Algebraic varieties over $\mathbb{F}_1$

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Define a *gadget over* $\mathbb{F}_1$ to be the pair $X = (\underline{X}, \mathcal{A}_X)$ where $\underline{X}$ is a covariant functor from the category $\mathcal{F}$ of finite abelian groups to the category of sets, and $\mathcal{A}_X$ is a complex algebra. Given a finite abelian group μ, a point $x \in \underline{X}(\mu)$, and a multiplicative map σ from μ to the multiplicative group of non zero complex numbers, we assume given a character, called *evaluation*

$$e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}\,.$$

If $f : \mu' \to \mu$ is a morphism in $\mathcal{F}$ and if $y \in \underline{X}(\mu')$, the following equality is supposed to be satisfied :

$$e_{f(y),\sigma} = e_{y,\sigma\circ f} \tag{1}$$

for any morphism $\sigma : \mu \to \mathbb{C}^*$.

An affine variety V over $\mathbb{Z}$ defines a gadget X over $\mathbb{F}_1$ by letting $\underline{X}(\mu)$ be the set of points of V in the group algebra of μ and by defining the algebra $\mathcal{A}_X$ to be the ring of regular functions on the complex points of V (with the obvious evaluation maps).

A *morphism* $\varphi : X \to Y$ between two gadgets over $\mathbb{F}_1$ consists of a natural transformation

$$\underline{\varphi} : \underline{X} \to \underline{Y}$$

and a morphism of algebras

$$\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$$

compatible with evaluation maps. It is called an *immersion* when both $\underline{\varphi}$ and φ^* are injective.

An *affine variety over* $\mathbb{F}_1$ is a gadget X such that

- For every μ in $\mathcal{F}$ the set $\underline{X}(\mu)$ is finite;
- The complex algebra $\mathcal{A}_X$ is a commutative Banach algebra;
- There exists an affine variety $X_{\mathbb{Z}}$ over $\mathbb{Z}$ and an immersion $i : X \to X_{\mathbb{Z}}$ of gadgets satisfing the following property:

for any affine variety V over $\mathbb{Z}$ and any morphism of gadgets $\varphi : X \to V$, there exists a unique algebraic morphism

$$\varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \to V$$

such that $\varphi = \varphi_{\mathbb{Z}} \circ i$.

Examples of varieties $X_{\mathbb{Z}}$, where X is an affine variety over $\mathbb{F}_1$, include smooth toric varieties and the algebraic group-schemes GL_2 and GL_3.