Foliations

Manifold V and smooth subbundle F of TV

a) Every $x \in V$ is contained in a submanifold L of V such that

$$T_y(L) = F_y \qquad \forall\, y \in L\,.$$

b) Every $x \in V$ is in the domain $U \subset V$ of a submersion $p : U \to \mathbb{R}^q$ ($q = \mathrm{Codim}\, F$) with

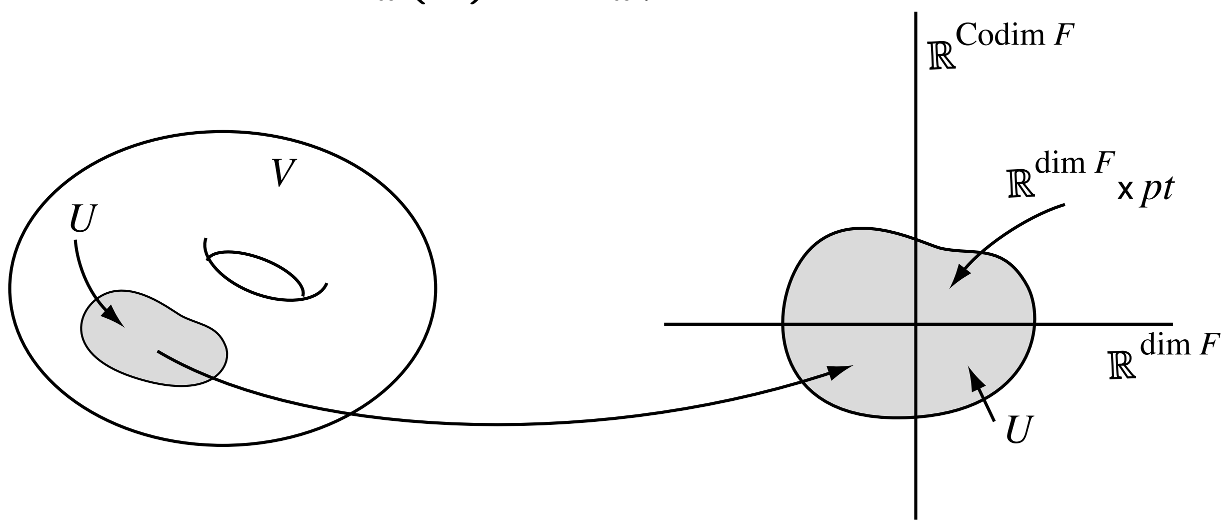
$$F_y = \mathrm{Ker}(p_*)_y \qquad \forall\, y \in U\,.$$

c) $C^\infty(F) = \{X \in C^\infty(TV)\,,\ X_x \in F_x \quad \forall\, x \in V\}$ is a Lie algebra.

d) The ideal $J(F)$ of smooth exterior differential forms which vanish on F is stable by exterior differentiation.

Leaves

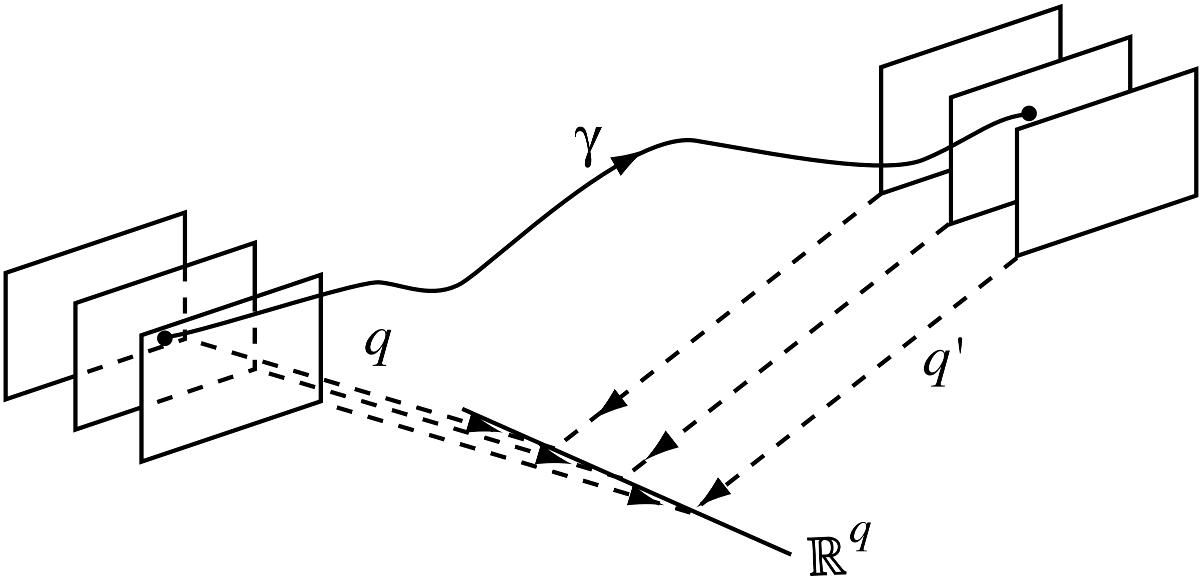
The leaves of the foliation (V, F) are the maximal connected submanifolds L of V with $T_x(L) = F_x$, $\forall\, x \in L$

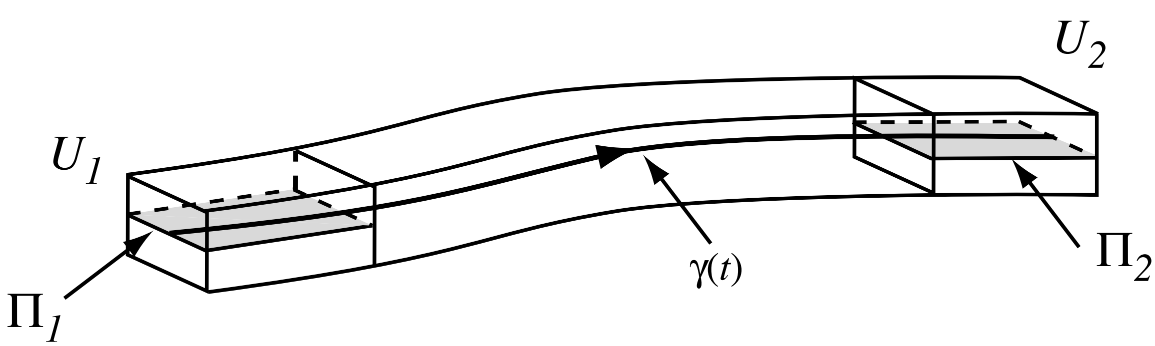


1) Though V is compact, the leaves L_α, $\alpha \in X$ can fail to be compact.

2) The space X of leaves L_α, $\alpha \in X$ can fail to be Hausdorff and in fact the quotient topology can be trivial (with no non trivial open subset).

Holonomy





Invariant measure for flows

Assume $\dim F = 1$ and F is oriented.

Take $X \in C^\infty(F^+)$. The leaves are then the orbits of the flow $\psi_t = \exp t\,X$.

Measure $=$ 0-dimensional current

$\langle \mu, \omega \rangle = \int \omega \, d\mu$, $\forall\, \omega \in C^\infty(V)$

Invariance for X

$$\psi_t \, \mu = \mu \Longleftrightarrow \partial_X \, \mu = 0 \Longleftrightarrow d(i_X \, \mu) = 0$$

Transverse measure for 1-dimensional foliations

Assume $\dim F = 1$ and F is oriented.

$X, X' \in C^\infty(F^+)$ are related by $X' = \varphi\, X$, $\varphi \in C^\infty(V)^+$.

$$\psi'_t(v) = \psi_{T(t,v)}(v) \qquad \forall\, t \in \mathbb{R}\,,\ v \in V\,.$$

$$\psi_t\, \mu = \mu \Longleftrightarrow \psi'_t\, \mu' = \mu' \qquad \mu' = \varphi^{-1} \mu$$

$$C = i_X\, \mu$$

$$i_{X'} \mu' = i_X\, \mu$$

Ruelle-Sullivan Current

1) C is closed, *i.e.* $dC = 0$

2) C is positive in the leaf direction, *i.e.* if ω is a smooth 1-form whose restriction to leaves is positive then $\langle C, \omega \rangle \geqq 0$.

$$\langle \mu, f \rangle = \langle C, \omega \rangle \,, \quad \forall\, \omega \in C^\infty(\Lambda^1\, T^*_{\mathbb{C}}) \,, \quad \omega(X) = f \,.$$

Transverse Measure

A transverse measure Λ for the foliation (V, F) is a σ-additive map $B \to \Lambda(B)$ from Borel transversals (*i.e.* Borel sets in V with $V \cap L$ countable for any leaf L) to $[0, +\infty]$ such that

1) If $\psi : B_1 \to B_2$ is a Borel bijection and $\psi(x)$ is on the leaf of x for any $x \in B_1$, then $\Lambda(B_1) = \Lambda(B_2)$.

2) $\Lambda(K) < \infty$ if K is a compact subset of a smooth transversal.

The Ruelle-Sullivan cycle and the Euler number of a measured foliation

(V, F) equipped with a transverse measure Λ

$dC = 0$ thus C defines a cycle $[C] \in H_k(V, \mathbb{R})$

Let now $e(F) \in H^k(V, \mathbb{R})$ be the Euler class of the oriented real vector bundle F on V

$$\chi(F, \Lambda) = \langle e(F), [C] \rangle$$

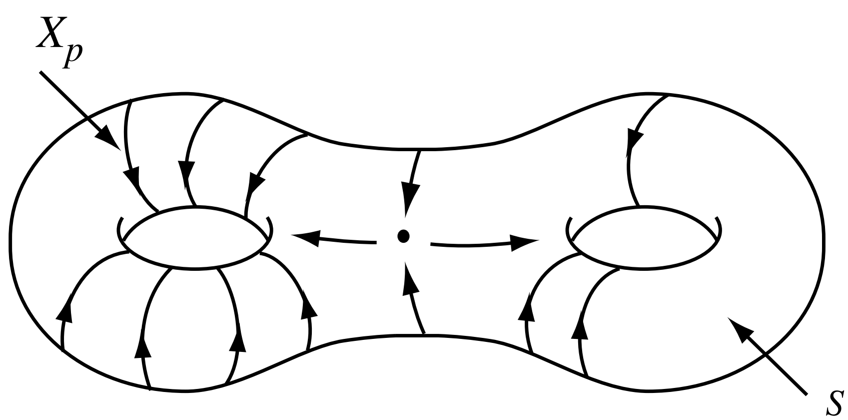
Poincaré and H. Hopf:

$$\chi(M) = \sum_{p \in \text{Zero}\, X} \omega(X, p)$$

$$\chi(F, \Lambda) = \int_{p \in \text{Zero}\, X} \omega(X, p)\, d\Lambda(p)$$

The degree of a vector field

In local coordinates, $X = \Sigma\, a^i \frac{\partial}{\partial x^i}$, the matrix $\frac{\partial\, a^i}{\partial x^j}(p)$ is non degenerate and the local degree is the sign of its determinant



Continuous Dimensions

Let B, B' be Borel transversals, then if the two bundles of Hilbert spaces $(H_L)_{L\in V/F}$, $H_L = \ell^2(L\cap B)$; $H'_L = \ell^2(L\cap B')$ are measurably isomorphic, one has $\Lambda(B) = \Lambda(B')$.

Type II von Neumann algebra

It is the algebra of random operators (modulo null sets for Λ):

$$M = \{(T_L)_{L\in V/F} \,|\, T_L \in \text{End } L^2(\tilde{L})\,, \quad \forall L \in V/F\}$$

$\tilde{L}$ is the holonomy covering of L.

Real Betti numbers

Theorem (ac 1978)

a) For each $j = 0, 1, 2, \ldots, \dim F$, there exists a Borel transversal B_j such that the bundle $(H^j(L, \mathbb{C}))_{L \in V/F}$ of j-th square integrable harmonic forms on L is measurably isomorphic to $(\ell^2(L \cap B))_{L \in V/F}$.

b) The scalar $\beta_j = \Lambda(B_j)$ is finite, independent of the choice of B_j, of the choice of the Euclidean structure on F.

c) One has $\Sigma\,(-1)^j\,\beta_j = \chi(F, \Lambda)$.

Dimension two leaves

dim$F = 2$

$H_L^{(0)} = \{$square integrable harmonic 0-forms on $L\}$

As harmonic 0-forms are constant, there are two cases:

If L is not compact, one has $H_L^{(0)} = \{0\}$.

If L is compact, one has $H_L^{(0)} = \mathbb{C}$.

Corollary

If the set of compact leaves of (V, F) is Λ-negligible then the integral $\int K \, d\Lambda$ of the intrinsic Gaussian curvature of the leaves is $\leqq 0$.

Proof $\frac{1}{2\pi} \int K \, d\Lambda = \beta_0 - \beta_1 + \beta_2 = -\beta_1 \leqq 0$. Q.E.D.

Longitudinal elliptic operators

One starts with a pair of smooth vector bundles E_1, E_2 on V together with a differential operator D on V from sections of E_1 to sections of E_2 such that:

1) D restricts to leaves, *i.e.* $(D\xi)_x$ only depends on the restriction of ξ to a neighborhood of x in the leaf of x (*i.e.* D only uses partial differentiation in the leaf direction).

2) D is elliptic when restricted to any leaf.

The index theorem for measured foliations

Theorem (ac 1978)

a) There exists a Borel transversal B (resp. B') such that the bundle $(\ell^2(L \cap B))_{L\in V/F}$ is measurably isomorphic to the bundle $(\mathrm{Ker}\, D_L)_{L\in V/F}$ (resp. to $(\mathrm{Ker}\, D_L^*)_{L\in V/F}$).

b) The scalar $\Lambda(B) < \infty$ is independent of the choice of B and noted $\dim_\Lambda(\mathrm{Ker}\,(D))$.

c) $$\dim_\Lambda(\mathrm{Ker}\,(D)) - \dim_\Lambda(\mathrm{Ker}\,(D^*)) =$$
. $$\varepsilon\,\langle \mathrm{ch}\,\sigma_D\,\mathrm{Td}\,(F_{\mathbb{C}}), [C]\rangle$$

($\varepsilon = (-1)\,\frac{k(k+1)}{2}$, $k = \dim F$, $\mathrm{Td}\,(F_{\mathbb{C}}) =$ Todd genus, $\sigma_D =$ symbol of D) .

ℓ^2 Betti numbers for measured equivalence relations

Theorem (D. Gaboriau 2000)

The Betti numbers $\beta_j(F, \Lambda)$ of a foliation with contractible leaves are invariants of the measured equivalence relation $\mathcal{R} = \{(x, y) \mid y \in \text{leaf}(x)\}$.

$$0 \overset{\partial_0}{\leftarrow} C_0^{(2)} \overset{\partial_1}{\leftarrow} C_1^{(2)} \overset{\partial_2}{\leftarrow} \cdots \overset{\partial_n}{\leftarrow} C_n^{(2)} \overset{\partial_{n+1}}{\leftarrow} C_{n+1}^{(2)} \overset{\partial_{n+2}}{\leftarrow} \cdots$$

$$H_n^{(2)}(\Sigma, \mathcal{R}, \mu) := Ker\partial_n / \overline{Im\partial_{n+1}}$$

$$\beta_n(\Sigma, \mathcal{R}) := \dim_\Lambda(H_n^{(2)}(\Sigma, \mathcal{R}, \mu))$$

is independent of the choice of a bounded n-connected $\mathcal{R}$-complex Σ.

(Cheeger-Gromov in the case of discrete groups after Atiyah's ℓ^2 Betti numbers for covering spaces).

$\mathbf{II_1}$ Factors and discrete groups

Group Γ	$\mathbf{II_1}$ Factor M
Representation	Correspondence (M-bimodule)
Trivial rep.	Standard form $L^2(M)$
Regular rep.	Coarse : $L^2(M)_{HS}$ $= L^2(M)\bar{\otimes}L^2(M)$
Amenable	$L^2(M) \subset_{weakly} L^2(M)_{HS}$
Property T (k)	$L^2(M)$ isolated (ac + vj)

$\mathbf{II}_1$ Factors and Betti numbers (ac + ds)

By results of W. Luck for discrete groups Γ,

$$\beta_*^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_*(\Gamma; L(\Gamma))$$

where H_* stands for the algebraic group homology.

L^2-homology of a von Neumann algebra M:

$$H_k^{(2)}(M) = H_k(M; M\bar{\otimes}M^o).$$

Here H_k stands for the algebraic Hochschild cohomology of M.

$H_0^{(2)}(M, \tau) \neq 0$ if and only if M is hyperfinite.

One is led to consider the L^2-Betti numbers,

$$\beta_k^{(2)}(M) = \dim_{M\bar{\otimes}M^o} H_k^{(2)}(M)$$

Problem For which groups does one have

$$\beta_*^{(2)}(L(\Gamma)) = \beta_*^{(2)}(\Gamma)$$

Foliation $\mapsto$ von Neumann algebra

It is the algebra of random operators:

$$M = \{(T_L)_{L\in V/F}\,|\,T_L \in \text{End}\, L^2(\tilde{L})\,, \quad \forall L \in V/F\}$$

$$||T|| = \text{EssSup}\, ||T_L||$$

(modulo null sets for the smooth (Lebesgue) measure)

Examples

- Reeb Foliation $\mapsto$ Type I_∞
- Kronecker Foliation $\mapsto$ Type II_∞ hyperfinite
- Anosov Foliation $\mapsto$ Type III_1 hyperfinite
- Flat connection $\mapsto$ Type II_∞ non-hyperfinite (genus > 1)

Invariants of von Neumann algebras

TT : $\phi \to \sigma_t^\phi$

Theorem (ac 1972)

$$\delta : \mathbb{R} \to \mathrm{Out}M = \mathrm{Aut}M/\mathrm{Int}M$$

$(\sigma_t^\psi = \mathrm{Ad}u_t\, \sigma_t^\phi)$

Invariants

- $S(M) = \cap \,\mathrm{Spec}\, \Delta_\varphi \subset \mathbb{R}_+$
- $T(M) = \mathrm{Ker}\, \delta \subset \mathbb{R}$
- $W(M) =$ Flow of weights.

($\Rightarrow$ hyperfinite non-ITPFI factor)

Weights on foliation algebra

For manifolds the positive measures in the "smooth" class are given by 1-densities,

Theorem (ac 1976)

- Faithful normal weights φ on M correspond to positive (unbounded) random operator one-densities

  $$(T_{L,v})_{L\in V/F} \,|\, T_{L,\lambda v} = \lambda\, T_{L,v}\,, \quad \forall v \in \wedge^q T(V/F)$$

- Independently of v one has

  $$\sigma_t^\varphi = \mathrm{Ad}\, T_{L,v}^{it}$$

- Independently of v one has

  $$(D\psi : D\varphi)_t = S_{L,v}^{it}\, T_{L,v}^{-it} \in M$$

The Godbillon-Vey invariant

Let (V, F) be a transversally oriented compact foliated manifold, of codimension 1.

$$F = \text{Ker}\,\omega\,, \quad \omega \in C^\infty(V, T^*(V))$$

$$d\omega = \beta \wedge \omega\,, \quad \alpha = d\beta \wedge \beta$$

$$GV := \text{Class}\,\alpha \in H^3(V, \mathbb{R})$$

Theorem (ac 1983)(refining previous work of Heitsch-Hurder)

Let M be the associated von Neumann algebra, and $W(M)$ be its flow of weights. Then if the Godbillon-Vey *class* of (V, F) is different from 0, there exists an invariant probability measure for the flow $W(M)$.

($\Rightarrow$ type III)

Homology of traces

Let B be a unital Banach algebra, view the dual space B^* as a bimodule over B with $\langle a\varphi b, x\rangle = \langle \varphi, bxa\rangle$, $\forall\, a, x, b \in B$.

Lemma Let δ be a densely defined derivation of B with values in B^*. Assume that the unit 1_B belongs to the domain of the adjoint δ^* of δ, then:

a. $\tau = \delta^*(1)$ is a trace on B.

b. The map of $K_0(B)$ to $\mathbb{C}$ given by τ is equal to 0.

Homology ~ 0 gives 1-traces

Definition Let B be a Banach algebra. By a 1-trace on B we mean a densely defined derivation δ from B to B^* such that

$$\langle \delta(x), y \rangle = -\langle \delta(y), x \rangle \qquad \forall\, x, y \in \mathrm{Dom}\,\delta\,.$$

Lemma Let δ be a 1-trace on B, then:

a) δ is closable.

b) There exists a unique map of $K_1(B)$ to $\mathbb{C}$ such that, for any $u \in \mathrm{GL}_n(\mathrm{Dom}\,\bar{\delta})$ (closure of δ) one has:

$$\varphi(u) = \langle u^{-1}, \bar{\delta}(u) \rangle$$

$$\langle x, \delta(a) \rangle = \int \sum_{\Gamma} x_g\, g(da_{g^{-1}}) \qquad \forall\, x \in C(S^1) \rtimes \Gamma$$

implies $K_1(C(S^1)) \subset K_1(C(S^1) \rtimes \Gamma)$.

Higher jet bundle $J_k^+(S^1)$

$J_k^+ =$ bundle of positive frames of order k on the oriented manifold S^1.

Jet or order k, $j^k(f)$, at $0 \in \mathbb{R}$ of a local orientation preserving diffeomorphism f of a neighborhood of 0 in $\mathbb{R}$ to a neighborhood of $y = f(0)$

$S^1 = \mathbb{R}/\mathbb{Z}$ with y the corresponding coordinate. Then natural coordinates in J_k^+ are $(y, y_1, \ldots, y_k)$, $y \in \mathbb{R}/\mathbb{Z}$, $y_\ell \in \mathbb{R}$, $y_1 > 0$.

$$f(t) = y + ty_1 + t^2 y_2 + \ldots + t^k y_k\,, \quad t \in \mathbb{R}\,.$$

J_k^+ is a principal G^k bundle over S^1, where G^k is the Lie group of k jets of orientation preserving diffeomorphisms of $\mathbb{R}$ which fix $0 \in \mathbb{R}$.

For any $f \in J_k^+$, $g \in G^k$, the product is just the composition $f \circ g$ of the jets.

The one-trace $d\,\mathrm{Log}(\varphi')$ on $A = C_0(J_1^+) \rtimes \Gamma$

Γ acts on S^1 by diffeomorphisms.

$\ell(g) \in C^\infty(S^1)$ is the logarithm of the Jacobian of the diffeomorphism associated to $g \in \Gamma$:

$$\ell(g) = \mathrm{Log}\, \frac{dg(y)}{dy}$$

1-cocycle $\omega \in Z^1(\Gamma, \Omega^2_{J_1^+})$ given by:

$$\omega(g) = d\ell(g) \wedge \frac{dy_1}{y_1}$$

$$\delta\left(\sum U_g\, y_g\right) = \sum U_g\, y_g\, \omega_g$$

where for $f \in C_c^\infty(J_1^+)$, $f\omega_g$ is the element of A^* given by

$$\langle h, f\omega_g \rangle = \int_{J_1^+} h_1\, f\omega_g\,, \quad \forall\, h = \sum h_g\, U_g \in A$$

Anabelian one-traces

Definition Let δ be a 1-trace on B, then δ is *anabelian* iff the domain of the adjoint δ^* contains the center $Z(B^{**})$ and $\delta^* = 0$ on $Z(B^{**})$.

Then for any $h \in Z(B^{**})$ the product $h\,\delta$ is also an anabelian one-trace.

Let $\delta = d\,\mathrm{Log}(\varphi')$ on $A = C_0(J_1^+) \rtimes \Gamma$, then

a) δ is an anabelian one-trace.

b) δ is invariant under the action of $G^1 = \mathbb{R}_+^*$.

c) Any $u \in K_1(A)$ defines a G^1-invariant normal linear form on $W(M)$ by

$$L(h) = \langle u^{-1}, h\,\bar{\delta}(u)\rangle\,, \quad \forall h \in W(M) \subset Z(A^{**})$$

Assembly map (ac + pb)

Geometric group

$K^*(V,\Gamma)$ is the K homology of the pair $(B\tau, S\tau)$ of the unit ball, unit sphere bundle of $\tau = TV$ over V_Γ.

A cycle (N, F, g) is a triple where N is a compact manifold without boundary, $F \in K^*(N)$ a K-theory class, and g is a continuous map from N to $V_\Gamma = V \times_\Gamma E\Gamma$, which is K-oriented, *i.e.*, such that the bundle $TN \oplus g^*\tau$ is gifted with a Spin^c structure.

$$\mu : K^*(V,\Gamma) \to K_*(C_0(V) \times \Gamma)$$

is the basic *index* construction of K-theory classes.

Geometric examples

Let (V, F) be a foliated manifold, special cases of the assembly map include

- Closed transversals $N \subset V$ give elements of $K_0(C^*(V, F))$.

- Analytic longitudinal index

$$\text{Index}(D_L) \in K(C^*(V, F))$$

(is well defined in full generality, no transverse measure (ac + gs)).

Chern Character

Let $\tau = TV$ and $H_*^\tau(V_\Gamma, \mathbb{Q})$ be the ordinary singular homology of the pair $(B\tau, S\tau)$ over V_Γ, and with coefficients in $\mathbb{Q}$. The Chern character

$$\mathrm{ch} : K^*(V, \Gamma) \to H_*^\tau(V_\Gamma, \mathbb{Q})$$

is a rational isomorphism.

Thom isomorphism

$$\Phi : H_{q+n}^\tau(V_\Gamma, \mathbb{Q}) \to H_q(V_\Gamma, \mathbb{Q}) \qquad (n = \mathrm{dim} V)$$

(where $\Phi(z) = p_*(U \cap z)$, $\forall\, z \in H_{q+n}((B\tau, S\tau), \mathbb{Q})$ and where p is the projection from $B\tau$ to the base V_Γ).

Thus $\Phi \circ \mathrm{ch}$ is a rational isomorphism:

$$\Phi \circ \mathrm{ch} : K^*(V, \Gamma) \to H_*(V_\Gamma, \mathbb{Q}) \,.$$

Index pairing

Let $\delta = d\,\mathrm{Log}(\varphi')$ on $A = C_0(J_1^+) \rtimes \Gamma$, then for any $x \in K^*(J_1^+, \Gamma)$ one has

$$\langle \mu(x), \delta \rangle = \langle \Phi\,\mathrm{ch}\,x, (B\pi)^* \mathrm{GV} \rangle$$

where $\mathrm{GV} \in H^3(\mathrm{WO}_1)$ is the Godbillon-Vey class, and $\pi : J_1^+ \rtimes \Gamma \to \mathcal{G}_1$ the natural homomorphism to the Haefliger groupoid.

One has the same formula replacing everywhere J_1^+ by S^1 and the 1-trace δ by the following cyclic 2-cocycle τ on $C_c^\infty(S^1 \rtimes \Gamma)$:

$$\tau(f^0, f^1, f^2) = \sum \int f^0(\gamma_0)\, f^1(\gamma_1)\, f^2(\gamma_2)\, \omega(g_1, g_2)$$

$$\omega(g_1, g_2) = d\ell(g_1 g_2)\, \ell(g_2) - \ell(g_1 g_2)\, d\ell(g_2)$$

Hochschild cohomology $H^*(\mathcal{A}, \mathcal{A}^*)$

$b\tau(a^0, a^1, \ldots, a^{n+1}) =$

$\sum(-1)^j \, \tau(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) +$

$(-1)^{n+1} \, \tau(a^{n+1} a^0, \ldots, a^n)$

Cyclic cohomology $HC^n(\mathcal{A})$

$$\tau(a^1, \ldots, a^n, a^0) = (-1)^n \, \tau(a^0, \ldots, a^n) \, , \quad \forall \, a^i \in \mathcal{A}$$

Exact Triangle

$$\begin{array}{ccccc} & & H^*(\mathcal{A}, \mathcal{A}^*) & & \\ & B \swarrow & & \nwarrow I & \\ HC^*(\mathcal{A}) & & \xrightarrow{S} & & HC^*(\mathcal{A}) \end{array}$$

Densely defined cocycles on Banach algebras

Definition

Let B be a Banach algebra. By an n-trace on B we mean an $n+1$ linear functional τ on a dense subalgebra $\mathcal{A}$ of B such that

a) τ is a cyclic cocycle on $\mathcal{A}$.

b) For any $a^i \in \mathcal{A}$, $i = 1, \ldots, n$ there exists $C = C_{a^1,\ldots,a^n} < \infty$ such that for all $x^j \in \mathcal{A}$,

$$|\hat{\tau}((x^1 da^1)(x^2 da^2) \ldots (x^n da^n))| \leq C \, \|x^1\| \ldots \|x^n\|$$

Chern character for Banach algebras

Lemma

Let τ be an n-trace on a Banach algebra B. Then there exists a map φ of $K_i(B)$ $(i = n(2))$ to $\mathbb{C}$ such that:

a) If n is even and $e \in \text{Proj}\, M_q(\text{Domain}\,\tau)$ then

$$\varphi([e]) = \tau \otimes \text{Tr}\,(e, \ldots, e)\,.$$

b) If n is odd and $u \in \text{GL}_q(\text{Domain}\,\tau)$ then

$$\varphi([u]) = \tau \otimes \text{Tr}\,(u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u)\,.$$

Forms and Currents

Space X	Algebra $\mathcal{A}$
Vector bundle	Finite projective module
Differential form	Hochschild cycle
DeRham current	Hochschild cocycle
DeRham homology	Cyclic cohomology
Chern Weil theory	Pairing $\langle K(\mathcal{A}), HC(\mathcal{A})\rangle$

Hopf algebra $\mathcal{H}_1$ (ac + hm)

As an algebra $\mathcal{H}_1$ is the universal enveloping algebra of the Lie algebra $\{X, Y, \delta_n\,; n \geq 1\}$ and brackets

$$[Y, X] = X\,,\ [Y, \delta_n] = n\,\delta_n\,,\ [X, \delta_n] = \delta_{n+1}$$

$$[\delta_k, \delta_\ell] = 0\,, \quad n, k, \ell \geq 1\,.$$

As a Hopf algebra, the coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\Delta\, Y = Y \otimes 1 + 1 \otimes Y\,, \quad \Delta\, \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1$$

$$\Delta\, X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y$$

and the multiplicativity property

$$\Delta(h^1\, h^2) = \Delta h^1 \cdot \Delta h^2\,, \quad h^1, h^2 \in \mathcal{H}_1$$

Action of $\mathcal{H}_1$ on $A = C_0(J_1^+) \rtimes \Gamma$

$J_+^1(S^1) \simeq S^1 \times \mathbb{R}^+$

$$j(t) = y + t\, y_1 + \cdots\,, \qquad y_1 > 0\,,$$

$$\varphi(y, y_1) = (\varphi(y),\, \varphi'(y) \cdot y_1)\,.$$

The action of $\mathcal{H}_1$ is then given as follows:

$$Y(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y_1} U_\varphi^*\,, \qquad X(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y} U_\varphi^*$$

$$\delta_n(fU_\varphi^*) = y_1^n \frac{d^n}{dy^n} \left(\log \frac{d\varphi}{dy}\right) fU_\varphi^*$$

The volume form $\dfrac{dy \wedge dy_1}{y_1^2}$ on $J_+^1(S^1)$ is invariant under $\mathrm{Diff}^+(S^1)$ and gives rise to the following trace $\tau : A \to \mathbb{C}$,

$$\tau(fU_\varphi^*) = \begin{cases} \int_{J_+^1(S^1)} f(y, y_1) \frac{dy \wedge dy_1}{y_1^2} & \text{if } \varphi = 1\,, \\ 0 & \text{if } \varphi \neq 1\,. \end{cases} \tag{1}$$

The trace τ is ν-invariant under $\mathcal{H}_1$

The trace τ is ν-invariant with respect to the action $\mathcal{H}_1 \otimes A \to A$ and with the modular character $\nu \in \mathcal{H}_1^*$, determined by

$$\nu(Y) = 1, \quad \nu(X) = 0, \quad \nu(\delta_n) = 0\,;$$

The invariance property is given by the identity

$$\tau(h(a)) = \nu(h)\,\tau(a)\,, \quad \forall h \in \mathcal{H}_1$$

The fact that $S^2 \neq 1$ is automatically corrected by twisting with ν. Indeed, $\widetilde{S} = \nu * S$ satisfies

$$\widetilde{S}^2 = 1\,.$$

Hopf cyclic cohomology $HC^*_{\text{Hopf}}(\mathcal{H})$

Hopf algebra $\mathcal{H}$, character ν with

$$\widetilde{S}^2 = 1 \qquad \widetilde{S} = \nu * S$$

$$C^n(\mathcal{H}) = \mathcal{H}^{\otimes n}$$

Face operators $\partial_i : C^{n-1}(\mathcal{H}) \to C^n(\mathcal{H})$ are

$$\begin{aligned}
\partial_0(h^1 \otimes \ldots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{n-1}, \\
\partial_j(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1}, \\
\partial_n(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes h^{n-1} \otimes 1 ;
\end{aligned}$$

Degeneracy operators $\sigma_i : C^{n+1}(\mathcal{H}) \to C^n(\mathcal{H})$, are

$$\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} ;$$

Cyclic operator $\tau_n : C^n(\mathcal{H}) \to C^n(\mathcal{H})$

$$\tau_n(h^1 \otimes \ldots \otimes h^n) = (\Delta^{n-1}\widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes 1 .$$

Normalized bicomplex $(CC^{*,*}(\mathcal{H}), b, B)$

The Hopf cyclic cohomology is computed from the bicomplex $CC^{p,q}$, where:

$$\begin{aligned} CC^{p,q}(\mathcal{H}) &= \bar{C}^{q-p}(\mathcal{H}), \quad q \geq p, \\ CC^{p,q}(\mathcal{H}) &= 0, \quad q < p\,; \end{aligned}$$

with

$$\bar{C}^n(\mathcal{H}) = \cap \operatorname{Ker} \sigma_i\,, \quad \forall n \geq 1, \qquad \bar{C}^0(\mathcal{H}) = \mathbb{C};$$

$$b : \bar{C}^{n-1}(\mathcal{H}) \to \bar{C}^n(\mathcal{H}), \qquad b = \sum_{i=0}^{n} (-1)^i \partial_i$$

$$B = A \circ B_0\,, \quad n \geq 0\,,$$

where

$$A = 1 + (-1)^n \tau_n + \ldots + (-1)^{n^2} {\tau_n}^n\,.$$

$$B_0(h^1 \otimes \ldots \otimes h^{n+1}) = (\Delta^{n-1} \widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^{n+1}$$

$$B_0(h) = \nu(h),\ h \in \mathcal{H}$$

Characteristic map $HC^*_{\text{Hopf}}(\mathcal{H}_1) \to HC^*(\mathcal{A})$

One lets $\mathcal{A} \subset A$ be the smooth subalgebra.

The map

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0 \, h^1(a^1) \ldots h^n(a^n))$$

where $h^1, \ldots, h^n \in \mathcal{H}_1$ and $a^0, a^1, \ldots, a^n \in \mathcal{A}$, induces a characteristic homomorphism

$$\chi_\tau^* : HC^*_{\text{Hopf}}(\mathcal{H}_1) \to HC^*(\mathcal{A}) .$$

Characteristic map and Gelfand Fuchs cohomology

$$\kappa_1^* \,:\, H^*(\mathfrak{a}_1, \mathbb{C}) \stackrel{\simeq}{\longrightarrow} PHC^*_{\mathrm{Hopf}}(\mathcal{H}_1),$$

The element $\delta_1 \in \mathcal{H}_1$ is a Hopf cyclic cocycle, which gives a nontrivial class

$$[\delta_1] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1).$$

Moreover, $[\delta_1]$ is a generator for $PHC^{\mathrm{odd}}_{\mathrm{Hopf}}(\mathcal{H}_1)$ and corresponds to the Godbillon-Vey class in the isomorphism κ_1^* with the Gelfand-Fuchs cohomology.

Schwarzian derivative

Schwarzian derivative

$$\{y\,;x\} \;:=\; \frac{d^2}{dx^2}\left(\log\frac{dy}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log\frac{dy}{dx}\right)\right)^2 \;.$$

The element $\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ is a Hopf cyclic cocycle, whose action on the crossed product algebra $\mathcal{A} = C_c^\infty(J_+^1(S^1)) \rtimes \Gamma$ is given by the Schwarzian derivative

$$\delta_2'(fU_\varphi^*) = y_1^2\,\{\varphi(y)\,;y\}\,fU_\varphi^*$$

and whose class

$$[\delta_2'] \in HC^1_{\mathrm{Hopf}}(\mathcal{H}_1)$$

is equal to $B(c)$, where c is the following Hochschild 2-cocycle,

$$c \;:=\; \delta_1 \otimes X + \frac{1}{2}\delta_1^2 \otimes Y \;.$$

Transverse fundamental class

The generator of $PHC^{\text{even}}_{\text{Hopf}}(\mathcal{H}_1)$ is the class of the cyclic 2-cocycle

$$F := X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y ,$$

which in the foliation context represents the 'transverse fundamental class'.

It is "integral" and is the Chern character in K-homology of the spectral triple given by the hypoelliptic transverse signature operator:

$$Q = \begin{pmatrix} 0 & Y^2 + X \\ Y^2 - X & 0 \end{pmatrix}$$

Local index formula (ac + hm)

- The equality

$$⨍ P = \mathrm{Res}_{z=0}\,\mathrm{Trace}\,(P|D|^{-z})$$

  defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.

- Cocycle in the bicomplex (b, B) of $\mathcal{A}$,

$$\varphi_n(a^0, \dots, a^n) = \sum_k c_{n,k}$$

$$⨍ a^0 [D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)}\,|D|^{-n-2|k|}$$

- The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.

Modular Hecke algebra (ac + hm)

$$\alpha \cdot z = \frac{az+b}{cz+d},\ j(\alpha, z) = cz+d$$

$$f|_k \alpha(z) = \mathrm{det}(\alpha)^{k/2} f(\alpha\cdot z)\, j(\alpha,z)^{-k}$$

Let Γ be a congruence subgroup. By a *Hecke operator form of level* Γ we mean a map

$$F : \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \to \mathcal{M}\,, \qquad \Gamma\alpha \mapsto F_\alpha \in \mathcal{M}\,,$$

with *finite support* and satisfying the *covariance condition*

$$F_{\alpha\gamma} = F_\alpha|\gamma\,, \quad \forall \alpha \in \mathrm{GL}_2^+(\mathbb{Q}), \gamma \in \Gamma\,.$$

$$(F^1 * F^2)_\alpha := \sum_{\beta \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta$$

turns the vector space $\mathcal{A}(\Gamma)$ of all Hecke operator forms of level Γ into an associative algebra.

Hopf action of $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$

The Hopf algebra $\mathcal{H}_1$ admits a canonical action on $\mathcal{A}(\Gamma)$.

$$Y(f) = \frac{k}{2} \cdot f \,, \quad \forall\, f \in \mathcal{M}_k \,.$$

$$X := \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\,\frac{d}{dz}(\log \Delta) \cdot Y$$

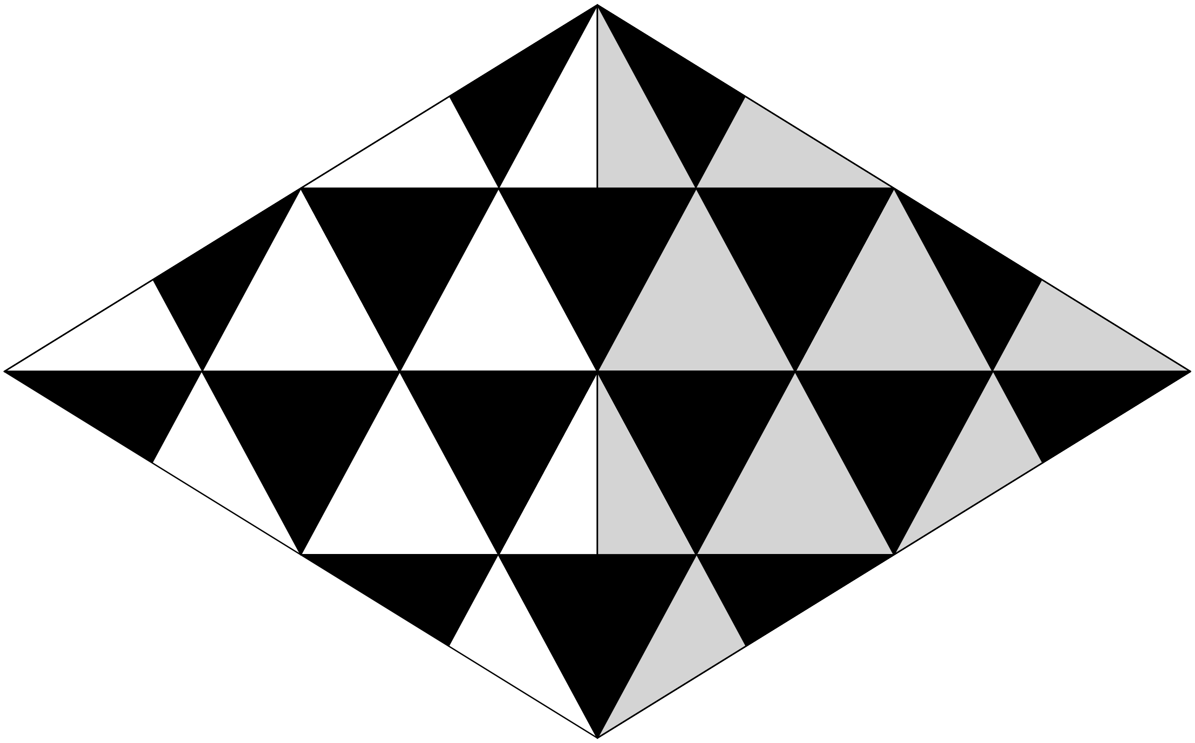
(where $\Delta(z) = \eta^{24} = q \prod_{n=1}^{\infty}(1-q^n)^{24}$ with $q = e^{2\pi i z}$)

$$\begin{aligned} Y(F)_\alpha &:= Y(F_\alpha)\,, \qquad \forall\, F \in \mathcal{A}(\Gamma)\,, \alpha \in G^+(\mathbb{Q})\,, \\ X(F)_\alpha &:= X(F_\alpha)\,, \\ \delta_n(F)_\alpha &:= \mu_{n,\alpha} \cdot F_\alpha\,, \end{aligned}$$

where

$$\mu_{n,\,\alpha} := X^{n-1}(\mu_\alpha)\,, \quad \forall n \in \mathbb{N}\,.$$

$$\mu_\gamma(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\gamma}{\eta^4}\,.$$



$$Z(z) := \frac{2\pi i}{3} \int_{i\infty}^{z} \eta^4 dz \,,$$

$$dZ := \frac{1}{3} \eta^4 \frac{dq}{q} = \frac{2\pi i}{3} \eta^4 \, dz \,.$$

$J_1^+(S^1)$	$\mathcal{L}^{-2}$
S^1	$\Gamma'(1)\backslash H^*$
$\theta \in \mathbb{R}/2\pi\mathbb{Z}$	$Z \in \mathbb{C}/\Lambda$
$e^{i\theta} = \cos\theta + i\sin\theta$	$(\wp_\Lambda(Z),\, \wp'_\Lambda(Z)) =$ $(\sqrt[3]{j},\, -\frac{2\,E_6}{\eta^{12}})$
$\phi(\theta)$	$Z\|\,\gamma$
$\phi'(\theta)$	$J(\gamma) = \frac{dZ\|\,\gamma}{dZ}$
$(\frac{d}{d\theta})^n \operatorname{Log}\,(\phi'(\theta))$	$(\frac{d}{dZ})^n \operatorname{Log}\,(J(\gamma))$

Hopf Symmetry of Modular Hecke Algebras

Let Γ be any congruence subgroup.

1^0. The above define a Hopf action of the Hopf algebra $\mathcal{H}_1$ on the algebra $\mathcal{A}(\Gamma)$.

2^0. The Schwarzian derivation $\delta_2' = \delta_2 - \frac{1}{2}\delta_1^2$ is inner and is implemented by $\omega_4 = -\frac{E_4}{72} \in \mathcal{A}(\Gamma)$, $E_4(q) := 1 + 240 \sum_1^\infty n^3 \frac{q^n}{1-q^n}$.

3^0. The image of the tranverse fundamental class $[F] \in HC^2_{\text{Hopf}}(\mathcal{H}_1)$ under the canonical map from the Hopf cyclic cohomology of $\mathcal{H}_1$ to the Hochschild cohomology of $\mathcal{A}(\Gamma)$, gives the natural extension of the first Rankin-Cohen bracket $\{\cdot\,,\cdot\}_1$ to the algebra $\mathcal{A}(\Gamma)$.

Some available papers

Here are relevant papers that can be downloaded on the site

http://www.alainconnes.org/downloads.html

1^0. A survey of foliations and operator algebras.

2^0. Cyclic cohomology and the transverse fundamental class of a foliation.

3^0. Hopf algebras, cyclic cohomology and the transverse index theorem (with Henri Moscovici).

4^0. Modular Hecke Algebras and their Hopf Symmetry (with Henri Moscovici).

5^0. L^2 Homology for von-Neumann Algebras (with D. Shlyakhtenko).