section 12, equation 10) on the finite projective module \mathcal{S} . It is not hard to see (using 17) that they are characterized by a pair of covariant differentials

$$\nabla_j : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}) \quad (25)$$

such that

$$\nabla_j(\xi b) = (\nabla_j \xi)b + \xi \delta_j(b) \quad \forall \xi \in \mathcal{S}, b \in \mathcal{B}. \quad (26)$$

One checks that, as in the usual case, the trace of the curvature $\Omega = \nabla_1 \nabla_2 - \nabla_2 \nabla_1$, is independent of the choice of the connection. Now the remarkable fact here is that (up to the correct powers of $2\pi i$) the integral curvature of \mathcal{S} is an integer. In fact for the following choice of connection the curvature Ω is constant, equal to $\frac{1}{\theta}$ so that the irrational number θ disappears in the integral curvature, $\theta \times \frac{1}{\theta}$

$$(\nabla_1 \xi)(s) = -\frac{2\pi i s}{\theta} \xi(s) \quad (\nabla_2 \xi)(s) = \xi'(s). \quad (27)$$

Whith this integrality, one could get the wrong impression that the noncommutative torus \mathbb{T}_θ^2 looks very similar to the ordinary 2-torus. A striking difference is obtained by looking at the range of Morse functions. These are of course connected intervals for the 2-torus. For the above noncommutative torus the spectrum of a real valued function such as

$$h = U + U^* + \mu(V + V^*) \quad (28)$$

can be a Cantor set, i.e. have infinitely many disconnected pieces. This shows that the one dimensional shadows of our space \mathbb{T}_θ^2 are considerably different from the commutative case. The above noncommutative torus is the simplest example of noncommutative manifold, it arises naturally not only from foliations but also from the Brillouin zone in the Quantum Hall effect as understood by J. Bellissard, and in M-theory as we shall see in section 14.

We shall now describe the natural moduli space (or more precisely, its covering Teichmüller space) for the noncommutative tori, together with a natural action of $SL(2, \mathbb{Z})$ on this space. The discussion parallels the description of the moduli space of elliptic curves but we shall find that our moduli space is the boundary of the latter space.

We first observe that as the parameter $\theta \in \mathbb{R}/\mathbb{Z}$ varies from 1 to 0 in the above labelling of finite projective modules $\mathcal{H}_{p,q}^\theta$ one gets a monodromy, using the isomorphism $\mathbb{T}_\theta^2 \sim \mathbb{T}_{\theta+1}^2$. The computation shows that this monodromy is given by the transformation $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ i.e., $x \rightarrow x - y, y \rightarrow y$ in terms of the (x, y) coordinates in the K group. This shows that in order to follow the θ -dependence of the K group, we should consider the algebra \mathcal{A} together with a choice of isomorphism,

$$K_0(\mathcal{A}) \stackrel{\rho}{\simeq} \mathbb{Z}^2, \quad \rho(\text{trivial module}) = (1, 0). \quad (29)$$

Exactly as the Jacobian of an elliptic curve appears as a quotient of the $(1, 0)$ part of the cohomology by the lattice of integral classes, we can associate canonically to \mathcal{A} the following data:

- 1) The ordinary two dimensional torus $\mathbb{T} = HC_{\text{even}}(\mathcal{A})/K_0(\mathcal{A})$ quotient of the cyclic homology of \mathcal{A} by the image of K theory under the Chern character map.
- 2) The foliation F (of the above torus) given by the natural filtration of cyclic homology (dual to the filtration of HC^{even}).
- 3) The transversal T to the foliation given by the geodesic joining 0 to the class $[1] \in K_0$ of the trivial bundle.

It turns out that the algebra associated to the foliation F , and the transversal T is isomorphic to \mathcal{A} , and that a purely geometric construction associates to every element $\alpha \in K_0$ its canonical representative from the transversal given by the geodesic joining 0 to α . (Elements of the algebra associated to the transversal T are just matrices $a(i, j)$ where the indices (i, j) are arbitrary pairs of elements i, j of T which belong to the same leaf. The algebraic rules are the same as for ordinary matrices. Elements of the module associated to another transversal T' are rectangular matrices, and the dimension of the module is the transverse measure of T' .)

This gives the correct description of the modules $\mathcal{H}_{p,q}$. The above is in perfect analogy with the isomorphism of an elliptic curve with its Jacobian. The striking difference is that we use the *even* cohomology and K group instead of the odd ones.

It shows that, using the isomorphism ρ , the whole situation is described by a foliation $dx = \theta dy$ of \mathbb{R}^2 where the exact value of θ (not only modulo 1) does matter now.

Now the space of translation invariant foliations of \mathbb{R}^2 is the boundary N of the space M of translation invariant conformal structures on \mathbb{R}^2 , and with $\mathbb{Z}^2 \subset \mathbb{R}^2$ a fixed lattice, they both inherit an action of $SL(2, \mathbb{Z})$. We now describe this action precisely in terms of the pair (\mathcal{A}, ρ) . Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$. Let $\mathcal{E} = \mathcal{H}_{p,q}$ where $(p, q) = \pm(d, -c)$, we define a new algebra \mathcal{A}' as the commutant of \mathcal{A} in \mathcal{E} , i.e. as

$$\mathcal{A}' = \text{End}_{\mathcal{A}}(\mathcal{E}). \quad (30)$$

It turns out (this follows from Morita equivalence) that there is a canonical map μ from $K_0(\mathcal{A}')$ to $K_0(\mathcal{A})$ (obtained as a tensor product over \mathcal{A}') and the isomorphism $\rho' : K_0(\mathcal{A}') \simeq \mathbb{Z}^2$ is obtained by

$$\rho' = g \circ \rho \circ \mu. \quad (31)$$

This gives an action of $SL(2, \mathbb{Z})$ on pairs (\mathcal{A}, ρ) with irrational θ (the new value of θ is $(a\theta + b)/(c\theta + d)$ and for rational values one has to add a point at ∞).

Finally another group $SL(2, \mathbb{Z})$ appears when we discuss the moduli space of flat metrics on \mathbb{T}_θ^2 . Provided we imitate the usual construction of Teichmüller space by fixing an isomorphism,

$$\rho_1 : K_1(\mathcal{A}) \rightarrow \mathbb{Z}^2 \quad (32)$$

of the *odd K* group with \mathbb{Z}^2 , the usual discussion goes through and the results of [61] show that for all values of θ one has a canonical isomorphism of the moduli space with the upper half plane M divided by the usual action of $SL(2, \mathbb{Z})$. Moreover, one shows that the two actions of $SL(2, \mathbb{Z})$ actually commute. The striking fact is that the relation between the two Teichmüller spaces,

$$N = \partial M \tag{33}$$

is preserved by the diagonal action of $SL(2, \mathbb{Z})$. Finally note that the above action of $SL(2, \mathbb{Z})$ on the parameter θ lies beyond the purely formal realm of deformation theory in which θ is treated as a formal deformation parameter. This is a key point in which noncommutative geometry should be distinguished from formal attempts to deform standard geometry.

XIV Noncommutative gauge Theory and String Theory

The analogue of the Yang-Mills action functional and the classification of Yang-Mills connections on the noncommutative tori was developed in [74], with the primary goal of finding a "manifold shadow" for these noncommutative spaces. These moduli spaces turned out indeed to fit this purpose perfectly, allowing for instance to find the usual Riemannian space of gauge equivalence classes of Yang-Mills connections as an invariant of the noncommutative metric. The next surprise came from the natural occurrence (as an unexpected guest) of both the noncommutative tori and the components of the Yang-Mills connections in the classification of the BPS states in M-theory [75]. In the matrix formulation of M-theory the basic equations to obtain periodicity of two of the basic coordinates X_i turns out to be the following variant of equation 1 of section 11,

$$U_i X_j U_i^{-1} = X_j + a \delta_i^j, i = 1, 2 \tag{1}$$

where the U_i are unitary gauge transformations.

The multiplicative commutator $U_1 U_2 U_1^{-1} U_2^{-1}$ is then central and in the irreducible case its scalar value $\lambda = \exp 2\pi i \theta$ brings in the algebra of coordinates on the noncommutative torus. The X_j are then the components of the Yang-Mills connections. It is quite remarkable that the same picture emerged from the other information one has about M-theory concerning its relation with 11 dimensional supergravity and that string theory dualities could be interpreted using Morita equivalence. The latter relates as we saw above in section 13 the values of θ on an orbit of $SL(2, \mathbb{Z})$ and this type of relation would be invisible in a purely deformation theoretic perturbative expansion like the one given by the Moyal product.

In their remarkable paper, Nekrasov and Schwarz [76] showed that Yang-Mills gauge theory on noncommutative \mathbb{R}^4 gives a conceptual understanding of the nonzero B-field desingularization of the moduli space of instantons obtained by perturbing the ADHM equations. In their paper [77], Seiberg and Witten exhibited the unexpected relation between the standard gauge theory and the noncommutative one.

The question of renormalizability of quantum field theories on noncommutative spaces [78] [79] [80] [81] [82] which was the basis of [83] is generating remarkable similarities with string theory [84] which hopefully should yield a better formulation of M -theory than what is currently available. The rate at which progress is occurring in this interplay between noncommutative geometry and physics makes it rather futile to try and foresee what will happen even in the near future but there are a few issues on which I can't help to make brief comments (as a non-expert). The first has to do with locality, the expressions discussed in section 8 which involve the residue applied to multiple products of elements of the algebra and the operator D do generate the natural candidate for local cochains in the general case. This was the basic procedure used in [83] to generate *local* interactions.

Also the transformation from one standard gauge theory to the noncommutative one in [77] has the basic feature of respecting the foliations of gauge potentials by gauge equivalence and since gauge transformations are isospectral deformations of the corresponding Dirac operators (with potential) it is natural to wonder whether the Seiberg-Witten transformation can be interpreted in spectral terms.

String theory is a generalization of ordinary geometry whose onshell formulation is understood via conformal field theory. The corresponding mathematical question of existence of σ -models should benefit from the investigation of the Riemann-Hilbert problem attached to the renormalization of such a theory as in section 10.

Finally, one should probably also look for an offshell formulation of string-geometry. It is well known that the spectral information on a homogeneous Riemannian space can be grasped using Lie group representations but what we showed in section 11 is that even the nonhomogeneous metrics are accessible to such a Hilbert space representation treatment. The new feature is that the basic equations are no longer related to Lie group representations but to algebraic K-theory considerations. It is tempting to speculate that a similar adaptation of the Lie algebra representation theoretic approach to conformal field theory could yield the desired offshell formulation of stringy geometry.

XV References

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