

Transparents du deuxième cours

- Théorie quantique des champs**
- Graphes de Feynman**

Lagrangien → Hamiltonien

$$S = \int L(q, \dot{q}) dt$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$\lambda : T_q C \rightarrow T_q^* C$$

$$\lambda(\dot{q}) = \partial_{\dot{q}} L(q, \dot{q}) \in T_q^* C$$

$$\lambda^{-1}? \quad T_q^* C \rightarrow T_q C$$

$$\lambda(\dot{q}) = p \Leftrightarrow \dot{q} \rightarrow p \dot{q} - L(q, \dot{q})$$

stationnaire

Hamiltonien

$$H(p, q) = p \lambda^{-1}(p) - L(q, \lambda^{-1}(p)), \quad \forall p \in T_q^*C$$

$$L(q, \dot{q}) dt = p dq - H dt$$

$L \longleftrightarrow H$ symétrique

$$\{f, g\} = \left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial q} \right\rangle - \left\langle \frac{\partial f}{\partial q}, \frac{\partial g}{\partial p} \right\rangle$$

$$\frac{d}{dt} f = \{H, f\}$$

$$H(p, q) = \frac{p^2}{2m} + V(q)$$

Lagrangien en QFT

$$S(\phi) = \int \mathcal{L}(\phi) d^D x$$

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{\text{int}}(\phi).$$

$$(\partial\phi)^2 = g^{\mu\nu} \partial_\mu\phi \partial_\nu\phi$$

$$[g^{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Lagrangien libre

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$$

Intégrale de Feynman

Amplitude de probabilité

$$\exp\left(i\frac{S(\phi)}{\hbar}\right)$$

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{O}(\phi) e^{i\frac{S(\phi)}{\hbar}} D[\phi],$$

$$\mathcal{N}^{-1} = \int e^{i\frac{S(\phi)}{\hbar}} D[\phi]$$

Hamiltonien en QFT

$$t \rightarrow q(t) = \phi(t, \bullet) \in C$$

$$L(q, \dot{q}) = \frac{1}{2} \int \dot{\phi}(x)^2 d^{D-1}x - V(\phi)$$

$$V(\phi) = \int \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}(\phi) \right) d^{D-1}x$$

$$T^*C \sim TC, \quad \int a(x) b(x) d^{D-1}x$$

Hamiltonien

$$H(\pi, \phi) =$$

$$\int \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\partial_j \phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}(\phi) \right) d^{D-1}x$$

Quantification canonique

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = 0, \quad [\tilde{\pi}(x), \tilde{\pi}(y)] = 0,$$

$$[\tilde{\pi}(x), \tilde{\phi}(y)] = -i \hbar \delta(x - y)$$

$$\tilde{\phi}(t, x) = e^{itH} \tilde{\phi}(x) e^{-itH}$$

$$\sigma_t(T) = e^{itH} T e^{-itH}$$

Algèbre \mathcal{A} et dynamique $\sigma_t \in \text{Aut}\mathcal{A}$

Etat de “vide”

- ρ est une représentation irréductible
- Il existe un opérateur *positif* non-borné H sur \mathcal{H} tel que

$$\rho(\sigma_t(A)) = e^{itH} \rho(A) e^{-itH}, \quad \forall A \in \mathcal{A}$$

- Il existe un unique vecteur propre $|0\rangle \in \mathcal{H}$, $H|0\rangle = 0$ et $\rho(\mathcal{A})|0\rangle \subset \mathcal{H}$ est dense dans \mathcal{H}

$$\psi(A) = \langle 0 | \rho(A) | 0 \rangle, \quad \forall A \in \mathcal{A}$$

$$F(t) = \psi(A \sigma_t(B))$$

KMS $_{\infty}$ i.e. $T = 0$, Dirac coeff. A et B : $T > 0$.

$$\psi_{\beta}(A) = Z^{-1} \text{Tr}(A e^{-\beta H}), \quad Z = \text{Tr}(e^{-\beta H_b})$$

QFT

- Définition des observables et de la dynamique
- Classification des états de vide.

Trois propriétés

- Causalité
- Positivité de l'énergie
- Unitarité

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = 0, \quad \forall(x, y), \quad (x - y)^2 < 0$$

$$F(t) = \psi(A \sigma_t(B)) \rightarrow F(z) \quad \Im z \geq 0$$

$$\psi(A^* A) \geq 0, \quad \forall A \in \mathcal{A}$$

Exemple très simple

$$X = S^1 \times \mathbb{R}$$

$$\mathcal{L}(\phi) = \frac{1}{2} \left((\partial_0 \phi)^2 - (\partial_1 \phi)^2 - m^2 \phi^2 \right)$$

$$S(\phi) = \int \mathcal{L}(\phi) \, dx \, dt = \int L(t) \, dt$$

$$L(t) = \int_{S^1} \left(\frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 \right) \, dx$$

$$H(\phi, \pi) = \frac{1}{2} \int_{S^1} \left(\pi(x)^2 + (\partial \phi(x))^2 + m^2 \phi^2(x) \right) \, dx$$

$$\phi \rightarrow \int_{S^1} \phi(x) \, \pi(x) \, dx \in \mathbb{R}$$

Fourier

$$H = \sum_{k \in \mathbb{Z}} \frac{1}{2} \left(\pi_k \bar{\pi}_k + (k^2 + m^2) \phi_k \bar{\phi}_k \right)$$

Réalité \Rightarrow

$$\phi_{-k} = \bar{\phi}_k \quad \pi_{-k} = \bar{\pi}_k \quad \forall k \in \mathbb{Z}$$

$$[p, q] = -i \hbar$$

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) = \hbar \omega a^* a \quad \left(+\frac{1}{2}\right)$$

$$[a_k, a_k^*] = 1 \quad \forall k \in \mathbb{Z}$$

$$[a_k, a_\ell] = 0, \quad [a_k, a_\ell^*] = 0 \quad \forall k \neq \ell$$

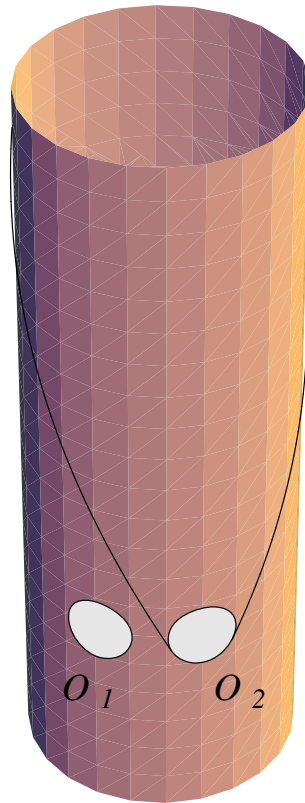
$$H_b = \sum_{k \in \mathbb{Z}} \hbar \omega_k a_k^* a_k$$

$$\omega_k = \sqrt{k^2 + m^2}$$

Champ Quantique

$$\phi(t, x) = e^{it \cdot H_b} \phi(0, x) e^{-it H_b}, \text{ où}$$

$$\phi(0, x) = \sum_{k \in \mathbb{Z}} \left(a_k e^{ikx} + a_k^* e^{-ikx} \right) (2\omega_k)^{-1/2}$$



Causalité

$$\phi(f) = \int f(x) \phi(0, x) dx, f \in C^\infty(S^1)$$

$$[\phi(f), \phi(g)] = 0$$

$$[\sigma_t \phi(f), \phi(g)] = \int c(x, y, t) f(x) g(y) dx dy,$$

$$c(x, y, t) = \sum_k e^{-ik(x-y)} \left(e^{-i\omega_k t} - e^{i\omega_k t} \right) \omega_k^{-1}$$

= $c(x - y, t)$ où c satisfait l'équation de Klein-Gordon

$$\left(\partial_0^2 - \partial_1^2 + m^2 \right) c = 0$$

avec $c(x, 0) = 0$, $\partial/\partial t c(x, 0) = \lambda \delta_0(x)$

Etat de vide

$$\mathcal{H}_b = \bigotimes_{k \in \mathbb{Z}} (\mathcal{H}_k, \Omega_k)$$

$\mathcal{U}(\mathcal{O})$ algèbre de von Neumann dans \mathcal{H}_b
engendrée par les fonctions des $\phi(f)$ avec
Support $f \subset \mathcal{O}$

- Causalité $\mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)'$
- Positivité de l'énergie $H_b \geq 0$
- Unitarité

KMS $_{\beta}$

$$\psi_{\beta}(A) = Z^{-1} \text{Tr}(A e^{-\beta H_b}), \quad Z = \text{Tr}(e^{-\beta H_b})$$

Fonctions de Green

$$G_N(x_1, \dots, x_N) = \langle 0 | T \tilde{\phi}(x_1) \dots \tilde{\phi}(x_N) | 0 \rangle$$

$$\text{Dyson} \quad t_1 \geq t_2 \geq \dots \geq t_N$$

$$G_N(x_1, \dots, x_N) = \mathcal{N} \int e^{i \frac{S(\phi)}{\hbar}} \phi(x_1) \dots \phi(x_N) D[\phi]$$

$$S(\phi) = S_0(\phi) + S_{\text{int}}(\phi)$$

$$d\Lambda = \exp(i S_0(\phi)) D[\phi],$$

$$G_N(x_1, \dots, x_N) =$$

$$\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \phi(x_1) \dots \phi(x_N) S_{\text{int}}(\phi)^n d\Lambda \right)$$

$$\cdot \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int S_{\text{int}}(\phi)^n d\Lambda \right)^{-1}$$

Gellman-Low

ϕ_F champs libres

$$G_N(x_1, \dots, x_N) =$$
$$\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T \phi_F(x_1) \dots \phi_F(x_N) \prod \mathcal{L}_{\text{int}}(y_j) | 0 \rangle \prod dy_j \right)$$
$$\cdot \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T \prod \mathcal{L}_{\text{int}}(y_j) | 0 \rangle \prod dy_j \right)^{-1}$$
$$G_2^F(x, y) = \int \phi(x) \phi(y) \exp(i S_0(\phi)) D[\phi]$$

Intégration sous $\exp -\frac{Q(X)}{2} D[X]$

V espace vectoriel réel, $Q \in V^* \otimes V^*$ forme quadratique non-dégénérée.

$$Q \in \text{Hom}(V, V^*) \rightarrow Q^{-1} \in \text{Hom}(V^*, V)$$

$$\partial_{Q^{-1}(L)} \frac{1}{2} Q = L$$

$$\int P(X) L(X) \exp -\frac{Q(X)}{2} D[X] =$$

$$- \int P(X) \partial_{Q^{-1}(L)} \left(\exp -\frac{Q(X)}{2} \right) D[X] =$$

$$\int \partial_{Q^{-1}(L)} (P(X)) \exp -\frac{Q(X)}{2} D[X]$$

$$Q^{-1} = \text{Propagateur}$$

Intégration sous $\exp(i S_0(\phi)) D[\phi]$

$$S_0(\phi) = (2\pi)^{-D} \int \frac{1}{2}(p^2 - m^2) \hat{\phi}(p) \hat{\phi}(-p) d^D p$$

$$\phi(x) = (2\pi)^{-D} \int \hat{\phi}(p) e^{ip \cdot x} d^D p$$

$$\begin{aligned} \mathcal{N}_0 \int \hat{\phi}(p_1) \hat{\phi}(p_2) \exp(i S_0(\phi)) D[\phi] = \\ i (2\pi)^D \delta(p_1 + p_2) (p_1^2 - m^2)^{-1} \end{aligned}$$

$$G_2^F(x, y) = i (2\pi)^{-D} \int \frac{e^{\pm ip \cdot (x-y)}}{p^2 - m^2} d^D p$$

Lemme

Soient $u > 0$ et $\omega > 0$ alors

$$\text{Lim}_{\epsilon \rightarrow 0+} \int_{\mathbb{R}} \frac{e^{\pm i p u}}{p^2 - \omega^2 + i\epsilon} dp = \frac{\pi}{i\omega} e^{-i u \omega}$$

$$F(\omega) = \int_{\mathbb{R}} \frac{e^{i p u}}{p^2 - \omega^2} dp$$

définit une fonction holomorphe de ω

$\Im(\omega) < 0$. Pour $a > 0$ on a

$$F(-ia) = \frac{\pi}{a} e^{-u a}$$

d'où

$$F(\omega) = \frac{\pi}{i\omega} e^{-i u \omega}, \quad \forall \omega \in \mathbb{C}, \Im(\omega) < 0.$$

On applique à la racine $\sim \omega$ de $\omega^2 - i\epsilon$

Feynman $p^2 - m^2 + i\epsilon$

$$\langle 0 | \phi_F(t, x) \phi_F(s, y) | 0 \rangle =$$

$$\sum_{k \in \mathbb{Z}} (2\omega_k)^{-1} e^{i(k(x-y) - (t-s)\omega_k)}$$

$t - s > 0 + \text{Lemme} \rightarrow e^{-i(t-s)\omega_k} = \dots$

$$\langle 0 | T \phi_F(t, x) \phi_F(s, y) | 0 \rangle =$$

$$\frac{i}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \frac{e^{i(k(x-y) - k_0(t-s))}}{k_0^2 - k^2 - m^2 + i\epsilon} dk_0$$

$$G_2^F(x, y) = i (2\pi)^{-D} \int \frac{e^{\pm ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} d^D p$$

Fonctions de Schwinger

$G_N(x_1, \dots, x_N)$ = valeur au bord de
 $S_N(x_1, \dots, x_N)$ fonctions holomorphes du
temps complexifié z_j de $x_j = (z_j, v_j)$ dans le
cône $\Im z_1 < \Im z_2 < \dots < \Im z_N$

$$S_N(x_1, \dots, x_N) =$$

$$\langle 0 | \tilde{\phi}(v_1) e^{i(z_2 - z_1)H} \dots \tilde{\phi}(v_{N-1}) e^{i(z_N - z_{N-1})H} \tilde{\phi}(v_N) | 0 \rangle$$

$$\langle 0 | \tilde{\phi}(v_1) e^{izH} \tilde{\phi}(v_2) | 0 \rangle, \quad \Im z \geq 0$$

Condition KMS_∞

Rotation de Wick

$$\tau = i t$$

$$\langle \mathcal{O} \rangle_E = \mathcal{N} \int \mathcal{O}(\phi_E) e^{-\frac{S_E(\phi_E)}{\hbar}} D[\phi_E]$$

$$\mathcal{N}^{-1} = \int e^{-\frac{S_E(\phi_E)}{\hbar}} D[\phi_E]$$

$$S_E(\phi_E) = \int \mathcal{L}_E(\phi_E) d^D x$$

$$\mathcal{L}_E(\phi_E) = \frac{1}{2}(\partial\phi_E)^2 + \frac{m^2}{2}\phi_E^2 + \mathcal{L}_{\text{int}}(\phi_E)$$

$$(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$g^{\mu\nu} = \delta^{\mu\nu}$$

Développement perturbatif

$$S_N(x_1, \dots, x_N) =$$
$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \phi_E(x_1) \dots \phi_E(x_N) S_{\text{int}}(\phi_E)^n d\Lambda \right)$$
$$\cdot \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int S_{\text{int}}(\phi_E)^n d\Lambda \right)^{-1}$$

↓

Graphes de Feynman

Graphes de Feynman

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

$$\int \phi_E(x_1) \phi_E(x_2) S_{\text{int}}(\phi_E)^2 d\Lambda$$

$$S_{\text{int}}(\phi_E) = \frac{g}{3!} \int \phi_E^3(x) d^D x = (2\pi)^{-2D} \frac{g}{3!}$$

$$\int \hat{\phi}(k_1) \hat{\phi}(k_2) \hat{\phi}(k_3) \delta(k_1 + k_2 + k_3) \prod d^D k_j$$

$$\phi_E(x_j) = (2\pi)^{-D} \int \hat{\phi}(p) e^{ipx_j} d^D p$$

$$\int \hat{\phi}(p_1) \hat{\phi}(p_2) \hat{\phi}(k_1) \hat{\phi}(k_2) \hat{\phi}(k_3) \hat{\phi}(q_1) \hat{\phi}(q_2) \hat{\phi}(q_3) d\Lambda$$

$$\delta(k_1 + k_2 + k_3) \delta(q_1 + q_2 + q_3) \prod d^D k_j \prod d^D q_j \prod d^D p_j$$

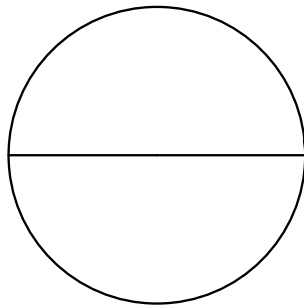
$$7 \times 5 \times 3 = 105$$

Accouplement $l_1 \longleftrightarrow l_2$

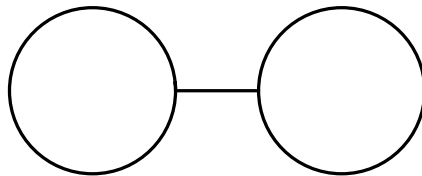


$$(2\pi)^D \delta(l_1 + l_2) \frac{1}{l_1^2 + m^2}$$

$$p_1 \longleftrightarrow p_2$$



$$k_1 \longleftrightarrow q_1, \quad k_2 \longleftrightarrow q_2, \quad k_3 \longleftrightarrow q_3$$

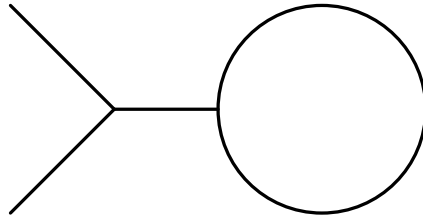


$$k_1 \longleftrightarrow k_2, \quad k_3 \longleftrightarrow q_1, \quad q_2 \longleftrightarrow q_3$$

$$15 = 6 + 9$$

Tadpole

$$p_1 \longleftrightarrow k_1, \quad p_2 \longleftrightarrow k_2$$



$$\Rightarrow k_3 \longleftrightarrow q_1, \quad q_2 \longleftrightarrow q_3$$

$$\delta(q_2 + q_3) \delta(q_1 + q_2 + q_3) \Rightarrow q_1 = 0$$

$$\Rightarrow k_3 = 0 \rightarrow \delta(p_1 + p_2)$$

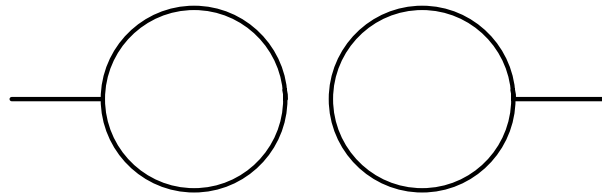
Graphe connexe avec une patte externe =
Tadpole

$$\langle 0 | \phi(x) | 0 \rangle = 0$$

$$6 \times 3 \times 2 = 36$$

Deux tadpoles

$$p_1 \longleftrightarrow k_1, \quad p_2 \longleftrightarrow q_1, \quad k_2 \longleftrightarrow k_3, \quad q_2 \longleftrightarrow q_3$$



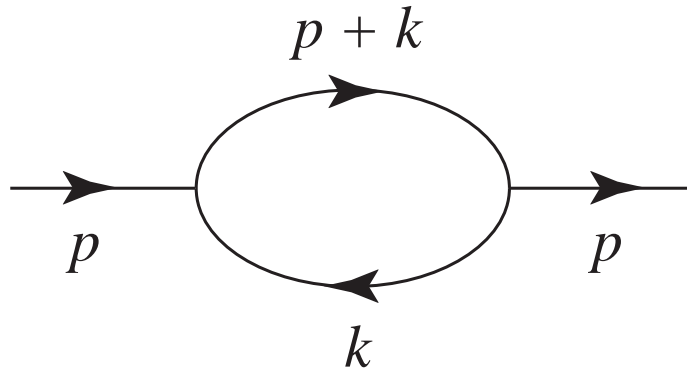
$$\delta(k_2 + k_3) \delta(k_1 + k_2 + k_3) \Rightarrow k_1 = 0$$

$$\Rightarrow p_1 = 0$$

$$9 \times 2 = 18$$

Self-Energie

$$p_1 \longleftrightarrow k_1, \quad p_2 \longleftrightarrow q_1, \quad k_2 \longleftrightarrow q_2, \quad k_3 \longleftrightarrow q_3$$



$$\frac{g^2}{2} \delta(p_1 + p_2) \frac{1}{p_1^2 + m^2} \frac{1}{p_2^2 + m^2}$$

$$\int \frac{1}{k^2 + m^2} \frac{1}{((p_1 + k)^2 + m^2)} d^D k$$

$$9 \times 2 \times 2 = 36$$

Facteurs $(2\pi)^{ND}$, $d^D k \rightarrow (2\pi)^{-D}$, $\delta(\bullet) \rightarrow (2\pi)^D$

Renormalisation de la Masse

$$\int e^{-\lambda k^2} d^D k = \lambda^{-D/2} \pi^{D/2}$$

↓

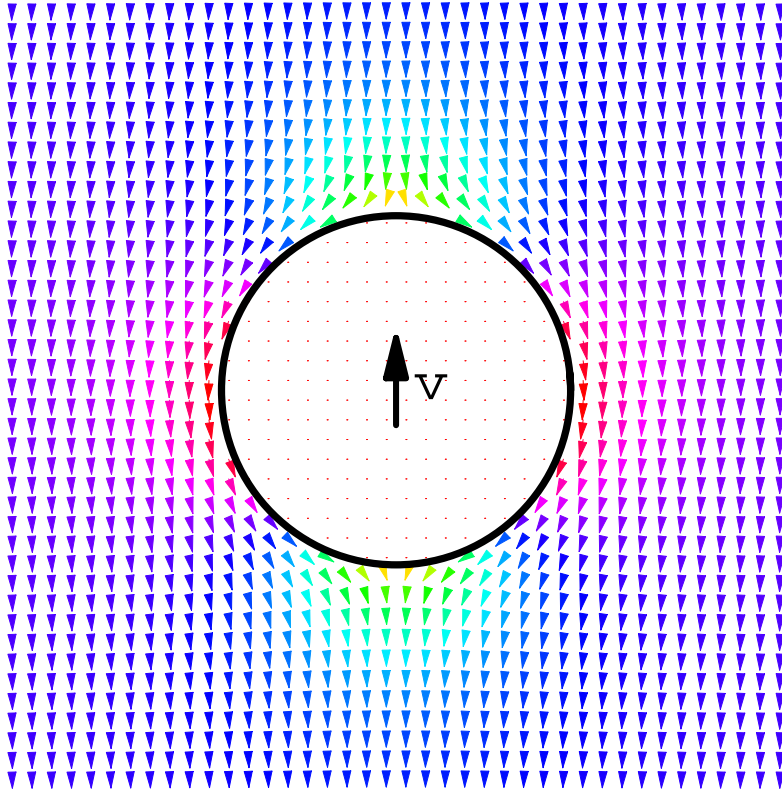
$$\int \frac{1}{k^2 + m^2} \frac{1}{((p + k)^2 + m^2)} d^D k =$$

$$\pi^{D/2} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 (x(1-x)p^2 + m^2)^{D/2-2} dx$$

Hydrodynamique Green 1830

$$F = m a$$

$$m \rightarrow m + \delta m$$



$$X = \text{Grad } h, \quad \Delta h = 0$$

$$h(x, y, z) = \frac{v}{2} (r^{-3} + 2) z, \quad r^2 = x^2 + y^2 + z^2$$

$$E(x, y, z) = \frac{v^2}{8} \frac{x^2 + y^2 + 4z^2}{(x^2 + y^2 + z^2)^4} \rho \, dx \, dy \, dz$$

$$\int E = \frac{1}{2} \delta m v^2, \quad \delta m = \frac{1}{2} M$$

J. Collins, *Renormalization*, Cambridge Monographs in Math. Physics, Cambridge University Press, 1984.

P. Dirac, *The quantum theory of the emission and absorption of radiation*. Proc. London Royal Soc. 114 (1927) 243–265.

R. Feynman, *The reason for antiparticles*, Elementary Particles and the Laws of Physics, Cambridge Univ. Press (1987).