Transparents du deuxième cours

– **Théorie quantique des champs**

– **Graphes de Feynman**

Lagrangien $\rightarrow$ Hamiltonien

$$S = \int L(q, \dot{q})\, dt$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0$$

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$\lambda \ : \ T_q C \rightarrow T_q^* C$$

$$\lambda(\dot{q}) = \partial_{\dot{q}} L(q, \dot{q}) \in T_q^* C$$

$$\lambda^{-1}? \quad T_q^* C \rightarrow T_q C$$

$$\lambda(\dot{q}) = p \Leftrightarrow \dot{q} \rightarrow p\, \dot{q} - L(q, \dot{q})$$

stationnaire

Hamiltonien

$$H(p,q) = p\,\lambda^{-1}(p) - L(q, \lambda^{-1}(p))\,, \qquad \forall p \in T_q^* C$$

$$L(q,\dot{q})\,dt = p\,dq - H\,dt$$

$L \longleftrightarrow H$ symétrique

$$\{f,g\} = \langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial q} \rangle - \langle \frac{\partial f}{\partial q}, \frac{\partial g}{\partial p} \rangle$$

$$\frac{d}{dt} f = \{H, f\}$$

$$H(p,q) = \frac{p^2}{2m} + V(q)$$

Lagrangien en QFT

$$S(\phi) = \int \mathcal{L}(\phi)\, d^D x$$

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{\text{int}}(\phi).$$

$$(\partial\phi)^2 = g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi$$

$$[g^{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Lagrangien libre

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$$

Intégrale de Feynman

Amplitude de probabilité

$$\exp\left(i\,\frac{S(\phi)}{\hbar}\right)$$

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{O}(\phi)\, e^{i\,\frac{S(\phi)}{\hbar}}\, D[\phi],$$

$$\mathcal{N}^{-1} = \int e^{i\,\frac{S(\phi)}{\hbar}}\, D[\phi]$$

Hamiltonien en QFT

$$t \to q(t) = \phi(t, \bullet) \in C$$

$$L(q, \dot{q}) = \frac{1}{2} \int \dot{\phi}(x)^2 \, d^{D-1}x - V(\phi)$$

$$V(\phi) = \int (\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}(\phi)) \, d^{D-1}x$$

$$T^*C \sim TC \,, \quad \int a(x) \, b(x) \, d^{D-1}x$$

Hamiltonien

$$H(\pi, \phi) =$$

$$\int (\frac{1}{2} \pi^2 + \frac{1}{2} (\partial_j \phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}(\phi)) \, d^{D-1}x$$

Quantification canonique

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = 0\,, \quad [\tilde{\pi}(x), \tilde{\pi}(y)] = 0\,,$$

$$[\tilde{\pi}(x), \tilde{\phi}(y)] = -\,i\,\hbar\,\delta(x-y)$$

$$\tilde{\phi}(t,x) = e^{i\,t\,H}\,\tilde{\phi}(x)\,e^{-i\,t\,H}$$

$$\sigma_t(T) = e^{i\,t\,H}\,T\,e^{-i\,t\,H}$$

Algèbre $\mathcal{A}$ et dynamique $\sigma_t \in \mathrm{Aut}\mathcal{A}$

Etat de "vide"

- ρ est une représentation irréductible
- Il existe un opérateur *positif* non-borné H sur $\mathcal{H}$ tel que

$$\rho(\sigma_t(A)) = e^{i\,t\,H}\,\rho(A)\,e^{-i\,t\,H}\,, \quad \forall A \in \mathcal{A}$$

- Il existe un unique vecteur propre $|\,0\,\rangle \in \mathcal{H}$, $H|\,0\,\rangle = 0$ et $\rho(\mathcal{A})|\,0\,\rangle \subset \mathcal{H}$ est dense dans $\mathcal{H}$

$$\psi(A) = \langle\,0|\,\rho(A)|\,0\,\rangle\,, \quad \forall A \in \mathcal{A}$$

$$F(t) = \psi(A\,\sigma_t(B))$$

KMS_∞ *i.e.* $T = 0$, Dirac coeff. A et B : $T > 0$.

$$\psi_\beta(A) = Z^{-1}\,\mathrm{Tr}(A\,e^{-\beta\,H})\,, \quad Z = \mathrm{Tr}(e^{-\beta\,H_b})$$

QFT

- Définition des observables et de la dynamique
- Classification des états de vide.

Trois propriétés

- Causalité
- Positivité de l'énergie
- Unitarité

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = 0\,, \quad \forall (x,y)\,, \quad (x-y)^2 < 0$$

$$F(t) = \psi(A\,\sigma_t(B)) \to F(z) \quad \Im z \geq 0$$

$$\psi(A^* A) \geq 0\,, \quad \forall A \in \mathcal{A}$$

Exemple très simple

$$X = S^1 \times \mathbb{R}$$

$$\mathcal{L}(\phi) = \frac{1}{2}\Big((\partial_0\phi)^2 - (\partial_1\phi)^2 - m^2\ \phi^2\Big)$$

$$S(\phi) = \int \mathcal{L}(\phi)\ dx\ dt = \int L(t)\ dt$$

$$L(t) = \int_{S^1} \left(\frac{1}{2}\ \left(\dot{\phi}\right)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2\right)\ dx$$

$$H(\phi,\pi) = \frac{1}{2}\int_{S^1}\Big(\pi(x)^2 + (\partial\ \phi(x))^2 + m^2\ \phi^2(x)\Big)\ dx$$

$$\phi \to \int_{S^1} \phi(x)\ \pi(x)\ dx \in \mathbb{R}$$

Fourier

$$H = \sum_{k\in\mathbb{Z}} \frac{1}{2}\Big(\pi_k \ \overline{\pi}_k + (k^2+m^2)\ \phi_k \ \overline{\phi}_k\Big)$$

Réalité $\Rightarrow$

$$\phi_{-k} = \overline{\phi}_k \quad \pi_{-k} = \overline{\pi}_k \quad \forall k \in \mathbb{Z}$$

$$[p,q] = -i\,\hbar$$

$$H = \frac{1}{2}(p^2+\omega^2 q^2) = \hbar\,\omega\, a^*\, a \quad (+\frac{1}{2})$$

$$[a_k, a_k^*] = 1 \qquad \forall k \in \mathbb{Z}$$

$$[a_k, a_\ell] = 0\ ,\ [a_k, a_\ell^*] = 0 \qquad \forall k \neq \ell$$

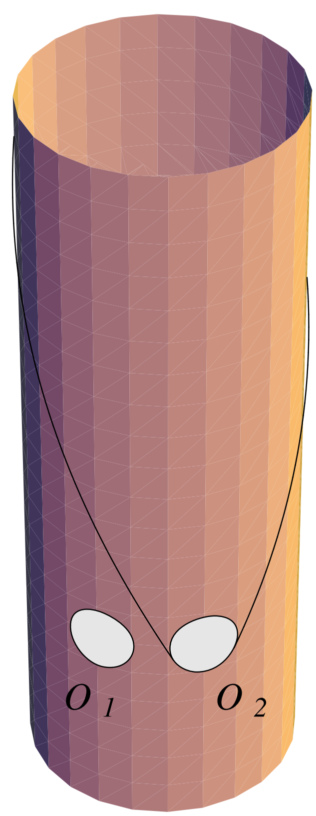
$$H_b = \sum_{k\in\mathbb{Z}} \hbar\ \omega_k\ a_k^*\ a_k$$

$$\omega_k = \sqrt{k^2+m^2}$$

Champ Quantique

$$\phi(t,x) = e^{it\cdot H_b}\ \phi(0,x)\ e^{-itH_b}, \text{ où}$$

$$\phi(0,x) = \sum_{k\in\mathbb{Z}} \left(a_k\ e^{ikx} + a_k^*\ e^{-ikx}\right)(2\omega_k)^{-1/2}$$



Causalité

$$\phi(f) = \int f(x)\ \phi(0,x)\ dx,\ f \in C^\infty(S^1)$$

$$[\phi(f), \phi(g)] = 0$$

$$[\sigma_t\ \phi(f), \phi(g)] = \int c(x,y,t)\ f(x)\ g(y)\ dx\ dy,$$

$$c(x,y,t) = \sum_k e^{-ik(x-y)} \left(e^{-i\omega_k t} - e^{i\omega_k t}\right)\ \omega_k^{-1}$$

$= c(x-y,t)$ où c satisfait l'équation de Klein-Gordon

$$\left(\partial_0^2 - \partial_1^2 + m^2\right) c = 0$$

avec $c(x,0) = 0$, $\partial/\partial t\ c(x,0) = \lambda\, \delta_0(x)$

Etat de vide

$$\mathcal{H}_b = \ \otimes_{k\in\mathbb{Z}} \ (\mathcal{H}_k, \Omega_k)$$

$\mathcal{U}(\mathcal{O})$ algèbre de von Neumann dans $\mathcal{H}_b$ engendrée par les fonctions des $\phi(f)$ avec Support $f \subset \mathcal{O}$

- Causalité $\quad \mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)'$
- Positivité de l'énergie $\quad H_b \geq 0$
- Unitarité

KMS$_\beta$

$$\psi_\beta(A) = \ Z^{-1} \, \mathrm{Tr}(A \, e^{-\beta \, H_b}) \, , \quad Z = \mathrm{Tr}(e^{-\beta \, H_b})$$

Fonctions de Green

$$G_N(x_1,\ldots,x_N) = \langle\, 0\,|T\,\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_N)|\,0\,\rangle$$

$$\text{Dyson} \qquad t_1 \geq t_2 \geq \ldots \geq t_N$$

$$G_N(x_1,\ldots,x_N) = \mathcal{N}\int e^{i\,\frac{S(\phi)}{\hbar}}\,\phi(x_1)\ldots\phi(x_N)\,D[\phi]$$

$$S(\phi) = S_0(\phi) + S_{\text{int}}(\phi)$$

$$d\Lambda = \exp\left(i\,S_0(\phi)\right)D[\phi],$$

$$G_N(x_1,\ldots,x_N) =$$

$$\left(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int\phi(x_1)\ldots\phi(x_N)\,S_{\text{int}}(\phi)^n\,d\Lambda\right)$$

$$\cdot\left(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int S_{\text{int}}(\phi)^n\,d\Lambda\right)^{-1}$$

Gellman-Low

ϕ_F champs libres

$$G_N(x_1, \ldots, x_N) =$$

$$\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle\, 0 \,| T \phi_F(x_1) \ldots \phi_F(x_N) \, \prod \mathcal{L}_{\mathrm{int}}(y_j) \,|\, 0 \,\rangle \prod dy_j \right)$$

$$\cdot \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle\, 0 \,| T \, \prod \mathcal{L}_{\mathrm{int}}(y_j) \,|\, 0 \,\rangle \prod dy_j \right)^{-1}$$

$$G_2^F(x, y) = \int \phi(x)\phi(y) \, \exp\left(i\, S_0(\phi)\right) D[\phi]$$

Intégration sous exp $-\frac{Q(X)}{2}\,D[X]$

V espace vectoriel réel, $Q \in V^* \otimes V^*$ forme quadratique non-dégénérée.

$$Q \in \mathrm{Hom}(V, V^*) \to Q^{-1} \in \mathrm{Hom}(V^*, V)$$

$$\partial_{Q^{-1}(L)}\,\frac{1}{2}\,Q = L$$

$$\int P(X)\,L(X)\ \exp -\frac{Q(X)}{2}\,D[X] =$$

$$-\int P(X)\,\partial_{Q^{-1}(L)}(\exp -\frac{Q(X)}{2})\,D[X] =$$

$$\int \partial_{Q^{-1}(L)}(P(X))\ \exp -\frac{Q(X)}{2}\,D[X]$$

$$Q^{-1} = \text{Propagateur}$$

Intégration sous $\exp(i\, S_0(\phi))\, D[\phi]$

$$S_0(\phi) = (2\pi)^{-D} \int \frac{1}{2}(p^2 - m^2)\, \widehat{\phi}(p)\widehat{\phi}(-p)\, d^D p$$

$$\phi(x) = (2\pi)^{-D} \int \widehat{\phi}(p)\, e^{ip.x}\, d^D p$$

$$\mathcal{N}_0 \int \widehat{\phi}(p_1)\, \widehat{\phi}(p_2)\, \exp(i\, S_0(\phi))\, D[\phi] =$$

$$i\,(2\pi)^D\, \delta(p_1 + p_2)\, (p_1^2 - m^2)^{-1}$$

$$G_2^F(x,y) = i\,(2\pi)^{-D} \int \frac{e^{\pm ip.(x-y)}}{p^2 - m^2}\, d^D p$$

Lemme

Soient $u > 0$ et $\omega > 0$ alors

$$\mathrm{Lim}_{\epsilon \to 0+} \int_{\mathbb{R}} \frac{e^{\pm\, ip\, u}}{p^2 - \omega^2 + i\epsilon}\, dp = \frac{\pi}{i\,\omega}\, e^{-i\,u\,\omega}$$

$$F(\omega) = \int_{\mathbb{R}} \frac{e^{ip\,u}}{p^2 - \omega^2}\, dp$$

définit une fonction holomorphe de ω $\Im(\omega) < 0$. Pour $a > 0$ on a

$$F(-ia) = \frac{\pi}{a}\, e^{-\,u\,a}$$

d'où

$$F(\omega) = \frac{\pi}{i\,\omega}\, e^{-i\,u\,\omega}, \quad \forall \omega \in \mathbb{C}\,,\ \Im(\omega) < 0.$$

On applique à la racine $\sim \omega$ de $\omega^2 - i\epsilon$

Feynman $p^2 - m^2 + i\epsilon$

$$\langle 0 |\, \phi_F(t,x)\phi_F(s,y)\, | 0 \rangle =$$

$$\sum_{k\in\mathbb{Z}} (2\omega_k)^{-1}\, e^{i(k(x-y)-(t-s)\omega_k)}$$

$$t - s > 0 \text{ + Lemme} \rightarrow e^{-i(t-s)\omega_k} = ...$$

$$\langle 0 |\, T\, \phi_F(t,x)\phi_F(s,y)\, | 0 \rangle =$$

$$\frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{i(k(x-y)-k_0(t-s))}}{k_0^2 - k^2 - m^2 + i\epsilon}\, dk_0$$

$$G_2^F(x,y) = i\,(2\pi)^{-D} \int \frac{e^{\pm ip.(x-y)}}{p^2 - m^2 + i\epsilon}\, d^D p$$

Fonctions de Schwinger

$G_N(x_1, \ldots, x_N) =$ valeur au bord de $S_N(x_1, \ldots, x_N)$ fonctions holomorphes du temps complexifié z_j de $x_j = (z_j, v_j)$ dans le cône $\Im z_1 < \Im z_2 < \ldots < \Im z_N$

$$S_N(x_1, \ldots, x_N) =$$

$$\langle 0 \,|\, \tilde{\phi}(v_1)\, e^{i\,(z_2 - z_1)\,H} \;\ldots\; \tilde{\phi}(v_{N-1})\, e^{i\,(z_N - z_{N-1})\,H}\, \tilde{\phi}(v_N) |\, 0 \,\rangle$$

$$\langle 0 \,|\, \tilde{\phi}(v_1)\, e^{i\,z\,H}\, \tilde{\phi}(v_2) |\, 0 \,\rangle\,, \quad \Im z \geq 0$$

Condition KMS_∞

Rotation de Wick

$$\tau = i\,t$$

$$\langle \mathcal{O} \rangle_E = \mathcal{N} \int \mathcal{O}(\phi_E)\, e^{-\frac{S_E(\phi_E)}{\hbar}}\, D[\phi_E]$$

$$\mathcal{N}^{-1} = \int \; e^{-\frac{S_E(\phi_E)}{\hbar}}\, D[\phi_E]$$

$$S_E\,(\phi_E) = \int \mathcal{L}_E\,(\phi_E)\, d^D x$$

$$\mathcal{L}_E\,(\phi_E) = \frac{1}{2}(\partial\phi_E)^2 + \frac{m^2}{2}\,\phi_E^2 + \mathcal{L}_{\text{int}}(\phi_E)$$

$$(\partial\phi)^2 = \; g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$g^{\mu\nu} = \delta^{\mu\nu}$$

Développement perturbatif

$$S_N(x_1, \ldots, x_N) =$$

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \phi_E(x_1) \ldots \phi_E(x_N) \, S_{\mathrm{int}}(\phi_E)^n \, d\Lambda \right)$$

$$\cdot \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int S_{\mathrm{int}}(\phi_E)^n \, d\Lambda \right)^{-1}$$

$$\downarrow$$

Graphes de Feynman

Graphes de Feynman

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

$$\int \phi_E(x_1)\,\phi_E(x_2)\,S_{\text{int}}(\phi_E)^2\,d\Lambda$$

$$S_{\text{int}}(\phi_E) = \frac{g}{3!}\int \phi_E^3(x)\,d^Dx = (2\pi)^{-2D}\,\frac{g}{3!}$$

$$\int \widehat{\phi}(k_1)\,\widehat{\phi}(k_2)\,\widehat{\phi}(k_3)\,\delta(k_1+k_2+k_3)\prod d^Dk_j$$

$$\phi_E(x_j) = (2\pi)^{-D}\int \widehat{\phi}(p)\,e^{ipx_j}\,d^Dp$$

$$\int \widehat{\phi}(p_1)\,\widehat{\phi}(p_2)\,\widehat{\phi}(k_1)\,\widehat{\phi}(k_2)\,\widehat{\phi}(k_3)\,\widehat{\phi}(q_1)\,\widehat{\phi}(q_2)\,\widehat{\phi}(q_3)\,d\Lambda$$

$$\delta(k_1+k_2+k_3)\,\delta(q_1+q_2+q_3)\prod d^Dk_j\prod d^Dq_j\prod d^Dp_j$$

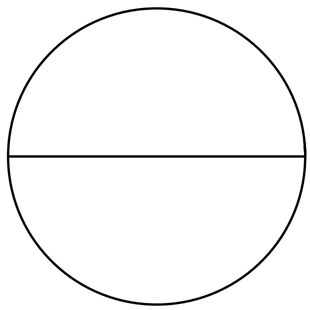
$$7\times 5\times 3 = 105$$

Accouplement $\ell_1 \longleftrightarrow \ell_2$

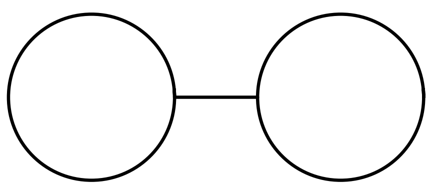
$$\downarrow$$

$$(2\pi)^D \, \delta(\ell_1 + \ell_2) \, \frac{1}{\ell_1^2 + m^2}$$

$$p_1 \longleftrightarrow p_2$$



$$k_1 \longleftrightarrow q_1 \,, \quad k_2 \longleftrightarrow q_2 \,, \quad k_3 \longleftrightarrow q_3$$

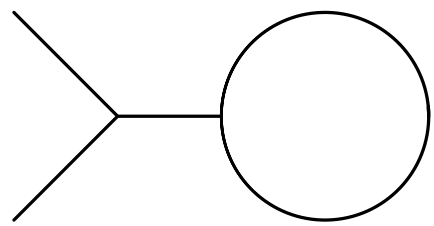


$$k_1 \longleftrightarrow k_2 \,, \quad k_3 \longleftrightarrow q_1 \,, \quad q_2 \longleftrightarrow q_3$$

$$15 = 6 + 9$$

Tadpole

$$p_1 \longleftrightarrow k_1\,, \quad p_2 \longleftrightarrow k_2$$



$$\Rightarrow \quad k_3 \longleftrightarrow q_1\,, \quad q_2 \longleftrightarrow q_3$$

$$\delta(q_2 + q_3)\,\delta(q_1 + q_2 + q_3) \Rightarrow \quad q_1 = 0$$

$$\Rightarrow \quad k_3 = 0 \to \delta(p_1 + p_2)$$

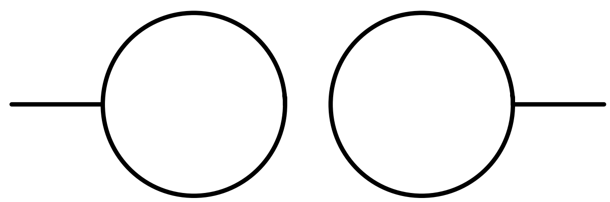
Graphe connexe avec une patte externe = Tadpole

$$\langle\, 0 \,|\, \phi(x) \,|\, 0 \,\rangle = 0$$

$$6 \times 3 \times 2 = 36$$

Deux tadpoles

$$p_1 \longleftrightarrow k_1\,, \quad p_2 \longleftrightarrow q_1\,, \quad k_2 \longleftrightarrow k_3\,, \quad q_2 \longleftrightarrow q_3$$



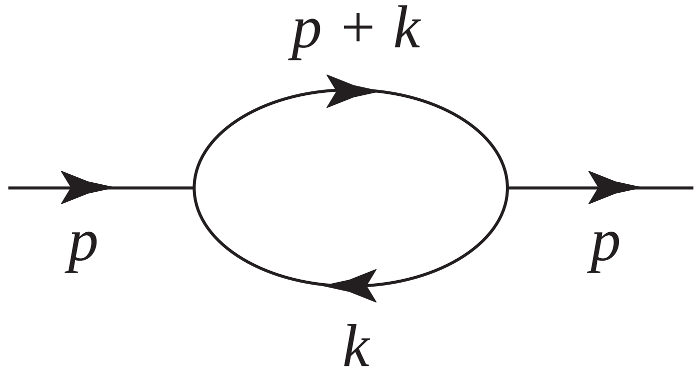
$$\delta(k_2+k_3)\,\delta(k_1+k_2+k_3) \Rightarrow \quad k_1 = 0$$

$$\Rightarrow \quad p_1 = 0$$

$$9 \times 2 = 18$$

Self-Energie

$$p_1 \longleftrightarrow k_1\,, \quad p_2 \longleftrightarrow q_1\,, \quad k_2 \longleftrightarrow q_2\,, \quad k_3 \longleftrightarrow q_3$$



$$\frac{g^2}{2}\,\delta(p_1 + p_2)\,\frac{1}{p_1^2 + m^2}\,\frac{1}{p_2^2 + m^2}$$

$$\int \frac{1}{k^2 + m^2}\,\frac{1}{((p_1 + k)^2 + m^2)}\,d^D k$$

$$9 \times 2 \times 2 = 36$$

Facteurs $(2\pi)^{N\,D}$, $d^D k \to (2\pi)^{-D}$, $\delta(\bullet) \to (2\pi)^{D}$

Renormalisation de la Masse

$$\int e^{-\lambda k^2} d^D k = \lambda^{-D/2} \pi^{D/2}$$

$$\downarrow$$

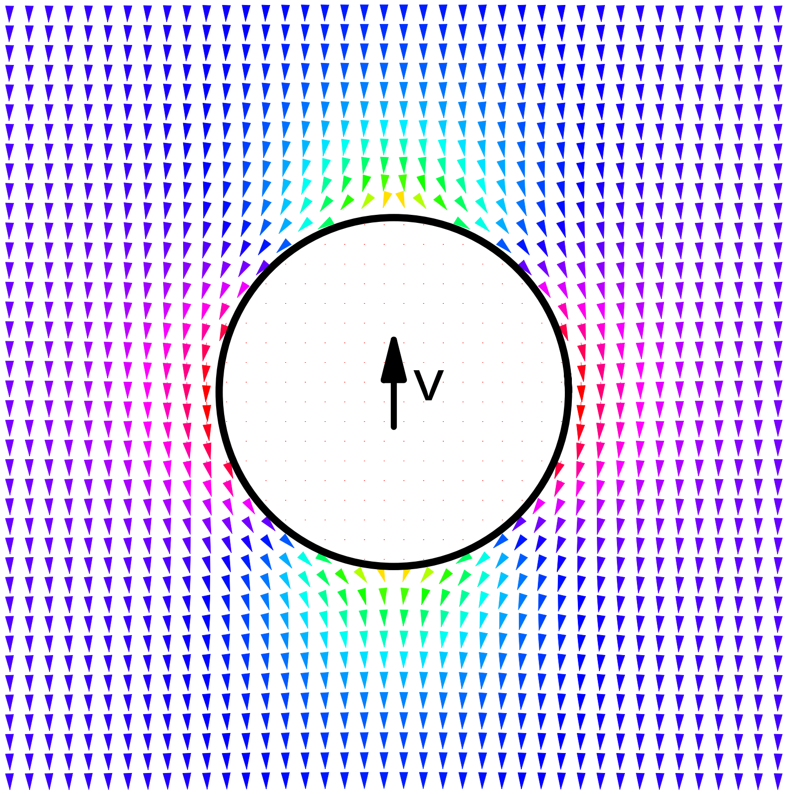
$$\int \frac{1}{k^2 + m^2} \frac{1}{((p+k)^2 + m^2)} d^D k =$$

$$\pi^{D/2} \Gamma(2 - \frac{D}{2}) \int_0^1 (x(1-x)p^2 + m^2)^{D/2-2} dx$$

Hydrodynamique Green 1830

$$F = m\, a$$

$$m \to m + \delta m$$



$$X = \text{Grad}\, h\,, \qquad \Delta h = 0$$

$$h(x,y,z) = \frac{v}{2}\,(r^{-3}+2)\,z\,, \qquad r^2 = x^2+y^2+z^2$$

$$E(x,y,z) = \frac{v^2}{8}\,\frac{x^2+y^2+4z^2}{(x^2+y^2+z^2)^4}\,\rho\, dxdydz$$

$$\int E = \frac{1}{2}\,\delta m\, v^2\,, \qquad \delta m = \frac{1}{2}\,M$$

J. Collins, *Renormalization*, Cambridge Monographs in Math. Physics, Cambridge University Press, 1984.

P. Dirac, *The quantum theory of the emission and absorption of radiation*. Proc. London Royal Soc. 114 (1927) 243–265.

R. Feynman, *The reason for antiparticles*, Elementary Particles and the Laws of Physics, Cambridge Univ. Press (1987).