Algebraic varieties over $\mathbb{F}_1$

Christophe Soulé

Define a gadget over $\mathbb{F}_1$ to be the pair $X = (\mathcal{X}, \mathcal{A}_X)$ where $\mathcal{X}$ is a covariant functor from the category $\mathcal{F}$ of finite abelian groups to the category of sets, and $\mathcal{A}_X$ is a complex algebra. Given a finite abelian group $\mu$, a point $x \in \mathcal{X}(\mu)$, and a multiplicative map $\sigma$ from $\mu$ to the multiplicative group of non zero complex numbers, we assume given a character, called evaluation

$$e_{x, \sigma} : \mathcal{A}_X \to \mathbb{C}.$$  

If $f : \mu' \to \mu$ is a morphism in $\mathcal{F}$ and if $y \in \mathcal{X}(\mu')$, the following equality is supposed to be satisfied:

$$e_{f(y), \sigma} = e_{y, \sigma \circ f}$$  \hspace{1cm} (1)

for any morphism $\sigma : \mu \to \mathbb{C}^*$.  

An affine variety $V$ over $\mathbb{Z}$ defines a gadget over $\mathbb{F}_1$ by letting $\mathcal{X}(\mu)$ be the set of points of $V$ in the group algebra of $\mu$ and by defining the algebra $\mathcal{A}_X$ to be the ring of regular functions on the complex points of $V$ (with the obvious evaluation maps).  

A morphism $\varphi : X \to Y$ between two gadgets over $\mathbb{F}_1$ consists of a natural transformation $\varphi : \mathcal{X} \to \mathcal{Y}$ and a morphism of algebras $\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$ compatible with evaluation maps. It is called an immersion when both $\varphi$ and $\varphi^*$ are injective.  

An affine variety over $\mathbb{F}_1$ is a gadget $X$ such that:
- For every $\mu$ in $\mathcal{F}$ the set $\mathcal{X}(\mu)$ is finite;
- The complex algebra $\mathcal{A}_X$ is a commutative Banach algebra;
- There exists an affine variety $X_\mathbb{Z}$ over $\mathbb{Z}$ and an immersion $i : X \to X_\mathbb{Z}$ of gadgets satisfying the following property:
  for any affine variety $V$ over $\mathbb{Z}$ and any morphism of gadgets $\varphi : X \to V$, there exists a unique algebraic morphism
  $$\varphi_\mathbb{Z} : X_\mathbb{Z} \to V$$
  such that $\varphi = \varphi_\mathbb{Z} \circ i$.

Examples of varieties $X_\mathbb{Z}$, where $X$ is an affine variety over $\mathbb{F}_1$, include smooth toric varieties and the algebraic group-schemes $GL_2$ and $GL_3$.  

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