Cyclic cohomology and the transverse fundamental class of a foliation

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In this paper we shall prove that for any transversally oriented foliated manifold (V, F), integration on the transverse fundamental class does yield a well defined map of $K^*(V/F)$ to $\mathbb{C}$. Here $K^*(V/F)$ is by definition the K theory of the C^* algebra $C^*(V, F)$ canonically associated to (V, F) (*cf.* [12]).

The proof puts together:

1) The algebraic ideas on cyclic cohomology that we have developed in [6], [7], and [8].

2) An analysis of densely defined cyclic cocycles on Banach algebras.

3) A geometric idea, which combines the techniques of R. Zimmer and G. Mackey [44], [30], with the reduction to maximal compact subgroup of G.G. Kasparov [27], to reduce the general problem to the special case when the foliation admits a holonomy invariant "almost isometric" transverse structure (*cf.* Section 4).

Let $K^*(V, F)$ be the geometrically defined K groups for (V, F) (*cf.* [11] and [3]), *i.e.*, when the holonomy groups are torsion free, the K homology of the classifying space of the graph of (V, F), twisted by the transverse bundle. Let $\mu : K^*(V, F) \to K_*(C^*(V, F))$ be the natural map ([11], [15] and [3]) from the geometric group to the analytical group. Our construction of the transverse fundamental class has the following strong implication on the rational injectivity of μ:

Theorem 0.1 *Let* $\mathrm{ch} : K^*(V, F) \to H_*(B\,\mathrm{Graph}\,(V, F))$ *be the Chern character and* $\mathcal{R} \subset H^*(B\,\mathrm{Graph}\,(V, F))$ *be the subring generated by the Pontrjagin classes of the transverse bundle, the Chern classes of any holonomy equivariant bundle on* V*, and the pull back of the Gel'fand-Fuchs characteristic classes:* $\gamma \in H^*(\mathrm{WO}_q)$*,* $q = \mathrm{Codim}\, F$*. Then if* $x \in K^*(V, F)$ *and* $\langle \mathrm{ch}\, x, \mathcal{R} \rangle \neq \{0\}$ *one has* $\mu(x) \neq \{0\}$ *in* $K_*(C^*(V, F))$*.*

In fact for any element P of $\mathcal{R}$ we construct an additive map of $K_*(C^*(V, F))$ to $\mathbb{C}$ which satisfies

$$\varphi(\mu(x)) = \langle \mathrm{ch}(x), P \rangle \qquad \forall\, x \in K^*(V, F)\,.$$

As an easy corollary we see that $K^q(V/F) = K_q(C^*(V, F))$ ($q = \mathrm{codim}\, F$ modulo 2), always contains a non trivial copy of $\mathbb{Z}$. The proof uses as a crucial tool the longitudinal index theorem of [15]. Using suitable combinations of Pontrjagin classes for P one gets a map φ from $K_*(C^*(V,F))$ to $\mathbb{C}$ which plays the role of integration in K theory. However we do not know if the image: $\varphi(K_*(C^*(V,F))) \subset \mathbb{C}$ of this map is contained in $\mathbb{Z}$. For elements of $K_*(C^*(V,F))$ which are in the range of μ one has $\varphi(y) \in \mathbb{Z}$, but to get the general result one would have to construct a corresponding element of $KK(C^*(V,F),\mathbb{C})$ which we have not done.

Since integration in K theory can now be done in two steps

a) $K^*(V)$ maps to $K^*(V/F)$ by taking the longitudinal index of the Dirac operator along the leaves,

b) $K^*(V/F)$ maps to $\mathbb{C}$ by φ,

one gets as an easy corollary of the ideas of [28], [37] and of our construction, the following generalization of a well known result of A. Lichnerowicz:

Theorem 0.2 *Let M be a compact oriented manifold and assume that the rational number $\hat{A}(M)$ is non-zero (since M is not assumed to be a spin manifold $\hat{A}(M)$ need not be an integer). Let then F be an integrable Spin subbundle of TM. There exists no metric on F for which the scalar curvature (of the leaves) is strictly positive* $(\geq \varepsilon > 0)$ *on M.*

To keep notations simple and to stress the role of the transverse direction we shall spend most of the paper dealing not with foliations but with actions of discrete groups on manifolds, by orientation preserving diffeomorphisms. Let Γ act on M then the reduced crossed product C^* algebra: $C_0(M) \rtimes \Gamma$ plays the role of $C^*(V,F)$, while the homotopy quotient $M \rtimes_\Gamma E\Gamma$ plays the role of the classifying space $B\,\mathrm{Graph}(V,F)$ of the Graph or holonomy groupoid of (V,F).

As a biproduct of our results we can analyze, in the case M *compact*, the following question:

> Is the unit 1_A of the C^* algebra $A = C(M) \rtimes \Gamma$ a torsion element of $K_0(A)$?

First, using as a tool the geometric group ([3]), we show that if Γ any subgroup of $\mathrm{PSL}(2,\mathbb{R})$ containing a cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$, the unit

1_A of $A = C(S^1) \rtimes \Gamma$ is a torsion element. Here Γ acts by homographic transformations on $S^1 = P_1(\mathbb{R})$.

Then, using Theorem 0.1, we get the following sharp contrast between the Fuchsian and Kleinian cases:

For any subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ acting on $S^2 = P(\mathbb{C})$ by homographic transformations the unit 1_A of $A = C(S^2) \rtimes \Gamma$ is a non-torsion element in $K_0(A)$.

As another corollary we prove that for any group Γ of diffeomorphisms of a compact manifold M having some nonzero Pontrjagin number (*cf.* [31]), the unit 1_A, $A = C(M) \rtimes \Gamma$, is a non torsion element in $K_0(A)$.

As a final point we note that Section 7 contains a description of the Godbillon-Vey class of codimension 1 foliations as a cyclic 2-cocycle on the C^* algebra $C^*(V,F)$, which is obtained from purely C^* algebra considerations from the cyclic 1-cocycle describing the fundamental class of V/F. Since the Godbillon-Vey invariant can assume any real value ([44]), the above cyclic 2-cocycle does not come from an element of $KK(C^*(V,F),\mathbb{C})$.

In a remarkable series of papers (see [23] for references) J. Heitsch and S. Hurder have analyzed the interplay between the vanishing of the Godbillon-Vey invariant of a compact foliated manifold (V,F) and the type of the von Neumann algebra of the foliation. Their work culminates in the following beautiful result of S. Hurder ([23]). If the von Neumann algebra is *semi-finite* then the Godbillon-Vey invariant *vanishes*.

We shall show that our method yields a stronger result, showing that if $\mathrm{GV} \neq 0$ the central decomposition of M contains necessarily factors M whose virtual modular spectrum is of finite covolume in $\mathbb{R}_+^*$.

Theorem 0.3 *Let (V,F) be a 3 dimensional oriented, transversally oriented compact foliated manifold,* $(\dim F = 2)$. *Let M be the associated von Neumann algebra, and $W(M)$ be its flow of weights* ([12]). *Then if the Godbillon-Vey invariant of (V,F) is different from 0, there exists an invariant probability measure for the flow $W(M)$.*

(The result still holds with $\dim V$ arbitrary, $\mathrm{Codim}\, F = 1$, if the Godbillon-Vey *class* is different from 0.)

1 Traces and unbounded derivations from A to A^*

If $\mathcal{A}$ is an algebra (over $\mathbb{C}$), any trace τ on $\mathcal{A}$ (*i.e.* a linear functional such that $\tau(xy) = \tau(yx)$, $\forall\, x, y \in \mathcal{A}$) determines an additive map from the algebraic K theory group $K_0(\mathcal{A})$ to $\mathbb{C}$. If $\tilde{\mathcal{A}}$ is obtained from $\mathcal{A}$ by adjoining a unit, and $\tilde{\tau}$ is extended to $\tilde{\mathcal{A}}$ by $\tilde{\tau}(1) = 0$, the above map assigns to an idempotent $e \in M_n(\tilde{\mathcal{A}})$ the scalar $(\tilde{\tau} \otimes \mathrm{Tr})(e)$, (where Tr is the usual trace on $M_n(\mathbb{C})$). In particular it follows that if $\mathcal{A}$ is unital and has a trace τ with $\tau(1) \neq 0$, then $K_0(\mathcal{A})$ contains a non-trivial copy of $\mathbb{Z}$, namely the multiples of the class of 1. This holds for instance for the C^* algebra $A = C(V)$ of continuous functions on a compact space V and $B = C^*(\Gamma)$, the norm closure of the group algebra $\mathbb{C}(\Gamma)$ of a discrete group Γ acting by the left regular representation in $\ell^2(\Gamma)$.

Even in the simplest cases: $A = C(V)$, with V a compact manifold, one sees that:

1. Different traces may give the same map: $K_0(A) \to \mathbb{C}$.

2. Not all additive maps from $K_0(A)$ to $\mathbb{C}$ come from traces. Indeed, in this example, a trace on A is a Radon measure μ on V or, in other words, a 0-dimensional current on V of order 0, and the map from $K_0(A) = K^0(V)$ to $\mathbb{C}$ given by a trace only depends upon the homology class of this 0-dimensional current, *i.e.*, if V is connected, its total mass $\mu(V)$.

Let μ be a Radon measure of 0 total mass on the connected compact manifold V and let us express the homology between the current μ and the current 0 in C^* algebra terms:

Lemma 1.1 *If $\mu(V) = 0$ there exists a densely defined derivation δ from the C^* algebra $A = C(V)$ to its dual A^* such that 1_A belongs to the domain of the adjoint δ^* and $\delta^*(1) = \mu$.*

Proof. Choose a Riemannian metric on V and assign in a Borel manner a geodesic path $\pi_{p,q} : [0,1] \to V$ to each pair p, q of elements of V. Assuming for simplicity that μ is real, let $\mu = \mu^+ - \mu^-$ be its Jordan decomposition, and put for $f, g \in C^\infty(V)$:

$$\tau(f, g) = \int \left(\int_0^1 \pi^*_{p,q}(f\, dg) \right) d\mu^+(p)\, d\mu^-(q)\,.$$

Then the equality $\langle \delta(g), f \rangle = \tau(f,g)$ gives a densely defined derivation δ from A to A^* (considered as a bimodule over A) and $\delta^*(1) = \mu^+(V)\,\mu$ since

$$\tau(1,g) = \int (g(p) - g(q))\, d\mu^+(p)\, d\mu^-(q) = \mu^+(V) \int g\, d\mu\,.$$

$\bullet$

Let us now prove in full generality that if B is a Banach algebra with unit any trace τ on B which is homologous to 0, *i.e.*, is of the form $\delta^*(1)$, gives the 0 map from $K_0(B)$ to $\mathbb{C}$.

Lemma 1.2 *Let B be a unital Banach algebra, δ a densely defined derivation of B with values in the dual space B^* (viewed as a bimodule over B) with $\langle a\varphi b, x\rangle = \langle \varphi, bxa\rangle$, $\forall\, a,x,b \in B$ and assume that the unit 1_B belongs to the domain of the adjoint δ^* of δ, then:*

a. *$\tau = \delta^*(1)$ is a trace on B.*

b. *The map of $K_0(B)$ to $\mathbb{C}$ given by τ is equal to 0.*

Proof. a) One has $\tau(xy) = \langle xy, \delta^*(1)\rangle = \langle \delta(xy), 1\rangle = \langle \delta(x), y\rangle + \langle \delta(y), x\rangle = \tau(yx)$, $\forall\, x,y \in \mathrm{Dom}\,\delta$.
b) The equality $\langle \delta(x), y\rangle + \langle \delta(y), x\rangle = \tau(xy)$ for $x,y \in \mathrm{Dom}\,\delta$ shows that $\mathrm{Dom}\,\delta \subset \mathrm{Dom}\,\delta^*$, with $\delta^*(x) = -\delta(x) + x\tau$ for any $x \in \mathrm{Dom}\,\delta$. This shows that δ is a closable operator from the Banach space B to the dual Banach space B^*. Let $\bar\delta$ be the closure of δ, then the domain of $\bar\delta : \mathcal{A} = \mathrm{Dom}\,\bar\delta$, is a subalgebra of B and $\bar\delta$ is a derivation from $\mathcal{A}$ to B^*.

If $a \in \mathcal{A}$ and a is invertible in B one has $a^{-1} \in \mathcal{A}$. Indeed, since $\mathcal{A}$ is dense in B, there exists $b \in \mathcal{A}$ with $\|1 - ab\| < 1$, $\|1 - ba\| < 1$, hence it is enough to show that if $a \in \mathcal{A}$, $\|a\|_n < 1$ then $(1-a)^{-1} \in \mathcal{A}$. This is clear since $\sum\limits_0^n a^k \to (1-a)^{-1}$, and $\delta\left(\sum\limits_0^n a^k\right)$ is norm convergent. This still holds if we replace $\mathcal{A}$ and B by $M_n(\mathcal{A})$, $M_n(B)$ (using the derivation $\bar\delta \otimes \mathrm{id} : \mathcal{A} \otimes M_n \to B^* \otimes M_n$). It follows (*cf.* [24]) that the inclusion $i : \mathcal{A} \to B$ is an isomorphism of $K_0(\mathcal{A})$ with $K_0(B)$. Now let e be an idempotent, $e \in M_n(\mathcal{A})$, and let us show that $(\delta^*(1) \otimes \mathrm{Tr})(e) = 0$. One can assume that $n = 1$, then

$$\begin{aligned} \langle e, \delta^*(1)\rangle &= \langle \delta(e), 1\rangle = \langle \delta(e^2), 1\rangle \\ &= \langle \delta(e), e\rangle + \langle \delta(e), e\rangle\,. \end{aligned}$$

But

$$\langle \delta(e), e\rangle = \langle \delta(e^2), e\rangle = 2\,\langle \delta(e), e\rangle\,.$$

•

We shall now show how to construct maps from $K_1(B)$ to $\mathbb{C}$ using instead of a trace a homology (in the sense of Lemma 1.2) between the trace $\tau = 0$ and itself:

Definition 1.3 *Let B be a Banach algebra. By a 1-trace on B we mean a densely defined derivation δ from B to B^* such that*

$$\langle \delta(x), y\rangle = -\langle \delta(y), x\rangle \qquad \forall\, x, y \in \mathrm{Dom}\,\delta\,.$$

Lemma 1.4 *Let δ be a 1-trace on B, then:*

a) *δ is closable.*

b) *There exists a unique map of $K_1(B)$ to $\mathbb{C}$ such that, for any $u \in \mathrm{GL}_n(\mathrm{Dom}\,\bar\delta)$ (closure of δ) one has:*

$$\varphi(u) = \langle u^{-1}, \bar\delta(u)\rangle \qquad (\delta \textit{ is extended to } B \otimes M_n(\mathbb{C})).$$

Proof. We can assume that B is unital with $\delta(1) = 0$.

a) Follows from the proof of Lemma 1.1.

b) We can assume that δ is closed, let $\mathcal{A}$ be its domain. As in the proof of Lemma 1.1, any element of $\mathcal{A}$ which is invertible in B is invertible in $\mathcal{A}$ and the same holds for $M_n(\mathcal{A}) \subset M_n(B)$. Thus, since the open set $M_n(B)^{-1}$ of invertible elements in $M_n(B)$ is locally convex, two elements u, v of $\mathrm{GL}_n(\mathcal{A})$ which are in the same connected component of $\mathrm{GL}_n(B)$ are connected by a piecewise affine path in $\mathrm{GL}_n(\mathcal{A})$.

On such a path $t \to u_t$ the function $f(t) = \langle u_t^{-1}, \delta(u_t)\rangle$ is constant, since its derivative is: $-\langle u_t^{-1}\,\mathring{u}_t\,u_t^{-1}, \delta(u_t)\rangle + \langle u_t^{-1}, \delta(\mathring{u}_t)\rangle = 0$ since $\langle u_t^{-1}, \delta(\mathring{u}_t)\rangle = -\langle \delta(u_t^{-1}), \mathring{u}_t\rangle = \langle u_t^{-1}\,\delta(u_t)\,u_t^{-1}, \mathring{u}_t\rangle$. The result then follows for instance as in [9]. •

We shall now give non-trivial examples of 1-traces on C^* algebras. The simplest example is to take a one parameter group α_t of automorphisms of

A, and an α-invariant trace τ on A. Let then D be the generator of (α_t), *i.e.* the closed derivation

$$D(x) = \lim_{t\to 0} \frac{1}{t}\,(\alpha_t(x) - x)\,.$$

The equality $x \in \operatorname{Dom} D \to \delta(x) = D(x)\,\tau \in A^*$ defines a 1-trace on A and the map of $K_1(A)$ to $\mathbb{C}$ given by this 1-trace coincides with the one defined in [9]. We shall now give other examples of 1-traces, not of the above form (the algebra A will have no nonzero traces in some examples) and use them to prove the non-triviality of $K_1(A)$ for crossed products $A = C(S^1)\times\Gamma$ of S^1 by a group of *orientation preserving* homeomorphisms. (Note that we shall use *reduced crossed products* so that, even if all homeomorphisms φ coming from the action of Γ are smooth, one cannot consider the pair $(L^2(S^1), P)$ (where P is the pseudo differential operator of order 0 with principal symbol $\sigma(x,\xi) = \operatorname{Sign}\xi$) as an element of $KK(A,\mathbb{C})$, since only the unreduced crossed product acts on $L^2(S^1)$ in general).

Theorem 1.5 *Let Γ be a countable group of orientation preserving homeomorphisms of S^1 and $A = C(S^1) \rtimes \Gamma$ be the reduced crossed product C^* algebra. Then the canonical homomorphism $i : C(S^1) \to A$ is an injection of $K_1(C(S^1)) = \mathbb{Z}$ in $K_1(A)$.*

Proof. On $B = C(S^1)$, we have a natural 1-trace obtained as the weak closure of the derivation δ, with domain $C^\infty(S^1)$, which assigns to each $f \in C^\infty(S^2)$, the differential df viewed as an element of B^*. (This uses the orientation.) This closure of δ is easily identified using distribution theory, as the derivation (also noted δ) with domain $BV(S^1)$, the space of functions $f \in C(S^1)$ with bounded variation, *i.e.* such that df is a measure, $\delta(f) = df$. A function is of bounded variation iff the sums $\sum\ |f(x_{i+1}) - f(x_i)|$ are bounded when the finite subset $(x_i)_{i=1,\ldots,m}$ of S^1 varies (with of course x_i in the same order on S^1 as i/m). It is then clear that if φ is any orientation preserving homeomorphism of S^1, $\varphi^* : C(S^1) \to C(S^1)$ leaves $\operatorname{Dom}\delta = BV(S^1)$ invariant and that $\delta(f\circ\varphi) = \varphi(\delta(f))$. Using this Γ-equivariant 1-trace on $C(S^1)$ we shall now construct a 1-trace on A. Let $\mathcal{A} = \big\{a \in A,\ a = \sum\limits_{\Gamma} a_g\, U_g$ with $a_g \neq 0$ only for finitely many $g \in \Gamma$, and $a_g \in BV(S^1)$, $\forall\, g \in \Gamma\big\}$. For any $a \in A$, let $\delta(a) \in A^*$ be the linear functional given by

$$\langle x, \delta(a)\rangle = \int \sum_{\Gamma} x_g\, g(da_{g^{-1}}) \qquad \forall\, x \in A\,.$$

Here, for any $g \in \Gamma$, $da_{g^{-1}}$ is a measure on S^1 and $g(da_{g^{-1}})$ is its image under the action of $g \in \Gamma$ on S^1. For any $x \in A$, $x = \sum x_g U_g$ one has $\|x_g\| \leq \|x\|$ where the latter is the norm in the reduced crossed product, since x_g is (in a faithful representation) a matrix element of x ([34]). Thus $\delta(a) \in A^*$. Let us check conditions 1, 2 (*cf.* Definition 1.3) for this 1-trace. For $a, b \in \mathcal{A}$ one has:

$$
\begin{aligned}
\langle b, \delta(a)\rangle &= \int \sum b_g \, g(da_{g^{-1}}) \\
&= -\int \sum db_{g^{-1}} \, g^* \, a_g \\
&= -\int \sum a_g \, g(db_{g^{-1}}) = -\langle a, \delta(b)\rangle \, .
\end{aligned}
$$

For $a, b, c \in A$ one has

$$
\begin{aligned}
\langle ab, \delta(c)\rangle - \langle a, \delta(bc)\rangle + \langle ca, \delta(b)\rangle &= \int \sum_{g_0 g_1 g_2 = 1} a_{g_0}((g_0^{-1})^* \, b_{g_1})(g_0 g_1) \, dc_{g_2} \\
&\quad - \int \sum_{g_0 g_1 g_2} a_{g_0} \, g_0(d(b_{g_1}(g_1^{-1})^* c_{g_2})) \\
&\quad + \int \sum_{g_0 g_1 g_2} c_{g_0}(g_0^{-1})^* a_{g_1}(g_0 g_1) \, db_{g_2} \\
&= -\sum_{g_0 g_1 g_2 = 1} \int a_{g_0} \, g_0(db_{g_1})(g_0 g_1)^{-1*} c_{g_2} \\
&\quad + \sum_{h_0 h_1 h_2 = 1} \int c_{h_0}(h_0^{-1})^* a_{h_1}(h_0 h_1) \, db_{h_2} \\
&= 0 \, .
\end{aligned}
$$

Thus δ is a 1-trace on A, and for any unitary $u \in C(S^1)$, $u \in BV(S^1)$, with winding number equal to 1 one has

$$
\langle u^{-1}, \delta(u)\rangle = \int u^{-1} \, du = 2i\pi \, .
$$

This shows that u^n is a non trivial element of $K_1(A)$ for any $n \in \mathbb{Z}$, $n \neq 0$.

$\bullet$

Remarks on Section 1.

1) Let A be a C^* algebra, then by [20] any bounded derivation δ from A to A^* is inner, hence $\delta^*(1) = 0$, so that bounded derivations give only trivial homologies.

2) If B is a non unital Banach algebra, Lemma 1.1 still holds if one replaces the equality $\tau = \delta^*(1)$ by:

$$\tau(xy) = \langle \delta(x), y \rangle + \langle \delta(y), x \rangle \qquad \forall\, x, y \in \mathrm{Dom}\, \delta\, .$$

3) Let A be a non unital C^* algebra, τ a densely defined semi-continuous weight which is a trace: $\tau(x^*x) = \tau(xx^*)$, $\forall\, x \in A$. Then using 2) above one can show that if δ is a densely defined derivation from A to A^* such that

a) $\mathrm{Dom}\, \delta \cap \mathrm{Dom}_{1/2}(\tau)$ [1] is dense in A.

b) $\langle \delta(x), y \rangle + \langle \delta(y), x \rangle = \tau(xy)$, $\forall\, x, y \in \mathrm{Dom}\, \delta \cap \mathrm{Dom}_{1/2}(\tau)$ then τ defines the 0-map from $K_0(A)$ to $\mathbb{C}$.

This happens if there exists a one parameter group of automorphisms θ_t of A such that $\tau \circ \theta_t = e^t\, \tau$, $\forall\, t \in \mathbb{R}$. Thus if $\varphi \in A^*$ is a K.M.S. state for a one parameter group t then the associated trace τ on the crossed product $\hat{A} = A \rtimes_\alpha \mathbb{R}$ (*cf.* [42]) is always homologous to 0.

4) The conclusion of Theorem 1.5 does not hold when Γ fails to preserve the orientation of S^1, in fact if $[u] \in K^1(D^1)$ is the generator, then $2i_*[u] = 0$ in $K_1(A)$.

2 Cyclic cohomology and Banach algebras

Let B be a (not necessarily unital) Banach algebra (over $\mathbb{C}$). Our aim in this section is to extend the results of Section 1 to cyclic cocycles $\tau(x^0, \ldots, x^n)$ with arbitrary n (not just 0 and 1). If $\mathcal{A}$ is an algebra and τ an $n+1$ linear functional on $\mathcal{A}$, we associate to τ the linear function $\hat{\tau}$ on $\Omega^n(\mathcal{A})$, with (*cf.* [7])

$$\hat{\tau}(a^0 da^1 da^2 \ldots da^n) = \tau(a^0, a^1, \ldots, a^n) \qquad a^i \in \mathcal{A}\, .$$

This gives a meaning to $\hat{\tau}((x^1 da^1)(x^2 da^2)(x^3 da^3) \ldots (x^n da^n))$, for $a^i, x^i \in \mathcal{A}$, but of course one could also define it directly by a formula. Thus, for $n = 2$, $\hat{\tau}((x^1 da^1)(x^2 da^2)) = \tau(x^1, a^1, x^2, a^2) - \tau(x^1 a^1, x^2, a^2)$. The crucial definition of this section is the following:

[1] $\mathrm{Dom}_{1/2}(\tau) = \{x \in A, \tau(x^*x) < \infty\}$.

Definition 2.1 *Let B be a Banach algebra. By an n-trace on B we mean an $n+1$ linear functional τ on a dense subalgebra $\mathcal{A}$ of B such that*

a) *τ is a cyclic cocycle on $\mathcal{A}$* [2].

b) *For any $a^i \in \mathcal{A}$, $i = 1, \ldots, n$ there exists $C = C_{a^1,\ldots,a^n} < \infty$ such that:*

$$|\hat{\tau}((x^1 da^1)(x^2 da^2)\ldots(x^n da^n))| \leq C\, \|x^1\| \ldots \|x^n\| \qquad \forall\, x^i \in \tilde{\mathcal{A}}\,.$$

Note that the conditions a), b) are still satisfied if we replace $\mathcal{A}$ by any subalgebra which is still dense in B.

Our aim is to show that an n-trace on B determines a map of $K_i(B)$, $i = n(2)$ to $\mathbb{C}$.

Let $\tilde{B}$ be obtained from B by adjoining a unit, then $\tilde{\mathcal{A}}$ is dense in $\tilde{B}$. Let $\tilde{\tau}$ be the natural extension of τ to $\tilde{\mathcal{A}}$:

$$\tilde{\tau}(a^0 + \lambda^0 1, a^1 + \lambda^1 1, \ldots, a^n + \lambda^n 1) = \tau(a^0, \ldots, a^n)\,.$$

Then $\tilde{\tau}$ is still a cyclic cocycle and is an n-trace on $\tilde{B}$. Thus from now on we shall assume that B is unital.

Let E be the locally convex vector space of multilinear functionals φ on $\underbrace{B \times B \times \ldots \times B}_{n\ \text{terms}} \times \underbrace{\mathcal{A} \times \ldots \times \mathcal{A}}_{n-1\ \text{terms}}$, which are continuous in the B variables. We endow E with the family of semi-norms $(p_\alpha)_{\alpha \in I}$ where $I = \mathcal{A}^{n-1}$, and:

$$p_\alpha(\varphi) = \sup_{x^j \in B, \|x^j\| \leq 1} |\varphi(x^1, \ldots, x^n, a^1, \ldots, a^{n-1})|\,.$$

Any bounded subset X of E is relatively compact in the topology of simple convergence: $\varphi_i \to \varphi$ if $\varphi_1(x,a) \to \varphi(x,a)$, $\forall\, x, a$. For $x \in B$ and $\varphi \in E$ put:

$$(x\varphi)(x^1, \ldots, x^n, a^1, \ldots, a^{n-1}) = \varphi(x^1 x, x^2, \ldots, x^n, a^1, \ldots, a^{n-1})$$

$$(\varphi x)(x^1, \ldots, x^n, a^1, \ldots, a^{n-1}) = \varphi(x^1, x x^2, \ldots, x^n, a^1, \ldots, a^{n-1})\,.$$

It is clear that E is now a bimodule over B and that, for each α, $p_\alpha(x\varphi)$ and $p_\alpha(\varphi x)$ are majorized by $p_\alpha(\varphi)\, \|x\|$.

[2] *i.e.* $\tau(a^1, \ldots, a^n, a^0) = (-1)^n\, \tau(a^0, \ldots, a^n)$, $\forall\, a^i \in \mathcal{A}$ and $\tau(a^0 a^1, \ldots, a^{n+1}) + \ldots + (-1)^j\, \tau(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + \ldots + (-1)^{n+1}\, \tau(a^{n+1} a^0, \ldots, a^n) = 0$.

Lemma 2.2 *For each $a \in \mathcal{A}$, let $\delta(a) \in E$ be such that*

$$\delta(a)(x^1, \ldots, x^n, a^2, \ldots, a^n) = \hat{\tau}((x^1 da)(x^2 da^2) \ldots (x^n da^n)),$$

then:

1) *δ is a derivation with values in the bimodule E.*

2) *δ is closable, for B gifted with norm topology and E with the topology of simple convergence.*

Proof. 1) $(\delta(a)b)(x^1, \ldots, x^n, a^2, \ldots, a^n) = \hat{\tau}((x^1 da)(bx^2 da^2) \ldots (x^n da^n))$ $(a\delta(b))(x^1, \ldots, x^n, a^2, \ldots, a^n) = \hat{\tau}((x^1 adb)(x^2 da^2) \ldots (x^n da^n))$, thus the equality $d(ab) = (da)b + adb$ shows that $\delta(ab) = \delta(a)b + a\delta(b)$.

2) We have to show that if $a_p \in \mathcal{A}$, and $\|a_p\| \to 0$ while $\delta(a_p)$ is weakly convergent to some $\varphi \in E$, then $\varphi = 0$. It is enough to show that $\varphi(b^1, \ldots, b^n, a^2, \ldots, a^n) = 0$ for $b^i \in \mathcal{A}$, $a^j \in \mathcal{A}$.

By hypothesis this value of φ is the limit of the net

$$\begin{aligned} \hat{\tau}((b^1 da_p)(b^2 da^2) \ldots (b^n da^n)) &= \hat{\tau}((da_p)(b^2 da^2) \ldots (b^n da^n) b^1) \\ &= -\hat{\tau}(a_p\, d((b^2 da^2) \ldots (b^n da^n) b^1)). \end{aligned}$$

As all b^i and a^j belong to $\mathcal{A}$ the last expression is, in modulus, smaller than $C\,\|a_p\|$ and hence goes to 0 as p goes to ∞. •

We shall now extend the cyclic cocycle τ to a subalgebra $\mathcal{B}$ of B, containing $\mathcal{A}$ and defined as follows: $b \in \mathcal{B}$ iff there exists finitely many elements $c_1, \ldots, c_k$ of $\mathcal{A}$ and a sequence $b_n \to b$, $b_n \in \mathcal{A}$ such that:

$$\sup_n\, p_\alpha(\delta(b_n)) \leq \sup_k\, p_\alpha(\delta(c_k)) \qquad \forall\, \alpha \in I = \mathcal{A}^{n-1}.$$

By hypothesis the sequence $\delta(b_n)$ is then bounded in E and by Lemma 2.2 the value of any weak limit of this sequence is:

$$\delta(b)(b^1, \ldots, b^n, a^2, \ldots, a^n) = -\hat{\tau}(bd(b^2 da^2) \ldots (b^n da^n) b) \qquad \forall\, b^i, a^i \in \mathcal{A}.$$

This shows that $\delta(b_n)$ is weakly convergent to an element of E, independent of the choice of the sequence b_n, and called $\delta(b)$. By construction one has $p_\alpha(\delta(b)) \leq \sup_k p_\alpha(\delta(c_k))$, $\forall\, \alpha \in I$.

Lemma 2.3

a) *$\mathcal{B}$ is a subalgebra of B.*

b) *For any q, $M_q(\mathcal{B})$ is stable under holomorphic functional calculus in $M_q(B)$.*

Proof. a) If $b_n \to b$, $b'_n \to b'$ then $b_n b'_n \to bb'$ and $p_\alpha(\delta(b_n b'_n)) = p_\alpha(\delta(b_n)b'_n + b_n\,\delta(b'_n)) \leq C(p_\alpha(\delta b_n)) + p_\alpha(\delta(b'_n)))$, thus the answer.

b) Let us first show that if $x \in M_q(\mathcal{B})$ and $\|x\| < 1$, then $1-x$ is invertible in $M_q(\mathcal{B})$. Here we choose on $B \otimes M_q(\mathbb{C})$ the norm given by the natural action in B^q (*i.e.* $\|(b_{ij})\| = \sup \|\sum b_{ij}\,c_j\|$ where $c_j \in B$, $\|c_j\| \leq 1$). We turn $M_q(E) = E \otimes M_q$ into a bimodule over $M_q(B) = B \otimes M_q$, in the obvious way

$$(b \otimes m)(\varphi \otimes m')(b'' \otimes m'') = b\varphi b'' \otimes mm', m'' \,.$$

The derivation δ extends to a derivation $\tilde{\delta}$: $\tilde{\delta}(b \otimes m) = \delta(b) \otimes m$ ($b \in \mathcal{A}$, $m \in M_q(\mathbb{C})$).

Now for each $(\varphi_{ij}) \in M_q(E)$, let $p_\alpha(\varphi) = \sup\limits_{i,j} p_\alpha(\varphi_{ij})$. For $b = (b_{ij}) \in M_q(B)$ one has $(b\varphi)_{ij} = \sum b_{ik}\,\varphi_{kj} \in E$ and $\|b_{ik}\| \leq \|b\|$, so that one has:

$$p_\alpha(b\varphi b') \leq q^2 \|b\|\,\|b'\|\,p_\alpha(\varphi) \qquad \forall\,\alpha \in I \,.$$

Next, let $a_n \in M_q(\mathcal{A})$, $a_n \to x$, with $\sup\limits_n p_\alpha(\delta(a_n)) \leq \sup\limits_k p_\alpha(\delta(c_k))$ for all $\alpha \in I$. We can assume that $\|a_n\| \leq C < 1$ for all $n \in \mathbb{N}$. Let $y_n = 1 + a_n + \ldots + a_n^n$. One has $y_n \in \mathcal{A}$, $y_{n_2} \to (1-x)^{-1}$ (in norm) and $\delta(y_n) = \delta(a_n) + (\delta(a_n)a_n + a_n\,\delta(a_n)) + \delta(a_n)a_n^2 + a_n\,\delta(a_n)a_n + a_n^2\,\delta(a_n) + \ldots + \left(\sum\limits_{j=0}^{q} a_n^j\,\delta(a_n)a_n^{q-j}\right) + \ldots$

Thus, for any $\alpha \in I$, $p_\alpha(\delta(y_n)) \leq a^2(1 + 2C + 3C^2 + \ldots)\,p_\alpha(\delta(a_n))$. Since $C < 1$ this shows that $\sup\limits_n p_\alpha(\delta(y_n)) \leq C' \sup\limits_k p_\alpha(\delta(c_k))$, and hence that $(1-x)^{-1} \in M_q(\mathcal{B})$.

This shows that if $x \in M_q(\mathcal{B})$ is invertible in $M_q(B)$ its inverse is in $M_q(\mathcal{B})$ (there exists, by density, an element y of $M_q(\mathcal{B})$ thus x has a left (and also right) inverse in $M_q(\mathcal{B})$). It follows that for any compact subset K of $\mathbb{C}$ disjoint from the spectrum of an element x of $M_q(\mathcal{B})$, there exist elements $c_1, \ldots, c_k$ of $\mathcal{A}$ and sequences $a_n(\lambda)$, $\lambda \in K$ of elements of $\mathcal{A}$ such that

1) For any $\alpha \in I$ one has

$$\sup_{n,\lambda} p_\alpha(\delta(a_n(\lambda))) \leq \sup p_\alpha(\delta(c_k)) .$$

2)

$$a_n(\lambda) \to (x-\lambda)^{-1} \text{ uniformly in } \lambda \in K .$$

Then, one concludes that for any holomorphic function f in a neighbourhood of the spectrum of x, one has $f(x) \in M_q(\mathcal{B})$. •

Corollary 2.4

a) *The inclusion $\mathcal{B} \subset B$ is an isomorphism of $K_0(\mathcal{B})$ with $K_0(B)$.*

b) *Let $K_1(\mathcal{B})$ be the quotient of $\mathrm{GL}_\infty(\mathcal{B})$ by the equivalence relation $u \sim v$ when u is connected to v by a piecewise linear path, then the obvious map $K_1(\mathcal{B}) \to K_1(B)$ is an isomorphism.*

Proof. a) The surjectivity of $i_* : K_0(\mathcal{B}) \to K_0(B)$ follows from Lemma 2.3b) (*cf.* [24]). If $e, f \in \mathrm{Proj}\, M_q(\mathcal{B})$ are equivalent idempotents in $M_q(B)$, *i.e.*, if there exists $u, v \in M_q(\mathcal{B})$ with $uv = f$, $vu = e$, one can for any $\varepsilon > 0$ find $u', v' \in M_q(\mathcal{B})$ such that $\|u'v' - f\| < \varepsilon$, $\|v'u' - e\| < \varepsilon$, $fu' = u'e = u'$ and $ev' = v'f = v'$. One can then conclude that e is equivalent to f in $M_q(\mathcal{B})$ using only the equality

$$M_q(\mathcal{B}) \cap M_q(B)^{-1} = M_q(\mathcal{B})^{-1} .$$

b) Again the surjectivity is clear. The injectivity follows by replacing an arc $u(t) \in \mathrm{GL}_n(B)$ with $u(0), u(1) \in \mathrm{GL}_n(\mathcal{B})$ by a piecewise linear arc (approximating each $u(i/p)$ by an element of $\mathrm{GL}_n(\mathcal{B})$, for p large enough the segment $\left[u(i/p), u\left(i + \frac{1}{p}\right)\right]$ is formed of invertible elements). •

To state the next lemma we shall adopt the following terminology: a subset X of $\mathcal{A}$ will be called δ-bounded when there exists a finite subset $c_1, \ldots, c_k$ of $\mathcal{A}$ such that

$$p_\alpha(\delta(a)) \leq \sup_j p_\alpha(c_j) \qquad \forall\, \alpha \in I\,,\ \forall\, a \in X\,.$$

Lemma 2.5 *For any $a^1, a^2, \ldots, a^n \in \mathcal{A}$, put $C(a^1, \ldots, a^n) = p_\alpha(\delta(a^1))$ where $\alpha = (a^2, \ldots, a^n) \in I = \mathcal{A}^{n-1}$, then*

a) $C(a^2, \ldots, a^n, a^1) = C(a^1, a^2, \ldots, a^n)$, $\forall\, a^i \in \mathcal{A}$.

b) *Let X be a δ-bounded subset of $\mathcal{A}$ then*

$$\sup_{a^j \in X} C(a^1, \ldots, a^n) < \infty \, .$$

Proof. a) By definition

$$C(a^1, \ldots, a^n) = \sup_{\|x^j\| \leq 1} |\hat{\tau}((x^1 da^1)(x^2 da^2) \ldots (x^n da^n))|$$

and since τ is a cyclic cocycle, C is invariant under cyclic permutations.
b) Let $c_1, \ldots, c_k$ with $p_\alpha(\delta(a)) \leq \sup_j p_\alpha\, \delta(c_j)$, $\forall\, \alpha \in I$, $\forall\, a \in X$. For $a^1, \ldots, a^n \in X$ one has: $C(a^1, \ldots, a^n) = p_\alpha(\delta(a^1)) \leq \sup p_\alpha\, \delta(c_j) = \sup_j C(c^j, a^2, \ldots, a^n)$. In the same way $C(c^j, a^2, \ldots, a^n) = C(a^2, \ldots, a^n, c^j)$ $\leq \sup_j C(c^{j_2}, a^3, \ldots, a^n, c^{j_1}) \ldots$. Thus we get

$$C(a^1, \ldots, a^n) \leq \sup C(c^{j_1}, c^{j_2}, \ldots, c^{j_n}) \, .$$

•

Lemma 2.6 *Let $b^1, \ldots, b^n \in B$. Then for any δ-bounded sequences $a_k^j \to b^j$, $a_k^j \in \mathcal{A}$, and any $x^1, \ldots, x^n \in B$, the sequence*

$$\hat{\tau}(x^1 da_k^1\, x^2 da_k^2 \ldots x^n da_k^n)$$

converges to a limit which depends only upon $b^1, \ldots, b^n, x^1, \ldots, x^n$.

Proof. Let X be the set of all a_k^j; it is δ-bounded by the hypothesis. Thus by Lemma 2.5, the multilinear functionals on $B \times \ldots \times B = B^n$ of the form $\varphi(x^1, \ldots, x^n) = \hat{\tau}(x^1 da^1 \ldots x^n da^n)$, $a^i \in X$, form a bounded set. Let $\varphi_k(x^1 \ldots x^n) = \hat{\tau}(x^1 da_k^1\, x^2 da_k^2 \ldots x^n da_k^n)$. To show the simple convergence of the sequence φ_k, one can assume that $x^1, \ldots, x^n \in \mathcal{A}$. Let us then replace X by $X' = X \cup \{x^j\}$. Applying Lemma 2.5b) with X' one gets $C_1 < \infty$ such that

$$|\hat{\tau}(ad(x^2 da^2 \ldots x^b da^n x^1))| \leq C_1 \|a\| \qquad \forall a^j \in X \, .$$

As $\hat\tau(x^1 da\, x^2 da^2 \dots x^n da^n) = -\hat\tau(ad(x^2 da^2 \dots x^n da^n\, x^1)$, we get

$$|\hat\tau(x^1 da^1 \dots x^n da^n) - \hat\tau(x^1 da'^1 x^2 da^2 \dots x^n da^n)| \le C_1 \|a^1 - a'^1\|$$

for any $a^1, \dots, a^n, a'^1 \in X$. Using cyclic permutations one gets:

$$|\hat\tau(x^1 da^1 \dots x^n da^n) - \hat\tau(x^1 da'^1 \dots x^n da'^n)| \le C_1 \sum \|a^j - a'^j\|$$

for $a^1, \dots, a^n, a'^1, \dots, a'^n \in X$.

This shows that the sequence $\varphi_k(x^1, \dots, x^n)$ is a Cauchy sequence. It also shows that its limit does not depend upon the choice of δ-bounded sequences $a_k^j \to b^j$, hence the result. •

Theorem 2.7 *Let τ be an n-trace on a Banach algebra B. Then there exists a map φ of $K_i(B)$ $(i = n(2))$ to $\mathbb{C}$ such that:*

a) *If n is even and $e \in \mathrm{Proj}\, M_q(\mathrm{Domain}\,\tau)$ then*

$$\varphi([e]) = \tau \otimes \mathrm{Tr}\,(e, \dots, e)\,.$$

b) *If n is odd and $u \in \mathrm{GL}_q(\mathrm{Domain}\,\tau)$ then*

$$\varphi([u]) = \tau \otimes \mathrm{Tr}\,(u^{-1}, u, u^{-1}, u, \dots, u^{-1}, u)\,.$$

Proof. By Corollary 2.4 it is enough to extend τ to a cyclic cocycle on $\mathcal{B}$. For this it is enough to show that for any δ-bounded sequences $a_k^j \to b^j$, the sequences $\tau(a_k^0, \dots, a_k^n)$ converge to a limit $\tau'(b^0, \dots, b^n)$. With the notations of Lemma 2.6, one has $\tau(a_k^0, \dots, a_k^n) = \varphi_k(a_k^0, 1, \dots, 1)$. As the sequence φ_k is bounded and as $a_k^0 \to b^0$ in norm one easily gets the conclusion. •

This theorem will be used in a crucial manner in Section 4. We shall now give a few examples of n-traces on C^* algebras.

Examples 2.8

a) Let V be a smooth manifold (not necessarily compact). Let $A = C_0(V)$ be the C^* algebra of continuous functions vanishing at ∞. Recall that a de Rham current C on V of dimension p, is a linear functional on the space $C_c^\infty(V, \wedge^p T_{\mathbb{C}}^* V)$ of differential forms of degree p on V, which is continuous in the following sense: for any compact $K \subset V$ and family $\omega_\alpha \in C_c^\infty(V, \wedge^p T_{\mathbb{C}}^*(V))$ with Support $\omega_\alpha \subset K$, $\forall\, \alpha$ converging to 0 in the C^k topology (on the coefficients of the forms ω_α) one has $C(\omega_\alpha) \to 0$. In other words C is of order k when for any $\omega \in C_c^\infty(V, \wedge^p T_{\mathbb{C}}^*(V))$ the linear functional $f \in C^\infty(V) \to C(f\omega)$ is continuous in the C^k topology.

Let then C be a *closed* current of dimension p and order 0 on V, put

$$\tau(f^0, \ldots, f^p) = C(f^0 df^1 \wedge \ldots \wedge df^p) \qquad \forall\, f^0, \ldots, f^p \in C_c^\infty(V)\,.$$

Let us check that it defines a p-trace on the C^* algebra $C_0(V)$. Its domain $C_c^\infty(V)$ is a dense subalgebra of $C_0(V)$ and one easily checks the cyclic cocycle property of τ using the closedness of C. One has $\hat\tau(x^1 da^1 \ldots x^p da^p) = C((x^1 da^1) \wedge \ldots \wedge (x^p da^p)) = C(x^1 \ldots x^p da^1 \wedge \ldots \wedge da^p)$. Thus, since $da^1 \wedge \ldots \wedge da^p = \omega$ belongs to $C_c^\infty(V, \wedge^p T_{\mathbb{C}}^*(V))$ there exists, as C is of order 0, a $C_{a_1 \ldots a_p} < \infty$ such that:

$$|\hat\tau(x^1 da^1 \ldots x^p da^p)| \le C_{a_1 \ldots a_p}\, \Pi \|x^j\|\,.$$

Using [17] one checks that the map φ of Theorem 2.7 from $K_i(C_0(V)) = K^i(V)$ to $\mathbb{C}$ is given by

$$\varphi([e]) = \langle \mathrm{ch}\, e, [C] \rangle$$

where ch is the usual Chern character: $K_c^*(V) \to H_c^*(V, \mathbb{R})$ and $[C] \in H_*(V, \mathbb{C})$ is the homology class of the closed current C [3]. Since there are always enough closed currents of order 0 to yield all of $H_*(V, \mathbb{C})$ we have not lost any information on the Chern character $\mathrm{ch}\, e \in H_c^*((V, \mathbb{R})$ in this presentation.

b) Let Δ be a locally finite simplicial complex, $x = |\Delta|$ the associated locally compact space. Let us construct enough p-traces on the C^* algebra $A = C_0(|\Delta|)$ to recover the usual Chern character, as in a). Let $\gamma = \sum \lambda_i\, s_i$ be a locally finite cycle of dimension p (*i.e.*, the s_i are all oriented p-simplexes of Δ and the λ_i's are complex numbers, with $b\gamma = \sum \lambda_i\, bd_i = 0$). Put

$$\tau(f^0, \ldots, f^p) = \sum \lambda_i \int_{s_i} f^0 df^1 \wedge \ldots \wedge df^p$$

[3] We use here the homology of locally finite chains, dual to the cohomology with compact support.

where the $f^j \in C_c(X)$ have the following property (*cf.* [39]): On each simplex s of Δ the restriction of f is equal to the restriction of a C^∞ function on the affine space of s. This space $C_c^\infty(\Delta) \subset C_0(X)$ is a dense subalgebra and by [39] one checks as in a) that τ is a p trace on $C_0(X)$. Again the map φ of Theorem 2.7 from $K_c^i(V)$ to $\mathbb{C}$ is given by

$$\varphi(x) = \langle \mathrm{ch}\, x, [\gamma] \rangle$$

where ch is the usual Chern character and $[\gamma]$ is the homology class of γ in the homology of locally finite chains on X, dual to the cohomology with compact support $H_c^*(X)$.

c) Let (A, G, α) be a C^* dynamical system, *i.e.* A is a C^* algebra on which the locally compact group G acts by automorphism $\alpha_g \in \mathrm{Aut}\, A$, $\forall\, g \in G$. Assume that G is a Lie group and let τ be an α-invariant trace on A. As in [14] let Ω be the graded differential algebra of right invariant differential forms (with complex coefficients) on G. By construction Ω is, as a graded algebra, identical with $\wedge_{\mathbb{C}} T_e^*(G)$ the exterior algebra on the dual of the Lie algebra of G. Let $t \in H_p(\Omega^*)$ be a p-homology class in the dual chain complex Ω^*. Considering t as a closed linear form on Ω^p, put:

$$\sigma(x^0, \ldots, x^p) = (\tau \otimes t)(x^0 dx^1 \ldots dx^p) \qquad x^j \in A^\infty .$$

Here, as in [14], A^∞ is the dense subalgebra of A formed of elements $x \in A$ for which $g \to \alpha_g(x)$ is a smooth function from G to the Banach space A. The differentials dx belongs to the algebra tensor product $A^* \otimes \Omega$ and for $x \in A^\infty$, $X_i \in \mathrm{Lie}\, G$ a basis of the Lie algebra of G, $\omega^i \in (\mathrm{Lie}\, G)^*$ the dual basis, $\delta_i \in \mathrm{der}(A)$ the unbounded derivations of A given by (α, X_i) one takes:

$$dx = \sum \delta_i(x) \otimes \omega^i \in A^\infty \otimes \Omega^1 .$$

One checks as in [14] that $\tau \otimes t$ is a closed graded trace on the differential algebra $A^\infty \otimes \Omega$ and it follows that σ is a cyclic cocycle on the algebra A^∞. Let us now show that it is a p-trace. For fixed $a^1, \ldots, a^p \in A^\infty$, the expression $\hat{\sigma}(x^1 da^1 \ldots x^p da^p)$ is a finite linear combination of terms of the form:

$$\tau(x^1 y^1 x^2 y^2 \ldots x^p y^p)$$

where $y^j \in A^\infty$ is of the form $\delta_k(a^j)$. Since by hypothesis $\tau \in A^*$ one has $|\tau(x^1 y^1 \ldots x^p y^p| \leq C \|x^1 y^1 \ldots x^p y^p\| \leq C' \|x^1\| \ldots \|x^p\|$ thus one gets the conclusion.

Applying Theorem 2.7 one recovers the Chern character that we introduced in [14], from $K(A)$ to $H^*(\Omega)$ (which is dual to $H_*(\Omega)$).

d) Let B be a Banach algebra, and $(H^{\pm}, F)$ be a Fredholm module over B, $p \in [1, \infty]$ be such that (*cf.* [6]) $\mathcal{A} = \{b \in B, [F, b] \in \mathcal{L}^p(H)\}$ is dense in B.

Let then $m \in \mathbb{N}$ be such that $n = 2m \leq p - 1$ and put:

$$\tau(a^0, \ldots, a^n) = \frac{1}{2} \, \mathrm{Trace}\,(\varepsilon \, F[iF, a^0] \ldots [iF, a^n]) \qquad \forall \, a^i \in \mathcal{A} \, .$$

As in [6] this defines a cyclic cocycle on $\mathcal{A}$, and we shall now check that it is an n-trace on B, if $n \geq p$. We can then rewrite τ as: $\tau(a^0, \ldots, a^n) = \mathrm{Trace}\,\varepsilon(a^0 da^1 \ldots da^n)$ using the notations of [6]. It follows that:

$$\hat{\tau}(x^1 da^1 \ldots x^n) = \mathrm{Trace}\,(\varepsilon \, x^1 da^1 \ldots x^n da^n) \, .$$

Thus from the Hölder inequality we get:

$$|\hat{\tau}(x^1 da^1 \ldots x^n da^n)| \leq \|x^1 da^1 \ldots x^n da^n\|_1 \leq \Pi \, \|x^j da^j\|_n < C \Pi \, \|x^j\|$$

since by hypothesis each da^j belongs to $\mathcal{L}^n$.

Remark 2.9

Combining the construction of n-traces in example b) with the known results on the Chern character, we see that if X is a locally compact space coming from a locally finite simplicial complex all maps from $K(C_0(X))$ to $\mathbb{C}$ which are additive come from n-traces on $A = C_0(X)$.

At this point it would be tempting to define for arbitrary n a homology relation between n-traces on an arbitrary Banach algebra extending that given by Lemma 1.1) and work out an analogue of homology in this context. We shall however see that this would be premature, because it would overlook a purely non-commutative phenomenon which prevents some very natural densely defined n-cocycles on C^* algebras to be n-traces, for $n > 2$.

As a simple example take $A = C^*(\Gamma)$ the reduced C^* algebra of the following solvable discrete group Γ. We let $\mathbb{Z}$ act by automorphisms on the group $\mathbb{Z}^2$ using the unimodular matrix $\alpha = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and let $\Gamma = \mathbb{Z}^2 \times_\alpha \mathbb{Z}$ be the semi-direct product. Now consider the group cocycle $c \in Z^2(\Gamma, \mathbb{R})$ given by the equality:

$$c(g_1, g_2) = v_1 \wedge \alpha^{n_1}(v_2) \qquad \forall \, g_j = (v_j, n_j) \in \Gamma \, .$$

(We represent an element of Γ as a pair (v,n) where $v \in \mathbb{Z}^2 \subset \mathbb{R}^2$ and $n \in \mathbb{Z}$, the $\wedge$ is the usual exterior product, with values in $\wedge^2 \mathbb{R}^2 = \mathbb{R}$.)

Given any discrete group and group cocycle c which is normalized so that $c(g_1, \ldots, g_n) = 0$ if any $g_i = 1$ or $g_1 \ldots g_n = 1$ one gets an n-cyclic cocycle on the group ring $\mathcal{A} = \mathbb{C}(\Gamma)$ by the equality:

$$\tau(f^0, \ldots, f^n) = \sum_{g_0 g_1 \cdots g_n = 1} f^0(f_0) \ldots f^n(g_n)\, c(g_1, \ldots, g_n)\,.$$

Thus there we get a cyclic 2-cocycle on $\mathcal{A} = \mathbb{C}(\Gamma)$, and thus a densely defined cyclic cocycle on $A = C^*(\Gamma)$. One has:

$$\begin{aligned} &\hat{\tau}(x^1 da^1 x^2 da^2) = \tau(x^1, a^1x^2, a^2) - \tau(x^1a^1, x^2a^2) \\ = &\sum_{g_0g_1g_2g_3=1} x^1(g_0)\, a^1(g_1)\, x^2(g_2)\, a^2(g_3)(c(g_1g_2, g_3) - c(g_2, g_3))\,. \end{aligned}$$

Let $a_1, a_2 \in \mathbb{C}(\Gamma)$. If for fixed g_1 and g_3 the function of g_2: $g_2 \to c(g_1g_2, g_3) - c(g_2, g_3)$ was bounded one would easily get the desired estimate:

$$|\hat{\tau}(x^1 da^1 x^2 da^2)| \leq C_{a^1,a^2} \|x^1\|\, \|x^2\|\,.$$

However, this precisely fails in our example, since with $g_1 = (v_1, 0)$, $g_3 = (v_2, 0)$ we get, for $g_2 = (0, n)$:

$$c(g_1g_2, g_3) - c(g_2, g_3) = v_1 \wedge \alpha^n(v_2)\,.$$

Moreover, fixing such a choice of v_1, v_2 and identifying Γ with a subgroup of the unitary group of $\mathbb{C}(\Gamma) \subset C^*(\Gamma)$ we get

$$\hat{\tau}(x^1 da^1 x^2 da^2) = v_1 \wedge \alpha^n(v^2)$$

for $x^1 = g_2^{-1} h^{-n} g_1^{-1}$, $a^1 = (v_1, 0)$, $x^2 = h^n$, $a^2 = (v^2, 0)$ with $h = (0,1) \in \Gamma$. This shows that τ is not a 2-trace.

Since, as one easily checks, there is for each pair a^1, a^2 of $\mathbb{C}(\Gamma)$ a constant C_{a^1,a^2} such that

$$|\tau(x da^1 a^2)| \leq C_{a^1,a^2} \|x\| \qquad \forall\, x \in \mathbb{C}(\Gamma)$$

the obstruction to τ being a 2-trace comes from the non-commutativity $da^1 x^2 \neq x^2 da^1$. The analysis of this lack of commutativity will occupy us for the next three sections, in the special case of the cyclic cocycle on $C_0(V) \times \Gamma$ coming from the Γ-equivariant fundamental class of the manifold V on which the discrete group Γ acts by orientation preserving diffeomorphisms. (The above 2-cocycle on $\mathbb{C}(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z})$ is a special case of this more general problem, since it is exactly the equivariant fundamental class of the 2-torus dual to $\mathbb{Z}^2$ on which $\mathbb{Z}$ acts by α.)

It hints towards an extension of the Tomita-Takesaki theory ([31]) to densely defined cyclic cocycles on a C^* algebra which satisfy the following reality condition:

Definition 2.10 *Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$, a cyclic cocycle τ is called* real *when, for any $a_0, \dots, a_n \in \mathcal{A}$ one has:*

$$\tau(a_0^*, a_n^*, \dots, a_1^*) = \tau(\overline{a_0, a_1, \dots, a_n})\,.$$

One checks that for any cyclic cocycle τ on $\mathcal{A}$ be equality

$$\tau^*(a_0, \dots, a_n) = \tau(\overline{a_0^*, a_n^*, \dots, a_1^*}) \qquad \forall\, a_1 \in \mathcal{A}$$

defines a new cyclic cocycle on $\mathcal{A}$, it follows that τ is always of the form $\tau = \tau_1 + i\,\tau_2$ with τ_1 and τ_2 real.

3 Modular theory and equivariant real vector bundles

Let X be a locally compact space and Γ be a discrete group acting by homeomorphisms on X. Let E be a *real* vector bundle of dimension n on X, which is Γ-equivariant. Even when $n = 1$ it is not in general possible to find a Γ-invariant Euclidean metric on E, in this section we shall interpret the obstruction in terms of the crossed product C^* algebra $A = C_0(X) \rtimes \Gamma$. Our aim is Theorem 3.7 below.

Let us first ignore the action of Γ on E, take an arbitrary Euclidean metric $\|\ \|$ on E and define a C^* module over A as follows: $\mathcal{E} = C_0(X, E_{\mathbb{C}}) \rtimes \Gamma$ is the completion of $C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$ (where $r : X \rtimes \Gamma \to X$ is the first projection), gifted with the norm: $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$, where $\langle \xi, \xi \rangle \in A$ is the element:

$$\langle \xi, \xi \rangle(x, g) = \sum_{h \in \Gamma} \langle \xi(xh, h^{-1}), \xi(xh, h^{-1}g) \rangle\,.$$

The C^* module structure on $\mathcal{E}$ is given by the equalities:

$$(\xi f)(x,g) = \sum_{h\in\Gamma} \xi(x,h)\, f(xh, h^{-1}g)$$

(for $\xi \in C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$, $f \in C_c(X \rtimes \Gamma))$ and, with the inner product in $E_{\mathbb{C}}$ antilinear in the first variable:

$$\langle \xi, \eta\rangle(x,g) = \sum_{h\in\Gamma} \langle \xi(xh, h^{-1}), \eta(xh, h^{-1}g)\rangle\,.$$

Let us now use the action of Γ on E to define an action of $C_c(X \rtimes \Gamma)$ on $\mathcal{E}$ by the equality:

$$(f\xi)(x,g) = \sum f(x,h)\, h\xi(xh, h^{-1}g)\,.$$

Note that $h\xi(xh, h^{-1}g)$ belongs to $E_{\mathbb{C}}(x)$.

Lemma 3.1 *For any $f \in C_c(X \rtimes \Gamma)$ the above equality defines an endomorphism of the C^* module $\mathcal{E}$, with adjoint given by*

$$(f^{\#}\xi)(x,g) = \sum f^{\#}(x,h)\, h\xi(xh, h^{-1}g)$$

where $f^{\#}(x,h) = f^{\#}(x,h)\,\Delta(x,h)$ and $\Delta(x,h) \in \mathrm{End}(E_{\mathbb{C}}(x)) =$:

$$\Delta(x,h)\,\xi = (h^{-1})^t\,(h^{-1})\,\xi\,, \qquad \forall\, \xi \in E_x\,.$$

Proof. Let us first show that f and $f^{\#}$ are bounded. Since they commute with the action of $C_c(X \rtimes \Gamma)$ on the right, and since the norm in $C_0(X) \rtimes \Gamma$ is defined as the sup norm $\sup\limits_{x\in X} \|\pi_x(a)\|$, where for each x, π_x is the left regular representation in $\ell^2(G_x)$, with $G_x = \{(y,g), y_g = x\}$, it is enough to estimate say $f^{\#}$ acting in $\mathcal{E} \otimes_A \ell^2(G_x)$ (where $A = C_0(X) \rtimes \Gamma$ acts in $\ell^2(G_x)$ by π_x). One checks that $\mathcal{E} \otimes_A \ell^2(G_x) = \ell^2(G_x, r^*E)$ and the action of $f^{\#}$ is still given by the same formula. Now we can consider the bijection q_x of Γ on G_x given by $q_x(g) = (xg^{-1}, g)$, and then our Hilbert space becomes $H_x = \ell^2(\Gamma, (r \circ q_x)^*E)$. The action of $f^{\#}$ now reads:

$$(T_x\,\xi)(g) = \sum_{h\in\Gamma} f^{\#}(xg^{-1}, h)\, h\xi(h^{-1}g)\,.$$

Thus it is enough to show that for any $h \in \Gamma$ and any continuous section θ with compact support of $\mathrm{End}(E_{\mathbb{C}})$, the following operator is bounded, independently of $x \in X$:

$$(S_x\,\xi)(g) = \theta(xg^{-1})\,h\xi(h^{-1}g) \qquad \forall\,\xi \in \ell^2(\Gamma, (r \circ q_x)^* E)\,.$$

For each $x \in X$ and $h \in \Gamma$, let $A(x,h)$ be the linear map from E_{xh} to E_x given by $\theta(x) \circ h$, then the above formula becomes:

$$(S_x\,\xi)(g) = A(xg^{-1}, h)\,\xi(h^{-1}g)\,.$$

Since θ has compact support, there exists a constant C such that $\sup\limits_{y \in X} \|A(y,h)\| \leq C < \infty$. It is then immediate that for all $x \in X$ one has $\|S_x\| \leq C$.

Let us now check that $f^{\#}$ is the adjoint of f. We may assume that $f(x,h) = 0$ if $h \neq h_0$. We get:

$$\begin{aligned}
\langle f\xi, \eta\rangle(x,g) &= \sum_{h \in \Gamma} \langle f\xi(xh, h^{-1}), \eta(xh, h^{-1}g)\rangle \\
&= \sum_{h \in \Gamma} \langle f(xh, h_0)\,h_0\,\xi(xhh_0, h_0^{-1}h^{-1}), \eta(xh, h^{-1}g)\rangle \langle \xi, f^{\#}\eta\rangle(x,g) \\
&= \sum_{h' \in \Gamma} \langle \xi(xh', h'^{-1}), f^{\#}\eta(xh', h'^{-1}g)\rangle \\
&= \sum_{h' \in \Gamma} \langle \xi(xh', h'^{-1}), f^*(xh', h^{-1})\,\Delta(xh', h_0^{-1})\,h_0^{-1}\eta(xh'h_0^{-1}g)\rangle\,.
\end{aligned}$$

With $h' = hh_0$, the last sum gives:

$$\sum_{h \in \Gamma} \langle \xi(xhh_0, h_0^{-1}h^{-1}), \bar{f}(xh, h^0)\,h_0^t\,\eta(xh, h^{-1}g\rangle\,.$$

•

We let λ be the unbounded homomorphism from $A = C_0(X) \rtimes \Gamma$ to $\mathrm{End}_A(\mathcal{E})$ which for $f \in C_c(X \rtimes \Gamma)$ is defined by:

$$\lambda(f)\,\xi = f\xi \qquad \forall\,\xi \in \mathcal{E}\,.$$

Lemma 3.2 *λ is a closable homomorphism of C^* algebras.*

Proof. We have to show that if $f_n \in C_c(X \rtimes \Gamma)$ and $\|f_n\| \to 0$, while $\lambda(f_n)$ converges in $\mathrm{End}_A(\mathcal{E})$, then it converges to 0. Let $T = \lim \lambda(f_n)$. For $\xi \in C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$, let us define $S\xi = \xi^*$ by:

$$S\xi(x,g) = g\,\bar{\xi}(xg, g^{-1}) \qquad \text{(where } \bar{\xi} \text{ has a meaning in } E_{\mathbb{C}}\text{)}.$$

One checks that $S(\lambda(f)\xi) = (S\xi)f^*$, for any $f \in C_c(X \rtimes \Gamma)$. Also $S^2 = 1$, thus it is enough to show that S is closable, *i.e.* that if $\xi_n \in C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$ and $\|\xi_n\| \to 0$, $S\xi_n \to \eta$ then $\eta = 0$. For each element ξ of the total space of the bundle E, $\xi \in E_x$, the equality $L_\xi^h(\eta) = \langle \xi, \eta(x,h)\rangle$ $(h \in \Gamma)$ determines a linear functional on $C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$ which extends to a continuous linear functional L_ξ^h on $\mathcal{E}$. Indeed, for any $\xi_0 \in C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}}))$ one has

$$\langle \xi_0, \eta\rangle(xh, e) = \sum_{h' \in \Gamma} \langle \xi_0(xhh'^{-1}, h'), \eta(xhh'^{-1}, h')\rangle$$

so that if $\xi_0(y, h') = 0$ for $h' \neq h$ and $\xi_0(x,h) = \xi$, one gets:

$$L_\xi^h(\eta) = \langle \xi_0, \eta\rangle(xh, e)\,.$$

The functionals L_ξ^h form a total subset of the Banach space $\mathcal{E}^*$ dual of $\mathcal{E}$ because they contain all functionals of the form:

$$\eta \to \langle \xi_0, \eta\rangle(x,e) \qquad \xi_0 \in C_c(X \rtimes \Gamma, r^*(E_{\mathbb{C}})) \qquad x \in X\,.$$

Thus if $L_\xi^h(\eta) = 0$, $\forall\, \xi, h$ one has $\langle \eta, \eta\rangle(x,e) = 0$, $\forall\, x \in X$ and hence, since $\langle \eta, \eta\rangle \geq 0$ one has $\langle \eta, \eta\rangle = 0$ *i.e.* $\eta = 0$. Finally, one has $\langle \xi, (S\eta)(x,h)\rangle = \langle \xi, h\,\bar{\eta}\,(xh, h^{-1})\rangle = \overline{\langle h^t\,\xi, \eta(xh, h^{-1})\rangle}$ so that each L_ξ^h is in the domain of the adjoint of S. $\bullet$

We let B be the domain of the closure of λ, it is by construction a subalgebra of $A = C_0(X) \rtimes \Gamma$, and it is a Banach algebra when we endow it with the graph norm:

$$|||x||| = \sup(\|x\|, \|\lambda(x)\|)\,.$$

Note however, that since λ is not a $*$ homomorphism, B is not in general a $*$ subalgebra of A.

It is not true in general that B is stable under holomorphic functional calculus in A or even, that $x \in \tilde{B}$, $x^{-1} \in \tilde{A} \Rightarrow x^{-1} \in \tilde{B}$.

We shall now see however that B is stable under holomorphic functional calculus in A, when the action of Γ on E is almost isometric in the sense of definition 3.3 below. If E is a *Euclidean* real vector bundle of dimension n on X, an action π of Γ on E is given by the linear maps $\pi(x,g) : E_{xg} \to E_x$, and thus can be thought of as a section of the bundle $s^*(E^*) \otimes r^*(E)$ on $X \times \Gamma$, where:

$$s(x,g) = xg\,, \quad r(x,g) = x \qquad \forall\,(x,g) \in X \rtimes \Gamma\,.$$

Of course an action of Γ on E satisfies the condition:

$$\pi(x,g_1)\,\pi(xg_1,g_2) = \pi(x,g_1g_2)\,.$$

Definition 3.3 *We shall say that an action π is almost isometric if there exists an isometric action π (of Γ on E), sections $p_1,\ldots,p_q$ of $s^*(E^*) \otimes r^*(E)$ and bounded automorphisms U_ε of E, $\varepsilon > 0$ such that:*

$$U_\varepsilon\,\pi(x,g)\,U_\varepsilon^{-1} = \pi_0(x,g) + \sum \varepsilon^k\,p_k(x,g)\,.$$

Note that if the action π of Γ on E is almost isometric then so is for instance the action $\pi(x,g) \otimes \pi(x,g)$ on $E \otimes E$.

The main example we shall use will come from:

Lemma 3.4 *Let π be an action of Γ on E, and assume that F is a π-invariant subbundle of E such that both the restriction of π to F and the action of Γ on E/F are isometric. Then π is almost isometric, as well as $\pi \otimes \ldots \otimes \pi = \pi^{\otimes q}$ for any q.*

Proof. Let $F^\perp$ be the orthogonal complement of F in E. It is not true in general that $F^\perp$ is invariant under π, but since $F^\perp$ is naturally isomorphic with E/F we can endow it with an isometric action π_1 of Γ and let $\pi_0 = \pi/F \oplus \pi_1$, and isometric action of Γ on $F \oplus F^\perp$. For $\varepsilon > 0$, let U_ε be the endomorphism of E given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$ in the decomposition $E = F \oplus F^\perp$. In this decomposition the matrix of $\pi(x,g)$ is of the form:

$$\pi(x,g) = \begin{bmatrix} \pi_F(x,g) & 0 \\ p(x,g) & \pi_1(x,g) \end{bmatrix}.$$

Thus one has $U_\varepsilon\,\pi(x,g)\,U_\varepsilon^{-1} = \pi_0(x,g) + \varepsilon \begin{bmatrix} 0 & 0 \\ p(x,g) & 0 \end{bmatrix}$. •

Proposition 3.5 *Let π be an almost isometric action of Γ on E then $M_q(\tilde{B})$ is stable under holomorphic functional calculus in $M_q(\tilde{A})$.*

Proof. With the notations of Definition 3.3, let V_ε be the endomorphism of $\mathcal{E}$ given by $(V_\varepsilon\,\xi)(x,g) = U_\varepsilon(\xi(x,g))$, π_ε be the action $U_\varepsilon\,\pi\,U_\varepsilon^{-1}$ of Γ on E and λ_ε the corresponding action of $C_c(X\times\Gamma)$ on $\mathcal{E}$. One has, for $f\in C_c(X\times\Gamma)$, $\lambda_\varepsilon(f) = V_\varepsilon\,\lambda(f)V_\varepsilon^{-1}$, and, with obvious notations $\lambda_\varepsilon(f) = \lambda_0(f) + \sum \varepsilon^k p_k(f)$. Since U_ε is bounded for each ε, as well as U_ε^{-1}, it follows that the domain of the closure of λ_ε is equal to B = domain of closure of λ, and that each p_k extends to a bounded linear map from B to $\mathrm{End}_A(\mathcal{E})$. Since π_0 is isometric, λ_0 is a $*$ representation of A in $\mathrm{End}_A(\mathcal{E})$ and we extend it to $\tilde{\lambda}_0 : \tilde{A}\to \mathrm{End}_A(\mathcal{E})$ by $\tilde{\lambda}_0(1) = 1$. We consider $\tilde{B}$ as a subalgebra of $\tilde{A}$ and extend λ_ε to $\tilde{B}$ by $\tilde{\lambda}_\varepsilon(1) = 1$. We have now: $\tilde{\lambda}_\varepsilon(f) = \tilde{\lambda}_0(f) + \sum \varepsilon^k p_k(f)$, $\forall\, f\in\tilde{B}$. Replacing $\tilde{\lambda}_\varepsilon$ by $\tilde{\lambda}_\varepsilon\otimes \mathrm{id}$ the same equality holds in $\mathrm{End}_A(\mathcal{E})\otimes M_q(\mathbb{C})$ for any $f\in\tilde{B}\otimes M_q(\mathbb{C})$. We just have to show that if the norm $\|f\|_{A\otimes M_q}$ is <1, $1-f$ is then invertible in $\tilde{B}\otimes M_q(\mathbb{C})$. We know that $M_q(\tilde{B})$ is a Banach algebra using the norm $|||f||| = \sup(\|f\|, \|\lambda_\varepsilon(f)\|)$. But this last expression is <1 for ε small enough, hence the conclusion. •

In the rest of this section we shall estimate multilinear functionals on $\mathcal{M}_E = C_c(X\rtimes\Gamma, r^*(E_\mathbb{C}))$ (considered as a bimodule over the algebra $\mathcal{A} = C_c(X\rtimes\Gamma)$) in terms of the Banach algebra norm $|||\;|||$ above. Let us begin by some algebraic remarks. We consider $C_c(X)$ as a subalgebra of $\mathcal{A}$ by $f\to\tilde{f}$: $\tilde{f}(x,g) = 0$ if $g\neq e$ and $\tilde{f}(x,e) = F(x)$. For any $a^1,\ldots,a^n\in\mathcal{A}$, $\xi^1,\ldots,\xi^m\in\mathcal{M}_E$ there exists $\rho\in C_c(X)$ such that $\rho\,a^j = a^j\rho = a^j$, $\xi^j\rho = \rho\,\xi^j = \xi^j$ for all j. Using this it is easy to show that, with obvious notations, the map $\xi\otimes\eta\to\xi\eta$:

$$\xi\eta(\gamma) = \sum_{\gamma_1\circ\gamma_2=\gamma} \xi(\gamma_1)\otimes\gamma_1\,\eta(\gamma_2) \qquad \forall\,\gamma\in X\rtimes\Gamma$$

gives an isomorphism of $\mathcal{A}$-bimodules of $\mathcal{M}_E\otimes_\mathcal{A}\mathcal{M}_F$ with $\mathcal{M}_{E\otimes F}$. Now, going back to the notations of Lemmas 3.1 and 3.2, and assuming E, F Euclidean (but not that the metric is Γ-invariant) one has:

Lemma 3.6

a) *Given $\xi,\eta\in\mathcal{M}_E$, $\xi',\eta'\in\mathcal{M}_F$ there exist $\xi'',\eta''\in\mathcal{M}_{E\otimes F}$ such that, for any $f\in\mathcal{A}$ one has:*

$$\langle\xi', \langle\xi, f\eta\rangle\,\eta'\rangle = \langle\xi'', f\eta''\rangle\,.$$

b) *Let $\rho \in C_c(X)$ and $\xi \in \mathcal{M}_E$. There exists finitely many $\xi^j \in \mathcal{M}_E$ and linear maps $\theta_j : \mathcal{A} \to \mathcal{A}$ such that*

1) $\rho f \xi = \sum \xi^j \, \theta_j(f)$, $\forall \, f \in \mathcal{A}$.

2) $\|\lambda_F(\theta_j(f))\| \leq C \|\lambda_{E\otimes F}(f)\|$, $\forall \, f \in \mathcal{A}$.

Proof. a) Let
$$\xi'', \xi''(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \xi(\gamma_1) \otimes (\gamma_1^{-1})^t \, \xi'(\gamma_2)$$
and
$$\eta''(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \eta(\gamma_1) \otimes \gamma_1 \, \eta'(\gamma_2)\,.$$
One checks that:
$$\langle \xi', \langle \xi, f\eta\rangle \, \eta'\rangle(x) = \sum_{\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5=\gamma} \langle \xi'(\gamma_1^{-1}), \langle \xi(\gamma_2^{-1}, f(\gamma_3)\gamma_3 \, \eta(\gamma_4)\rangle \, \gamma_2\gamma_3\gamma_4 \, \eta'(\gamma_5)\rangle\,,$$
while
$$\langle \xi'', f\eta''\rangle(\gamma) = \sum_{\gamma_1\cdots\gamma_5=\gamma} \langle \xi(\gamma_2^{-1}) \otimes \gamma_2^t \, \xi'(\gamma_1^{-1}), f(\gamma_3)\gamma_3 \, \eta(\gamma_4)) \otimes \gamma_3\gamma_4 \, \eta'(\gamma_5)\rangle\,.$$
Thus transposing γ_2^t one gets the equality.
b) We can assume that the support K of ρ is small enough so that we can find $\omega^1, \ldots, \omega^n \in C_c(X, E)$ which form an orthonormal basis of E_x for $x \in K$. Let $\xi^i \in \mathcal{M}$ be such that $\xi^i(x,h) = 0$ if $h \neq e$ and that $\xi^i(x,e) = \omega^i(x)$, $x \in X$. One has, with $\eta = \rho f \xi$:
$$\eta(x,g) = \rho(x) \sum_{h\in\Gamma} f(x,h) \, h\xi(xh, h^{-1}g)\,.$$
As $\eta(x,g) = 0$ for $x \notin K$ one gets that $\eta = \Sigma \, \xi^i \langle \xi^i, \eta\rangle$. Let $\theta_i(f) = \langle \xi^i, \eta\rangle = \langle \xi^i, \rho f \xi\rangle$. We have to estimate $\|\lambda_F(\theta_i(f))\|$. Let $\rho' \in C_c(X)$ be such that $\xi^i \rho' = \xi^i$, $\xi\rho' = \xi$, then for any $f \in \mathcal{A}$ one has $\theta_i(f) = \rho' \, \theta_i(f) \rho'$. Now since the restriction of F to the support of ρ' has a finite basis, it is enough to estimate $\|\langle \xi', \theta_i(f) \, \eta'\rangle\|$ for fixed $\xi', \eta' \in \mathcal{M}_F$. Now using a) there exists $\xi'', \eta'' \in \mathcal{M}_{E\otimes F}$ with $\langle \xi', \theta_i(f) \, \eta'\rangle = \langle \xi'', \rho f \eta''\rangle$. As $\|\langle \xi'', \rho f \eta''\rangle\| \leq C \|\lambda_{E\otimes F}(f)\mu\|$ we get the conclusion. •

We are now ready to prove the main result of this section (see also Remark 3.8). We let E be an *almost isometric* Γ-bundle on X. We fix $m \in \mathbb{N}$, and let B be the Banach algebra obtained as in Proposition 3.5 but using the almost isometric Γ bundle:

$$E' = E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus \ldots \oplus E^{\otimes m} .$$

For notational simplicity we put, for $f \in \mathcal{A}$, $\|f\|_k = \|\lambda_E \otimes_k (f)\|$. By Proposition 3.5, gifted with $|||\ ||| = \sup\limits_{k \leq m} \|\ \|_k$, the Banach algebra B satisfies the following conditions:

1) $\mathcal{A}$ is a dense subalgebra of B.

2) The canonical injection $\mathcal{A} \to A$ extends to a continuous homomorphism i of B in A, with dense range and such that:

$$i^{-1}(M_q(\tilde{A})^{-1}) = M_q(\tilde{B})^{-1} \qquad \forall\, q \in \mathbb{N} .$$

Theorem 3.7 *Let φ be an m-linear functional on $\mathcal{M}_E$ satisfying the following conditions:*

1) $\varphi(\xi_1, \ldots, \xi_j f, \xi_{j+1}, \ldots, \xi_m) = \varphi(\xi_1, \ldots, \xi_j, f\xi_{j+1}, \ldots, \xi_m)$ *for any* $j = 1, 2, \ldots, m-1$, $\xi_k \in \mathcal{M}_E$, $f \in \mathcal{A}$.

2) *For any $\xi_1, \ldots, \xi_m \in \mathcal{M}_E$ there exists $C < \infty$ such that:*

$$|\varphi(\xi_1, \ldots, \xi_m f)| \leq C\|f\|_A \qquad \forall\, f \in \mathcal{A} .$$

Then, for any $\rho \in \mathcal{A}$, and $\xi_1, \ldots, \xi_n \in \mathcal{M}_E$ there exists $C' < \infty$ with:

3) $|\varphi(\rho f_1\xi_1, f_2\xi_2, f_3\xi_3, \ldots, f_m\xi_m)| \leq C'|||f_1||| \ldots |||f_m||| \quad \forall\, f_i \in \mathcal{A}$.

Proof. Let $\rho' \in C_c(X)$ be such that $\rho'\rho = \rho$. We can replace ρ by ρ' and apply Lemma 3.6, to express $\varphi(\rho f_1\xi_1, \ldots, f_m\xi_m)$ as a sum of terms of the form: $\varphi(\xi_1'\theta(f_1), f_2\xi_2, \ldots, f_m\xi_m)$ where $\xi_1' \in \mathcal{M}_E$ and:

$$\|\theta(f_1)\|_{m-1} \leq C_1\|f_1\|_m .$$

Let $\rho_1 \in C_c(X)$ be such that $\xi_1'\,\rho_1 = \xi_1'$, then, using 1 one gets:

$$\varphi(\xi_1'\,\theta(f_1), f_2\xi_2, \ldots, f_m\xi_m) = \varphi(\xi_1', \rho_1\,\theta(f_1)f_2\xi_2, \ldots, f_m\xi_m) .$$

Applying Lemma 3.6 one has $\rho_1\,\theta(f_1)f_2\xi_2 = \sum \xi_2'\,\theta'(\theta(f_1)f_2)$ with

$$\begin{aligned}\|\theta'(\theta(f_1)f_2)\|_{m-2} &\leq C_2\|\theta(f_1)f_2\|_{m-1}\\ &\leq C_1C_2\|f_1\|_m\,\|f_2\|_{m-1}\,.\end{aligned}$$

This allows us to express $\varphi(\rho f_1\xi_1,\ldots,f_m\xi_m)$ as a sum of terms of the form:

$$\varphi(\xi_1',\xi_2'\,\theta(f_1,f_2),f_3\xi_3,\ldots,f_m\xi_m)$$

where $\|\theta(f_1,f_2)\|_{m-2}\leq C_2'|||f_1|||\,|||f_2|||$.

Iterating this, one writes $\varphi(\rho f_1\xi_1,\ldots,f_m\xi_m)$ as a sum of terms of the form: $\varphi(\xi_1',\xi_2',\ldots,\xi_m'\,\theta(f_1,f_2,\ldots,f_m))$, where $\|\theta(f_1,\ldots,f_m)\|\leq C'|||f_1|||\ldots|||f_m|||$.
•

Remark 3.8
Let us keep the notations of Theorem 3.7. For any $f\in C_c(X)\subset\mathcal{A}=C_c(X\rtimes\Gamma)$ and any $\xi\in\mathcal{M}_E$ one has:

$$(f\xi)(x,g)=f(x)\,\xi(x,g)\qquad\forall\,(x,g)\in X\rtimes\Gamma\,.$$

It follows that $\|\lambda_E(f)\|=\sup\limits_{x\in X}|f(x)|=\|f\|_A$, and that the restriction of the norm $|||\ |||$ of Theorem 3.7 to the algebra $C_c(X)\subset\mathcal{A}$ is the usual sup norm in $C_0(X)$. In particular if X is a smooth manifold, then $C_c^\infty(X\times\Gamma)$ is a dense subalgebra of $\mathcal{A}$ for the $|||\cdot|||$ norm.

4 The fundamental class of V/Γ in the almost isometric case

Let n and $p\leq n$ be positive integers and $G_p\subset\mathrm{SL}(n,\mathbb{R})$ be the subgroup of matrices of the form $g=\begin{bmatrix}g_{11} & 0\\ g_{21} & g_{22}\end{bmatrix}$, where $g_{11}\in\mathrm{SO}(p,\mathbb{R})$, $g_{22}\in\mathrm{SO}(n-p,\mathbb{R})$.

Let Γ be a discrete group of diffeomorphisms of a not necessarily compact manifold V of dimension n and let us assume that, for some p, the action of Γ preserves a G_p structure on V (*cf.* [44]). In particular, V is oriented and Γ preserves the orientation, and V has a smooth nowhere vanishing 1-density δ which is Γ-invariant. Our aim in this section is to prove Theorem 4.5 below, which shows that the crossed product of the fundamental cycle of V by Γ defines a map of $K(C_0(V) \rtimes \Gamma)$ (reduced crossed product C^* algebra) to the scalars.

By hypothesis, there exists a p-dimensional subbundle F of TV and a Riemannian metric $\| \ \|$ on V such that for any $x \in V$ and $g \in \Gamma$ the matrix of the tangent map at xg of the diffeomorphism $\varphi_g : \varphi_g(y) = yg^{-1}$ has the form: $\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix}$ where g_{11}, g_{22} are orthogonal, and in fact orientation preserving (F is oriented). For any differential form ω on V we put:

$$g(\omega) = (\varphi_g^{-1})^* \, \omega \qquad \forall \, g \in \Gamma \, .$$

Since $\varphi_{g_1 g_2} = \varphi_{g_1} \varphi_{g_2}$ one has $(g_1 g_2)\omega = g_1(g_2 \, \omega)$, $\forall \, g_1, g_2 \in \Gamma$. Let E be the Γ-equivariant bundle $E = T^*V$. The action of Γ on sections of T^*V, *i.e.* 1-forms ω is given by $\omega \to g\omega$ and the corresponding map: $T^*_{xg}(V) \to T^*_x(V)$ is the transpose of the above matrix $\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix}$. It follows that $E = T^*V$ is almost isometric in the sense of Definition 3.3 (using Lemma 3.4), as well as $\wedge^j \, T^*V = E_j$ for $j = 1, 2, \ldots, n$. For $j = n$, E_n is really isometric, it is equivariantly trivialized by the section δ.

Let $\mathcal{M}_j = C_c(V \rtimes \Gamma, r^*(E_{j\mathbb{C}}))$ be the corresponding bimodule over the algebra $\mathcal{A} = C_c(V \rtimes \Gamma)$.

Lemma 4.1 *For each j, k there exists a unique morphism of $\mathcal{A}$-bimodules of $\mathcal{M}_j \otimes_{\mathcal{A}} \mathcal{M}_k \xrightarrow{\pi} \mathcal{M}_{j+k}$ such that*

$$\pi(\omega \otimes \omega')(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} \omega(\gamma_1) \, \gamma_1 \, \omega'(\gamma_2) \qquad \forall \, \gamma \in V \rtimes \Gamma \, .$$

Here we have used the following groupoid notations, the groupoid is $V \rtimes \Gamma$ with source and range maps:

$$r(x,g) = x, \;\; s(x,g) = xg \qquad \forall \, (x,g) \in V \rtimes \Gamma$$

and composition $\gamma_1 \gamma_2 = (x_1, g_1 g_2)$ if $\gamma_i = (x_i, g_i)$, $s(\gamma_1) = r(\gamma_2)$. The value $\omega(\gamma_1) \wedge \gamma_1 \, \omega'(\gamma_2)$ makes sense because $\omega(\gamma_1)$ belongs to $\wedge^j \, T^*_x(V)$ and $\gamma_1 \, \omega'(\gamma_2)$ belongs to $\wedge^k \, T^*_x(V)$. One checks that for ω, ω' continuous with compact supports the above convolution is still continuous with compact support.

Let us check the $\mathcal{A}$-multilinearity, say that $\pi(\omega f \otimes \omega') = \pi(\omega \otimes f\omega')$. The left side is:

$$\sum_{\gamma_1\gamma_2\gamma_3=\gamma} \omega(\gamma_1) f(\gamma_2) \wedge \gamma_1\gamma_2\, \omega'(\gamma_3)$$

while the right side is

$$\sum_{\gamma_1\gamma_2\gamma_3=\gamma} \omega(\gamma_1) \wedge \gamma_1 f(\gamma_2)\gamma_2\, \omega(\gamma_3)\,.$$

Since the action of Γ on the fibers is linear we get the equality. •

We shall denote the above product by $\omega \wedge \omega'$, we of course have to remember that the graded commutativity rule *no longer holds*.

Lemma 4.2

a) *Let* $n = \dim V$*, then the equality* $\int \omega = \int_V \omega(x,e)$ *defines a trace on the* $\mathcal{A}$*-bimodule* $\mathcal{M}_n$*.*

b) *One has* $\int \omega_1 \wedge \omega_2 = (-1)^{jk} \int \omega_2 \wedge \omega_1$ *for any* $\omega_1 \in \mathcal{M}_j$*,* $\omega_2 \in \mathcal{M}_k$*,* $j+k=n$*.*

Proof. It is enough to prove b). One has $\int \omega_1 \wedge \omega_2 = \int_V \sum\limits_{\Gamma} \omega_1(x,h) \wedge h\,\omega_2(xh,h^{-1})$. For every $h \in \Gamma$ the invariance of integration of n-forms on V under the orientation preserving diffeomorphism h shows that:

$$\begin{aligned}\int_V \omega_1(x,h) \wedge h\,\omega_2(xh,h^{-1}) &= \int_V h^{-1}\omega_1(xh^{-1},h) \wedge \omega_2(x,h^{-1}) \\ &= (-1)^{jk} \int_V \omega_2(x,h^{-1}) \wedge h^{-1}\omega_1(xh^{-1},h)\,.\end{aligned}$$

•

Let us now take the notations of Theorem 3.7 of Section 3 with the almost isometric bundle $E = T^*V$. We let $|||\ |||$ be the corresponding norm on $C_c(V \rtimes \Gamma)$.

Lemma 4.3

a) *Let $\omega \in \mathcal{M}_n$. Then there exists a constant $C < \infty$ such that for any $f \in \mathcal{A} = C_c(V \rtimes \Gamma)$ one has*

$$\left| \int f\omega \right| \leq C \|f\|_A$$

where $\|f\|_A$ is the C^*-algebra norm of the reduced crossed product $C_0(V) \rtimes \Gamma = A$.

b) *Let $\omega_1, \dots, \omega_n \in \mathcal{M}_1$. There exists a constant $C < \infty$ such that, for any $f_1, \dots, f_n \in \mathcal{A}$ one has:*

$$\left| \int \omega_1 f_1 \wedge \omega_2 f_2 \wedge \dots \wedge \omega_n f_n \right| \leq C \Pi \, |||f_i||| \, .$$

Proof. a) Let $A = C_0(V) \rtimes \Gamma$. For any $f \in C_c(V \rtimes \Gamma)$ one has $\sup\limits_{x \in V, g \in \Gamma} |f(x,g)| \leq \|f\|_A$. To see this consider the representation of A in $\ell^2(G_x)$, $G_x = \{\gamma \in G, s(\gamma) = x\}$ $(G = V \rtimes \Gamma)$ given by $(\pi_x(f)\xi)(\gamma) = \sum\limits_{\gamma_1\gamma_2 = \gamma} f(\gamma_1)\, \xi(\gamma_2)$. Now to each $\gamma \in G_x$ corresponds a unit vector $\varepsilon_\gamma \in \ell^2(G_x)$ and one has:

$$f(\gamma) = \langle \pi_x(f)\, \varepsilon_x, \varepsilon_\gamma \rangle \, .$$

Next ω is by construction a differential form of degree n, with compact support, on $V \rtimes \Gamma$ so that it defines a finite Radon measure on $V \times \Gamma$. Thus the conclusion follows from the equality:

$$\int f\omega = \sum_\Gamma \int_V f(x,h)\, h\, \omega(xh, h^{-1}) \, .$$

b) Let φ be the n-linear functional on $\mathcal{M}_1 = \mathcal{M}_E$ given by:

$$\varphi(\omega_1, \dots, \omega_n) = \int \omega_1 \wedge \dots \wedge \omega_n \, .$$

Let us check conditions 1) 2) 3) of Theorem 3.7 of Section 3. Condition 1) follows from Lemma 4.1, condition 2) from Lemma 4.3a). Thus applying Theorem 3.7 and the tracial property (Lemma 3.2) we get the answer. $\bullet$

Let now Ω be the graded algebra of smooth differential forms with compact support on V. One has $\Omega^p = C_c^\infty(V, \wedge^p T_{\mathbb{C}}^*)$, and the differential $d : \Omega^p \to \Omega^{p+1}$ turns into a differential graded algebra. Moreover the linear functional $\tau(\omega) = \int_V \omega$ is a closed graded trace of degree n on Ω.

We let Γ act on Ω by $g\omega = \varphi^*\omega$, $\varphi(y) = yg$, $\forall\, y \in X$. Since the action of Γ preserves the structure of graded differential algebra of Ω and also the graded trace τ, it follows that the algebraic crossed product $\Omega \rtimes \Gamma$ is also a graded differential algebra with a closed graded trace of degree n. Let us describe it in more details.

Any element ω of $\Omega \rtimes \Gamma$ is a finite sum: $\omega = \sum\limits_{g \in \Gamma} \omega_g\, U_g$ where $\omega_g \in \Omega$, the U_g being symbols. The algebraic rules are:

1) $(\sum \omega_g\, U_g) + \left(\sum \omega'_g\, U_g\right) = \Sigma\,(\omega_g + \omega'_g)\, U_g$

2) $(\sum \omega_g\, U_g)\,(\sum \omega'_k\, U_k) = \sum \omega_g\, g(\omega'_k)\, U_{gk}$.

(In other words $U_g\, U_k = U_{gk}$ and $U_g\, \omega\, U_g^{-1} = g(\omega)$.)

3) $d\,(\sum \omega_g\, U_g) = \sum d\,\omega_g\, U_g$

4) $\int (\sum \omega_g\, U_g) = \int \omega_e$.

The equality $d(\omega_g\, g(\omega'_k)) = (d\,\omega_g) g(\omega'_k) + (-1)^{\partial \omega_g}\, \omega_g\, g(d\omega'_k)$ shows that d is a graded derivation of $\Omega \rtimes \Gamma$. One has $\int \omega\omega' = \sum\limits_g \int \omega_g\, g(\omega'_{g^{-1}}) = \sum\limits_g \int g(\omega_{g^{-1}})$ $\omega'_g = (-1)^{\partial\omega\partial\omega'} \int \omega'\omega$. Thus $\int$ is a (closed) graded trace on $\Omega \rtimes \Gamma$.

Hence it follows from [7] Prop. 1, that the following equality defines a cyclic cocycle on the algebra $C_c^\infty(V \rtimes \Gamma) = (\Omega \rtimes \Gamma)^0$:

$$\tau(f^0, \dots, f_n) = \int f^0 df^1 df^2 \dots df^n \,.$$

(Note here that the right hand side is a sum of integrals of ordinary differential forms on V, if f_g is the restriction of f to $V \times \{g\} : f_g(x) = f(x,g)$ one has:

$$\tau(f^0, \dots, f^n) = \sum_{g^0 \dots g^n = 1} \int f^0_{g_0}\, g_0\, d(f^1_{g_1}) \wedge g_0\, g_1\, d(f^2_{g_2}) \wedge \dots \wedge g_0 \dots g_{n-1}\, d(f^n_{g_n}) \,.$$

We extend τ to $C_c^\infty\widetilde{(V \rtimes \Gamma)}$ (obtained by adjoining a unit) by $\tau(f^0 + \lambda_0 1, \dots,$ $f^n + \lambda_n 1) = \tau(f^0, \dots, f^n)$ and note that this corresponds to adjoining a unit 1 to $\Omega \rtimes \Gamma$ satisfying $d1 = 0$.

We let B be the Banach algebra completion of $C_c^\infty(V \rtimes \Gamma)$ for the norm $|||\ |||$ of Theorem 3.7, and note that by Remark 3.8 *i.e.* the density of $C_c^\infty(V \rtimes \Gamma)$ in $\mathcal{A} = C_c(V \rtimes \Gamma)$, this Banach algebra is the same as that of Theorem 3.7.

Lemma 4.4 *τ is an n-trace on the Banach algebra B.*

Proof. We already know that on its domain: $C_c^\infty(V \rtimes \Gamma)$, τ is a cyclic cocycle, it remains to check that for any $f^1, \ldots, f^n \in C_c^\infty(V \rtimes \Gamma)$, $a^1, \ldots, a^n \in C_c^\infty(V \rtimes \Gamma)$ one has:

$$\hat{\tau}(f^1 da^1 \ldots f^n da^n) \leq c_{a^1 \ldots a^n} |||f^1||| \ldots |||f^n||| \, .$$

Using the tracial property of $\int$ this is exactly the content of Lemma 4.3b).

$\bullet$

Let A be a C^* algebra, $\mathcal{A}$ a subalgebra; we shall say that $\mathcal{A}$ is stable under holomorphic functional calculus when for any $q \in \mathbb{N}$, and $x \in M_q(\tilde{\mathcal{A}})$ one has $f(x) \in M_q(\tilde{\mathcal{A}})$ for any holomorphic function f on $\mathrm{Sp}\,(x)$.

Theorem 4.5 *Let Γ be a discrete group acting by diffeomorphisms preserving a G_p structure on the not necessarily compact manifold V. Let A be the reduced crossed product C^* algebra $A = C_0(V) \rtimes \Gamma$ and let $\mathcal{C}$ be the smallest subalgebra of A containing $C_c^*(V \rtimes \Gamma)$ and stable under holomorphic functional calculus.*
There exists a unique (additive) map φ of $K_i(A)$, $i = \dim V(2)$ to $\mathbb{C}$ such that:

1) *If $n = \dim V$ is even and e is an indempotent in $M_q(\tilde{\mathcal{C}})$ one has*

$$\varphi([e]) = \int e\, de \ldots de$$

2) *If $n = \dim V$ is odd and $u \in \mathrm{GL}_q(\tilde{\mathcal{C}})$ then*

$$\varphi([u]) = \int u^{-1}\, du\, du^{-1} \ldots du \, .$$

Proof. First the Banach algebra $B \subset A$ is stable under holomorphic functional calculus by Proposition 3.5. Thus by Theorem 2.7 the multilinear function τ on $C_c^\infty(V \times \Gamma)$ extends to an n-cyclic cocycle on $\mathcal{C}$. Since the inclusion $\mathcal{C} \to A$ is an isomorphism for both K_0 and $\pi_0\,\mathrm{GL}$ (*cf.* Theorem 2.7) we get the conclusion. $\bullet$

Corollary 4.6 *Let $I_V : C_0(V) \to A = C_0(V) \times \Gamma$ be the canonical homomorphism. There exists an additive map $\varphi : K_i(A) \to \mathbb{C}$ such that $\varphi(I_V(x)) = \langle \operatorname{ch} x, [V] \rangle$, $\forall\, x \in K^*(V)$.*

It shows in particular that if $\langle \operatorname{ch} x, [V] \rangle \neq 0$ then $I_V(x)$ is not a torsion element of $K_*(A)$.

Corollary 4.7 *Let $\mathcal{R}$ be the $\mathbb{C}$ subalgebra of $H^*(V,\mathbb{C})$ generated by Chern characters of complex vector bundles on V which can be endowed with an action of Γ preserving a hermitian metric. Then for any $P \in \mathcal{R}$ there exists an additive map Ψ of $K_i(A)$ to $\mathbb{C}$ such that:*

$$\Psi(I_V(x)) = \langle \operatorname{ch} x\, P, [V] \rangle \, .$$

Proof. By the multiplicativity of the Chern character: $\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch} E_1 \operatorname{ch} E_2$, we may assume that $P = \operatorname{ch}(E)$ for some Γ-equivariant hermitian bundle E on V. The $C_0(V)$ C^*-module $C_0(V,E)$ gifted with the natural action of Γ and also of $C_0(V)$ on the left, is an element of $KK_\Gamma(C_0(V), C_0(V))$ (*cf.* [27]). Let ε_E be the corresponding element of $KK(C_0(V) \rtimes \Gamma, C_0(V) \rtimes \Gamma)$ (*cf.* [27]) and $[E]$ be the class of E in $KK(C_0(V), C_0(V))$. Let us check that

$$[E] \otimes_{C_0(V)} I_V = I_V \otimes_A \varepsilon_E \, .$$

The left hand side is given by the $(C_0(V), A)$ C^*-bimodule: $C_0(V,E) \otimes_{C_0(V)} A$, and the right hand side by $C_0(V,E) \rtimes \Gamma$. These are canonically isomorphic so the answer follows:

Now with $\varphi : K_i(A) \to \mathbb{C}$ given by Corollary 4.6, put:

$$\Psi(z) = \varphi(z \otimes_A \varepsilon_E) \qquad \forall\, z \in K_i(A) \, .$$

Then $\Psi(I_V(x)) = \varphi(I_V(x) \otimes \varepsilon_E) = \varphi(x \otimes I_V \otimes_A \varepsilon_E) = \varphi((x \otimes [E]) \otimes I_V) = \varphi(I_V(x \otimes [E])) = \langle \operatorname{ch}(x \otimes [E]), [V] \rangle = \langle \operatorname{ch} x \operatorname{ch}(E), [V] \rangle = \operatorname{ch} x \cdot P, [V] \rangle$. •

5 Reduction to the almost isometric case

In this section we shall put together two techniques:

1) The notion of stable kernel of a homomorphism of groupoids, due to G. Mackey (see [30], [44] and [10]).

2) The reduction to the maximal compact subgroup of a Lie group, due to G.G. Kasparov (see [27]).

Our aim is to reduce a general action of discrete group Γ by orientation preserving diffeomorphisms of a manifold V to the almost isometric case, in so far as the problem of constructing the fundamental class of V/Γ is concerned.

Let us first explain how to use 1). Let us assume to simplify that the tangent bundle of V is trivial (we treat the general case below). Then the differential of the action of Γ on V yields a 1-cocycle $\pi(x,g)$ with values in the Lie group $\mathrm{GL}(n,\,\mathbb{R})^+$. Indeed if $\pi(x,g)$ is the matrix of the tangent map at xg of the map $y \to yg^{-1}$ one checks that:

$$\pi(x,g)\,\pi(xg,h) = \pi(x,gh) \qquad \forall\, x \in V,\ g,h \in \Gamma\,.$$

Then all the difficulty is that it is not possible, in general, to find an equivalent cocycle with values in the maximal compact subgroup $\mathrm{SO}(n,\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})^+$. In fact, it is already impossible to do it, in general, even among measurable cocycles (*cf.* [44]).

There is however a natural way to go around such an obstruction, which has already been fully exploited for the modular automorphism group of Type III factors (*cf.* [42], [16]). It amounts to replacing the groupoid $V \rtimes \Gamma$ by a new groupoid which is the inverse image, in the sense of virtual groups, of the subgroup $\mathrm{SO}(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R})^+$. By construction, this new groupoid is again of the form $W \rtimes \Gamma$, where W is a Γ-manifold which is the total space of a Γ-equivariant bundle with base B and fiber H: the symmetric space

$$H = \mathrm{GL}(n,\mathbb{R})^+/\mathrm{SO}(n,\mathbb{R})\,.$$

The advantage of W is that it now possesses a Γ-invariant almost isometric structure.

The use of 2) is to obtain a Thom map from the K theory of the crossed product $C_0(V) \rtimes \Gamma$ to the K theory of $C_0(W) \rtimes \Gamma$. This being said we can now proceed and describe in purely geometric terms the construction of the bundle W. This bundle depends functorially upon the tangent bundle TV, thus we shall begin by the description of a functor Γ from real finite dimensional vector spaces to Riemannian symmetric spaces.

Let first $H_n = \mathrm{GL}(n,\mathbb{R})^+/\mathrm{SO}(n,\mathbb{R})$, it is a homogeneous space which is the product $\mathbb{R} \times \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ through the map:

$$g \to (\mathrm{Log}\ \det(g), \det(g)^{-1/n} g)\,.$$

We endow it with the $\mathrm{GL}(n,\mathbb{R})^+$ left invariant metric which is the product of the standard metric on $\mathbb{R}$ by the standard $\mathrm{SL}(n,\mathbb{R})$ left invariant metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

In this way we get a Riemannian globally symmetric space which is:

a) Of non positive sectional curvature and contractible.

b) Equivariantly oriented for all n, and equivariantly Spin for n even.

We refer to [22] for a). To prove b) we have to check that the isotropy representation of $\mathrm{SO}(n,\mathbb{R})$, $\lambda : \mathrm{SO}(n,\mathbb{R}) \to \mathrm{SO}(q,\mathbb{R})$, $q = \dim \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, lifts to $\mathrm{Spin}\,(q)$ for n even. Let $n > 1$, then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ and we just have to show that the non trivial element of $\pi_1(\mathrm{SO}(n))$ is sent to $0 \in \pi_1(\mathrm{SO}(q))$ by λ. Let $q = 2p$ (resp. $2p+1$), then, expressed in terms of the weights: $\pm\mu_1, \ldots, \pm\mu_p$ of the standard representation of $\mathrm{SO}(q)$ (resp. $\pm\mu_1, \ldots, \pm\mu_p$, 0 for odd q), the weights of the Spin representation are:

$$\frac{1}{2}\left(\pm\mu_1 \pm \mu_2 \pm \ldots \pm \mu_p\right).$$

Thus it is enough to check that $\frac{1}{2}\sum_1^p \mu_j$, when pulled back to S^1 via a homomorphism $S^1 \xrightarrow{\rho} \mathrm{SO}(n)$ generating π_1, composed with λ: $\mathrm{SO}(n) \to \mathrm{SO}(q)$, gives a weight of S^1. Let us take ρ as follows:

$$\rho(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & \\ & & 0 \\ -\sin\theta & \cos\theta & \\ & 0 & 1 \end{bmatrix}.$$

The non zero weights of the pull-back of $\pm\mu_j$ to S^1 give:

$\underline{n = 2m}$, ± 2 with multiplicity 1, ± 1 with multiplicity $2 \times \frac{(m-1)(m-2)}{2}$

$\underline{n = 2m+1}$, ± 2 with multiplicity 1, ± 1 with multiplicity $1 + 2\,\underline{\frac{(m-1)(m-2)}{2}}$.

Thus the conclusion.

Now let E be a finite dimensional oriented real vector space, let $\Gamma(E)$ be the space of Euclidean metrics on E, *i.e.* of positive definite quadratic forms on E. Given a linear isomorphism $\varphi : E \to F$ set

$$(\Gamma(\varphi))(\xi) = q(\varphi^{-1}\xi) \qquad \forall\, \xi \in F,\ q \in \Gamma(E).$$

Thus the space $\Gamma(E)$ inherits all the $\mathrm{GL}(n,\mathbb{R})^+$ invariant structure of the space H_n and we can state:

Lemma 5.1

a) Γ *is a functor from the category of oriented vector spaces and linear isomorphisms to the category of oriented Riemannian manifold, with a Spin structure in the even dimensional case.*

b) *Any two points* P, Q *of* $\Gamma(E)$ *can be joined by a unique geodesic.*

c) *For any geodesic triangle* A, B, C *in* $\Gamma(E)$ *with sides* a, b, c *one has* $c^2 \geq a^2 + b^2 - 2ab \cos < C$.

See [22], [33] for the statements b) c).

Now let V be an oriented manifold, n its dimension. Then $\Gamma(TV)$ is a bundle with base V and fiber $p^{-1}\{x\} = \Gamma(T_x(V))$, $\forall\, x \in V$. We let W be the total space of this bundle, p the projection $W \to V$.

Let $q = \dim H_n = \frac{n(n+1)}{2}$, then $\dim W = q + n$. Let us endow W with the following G_q structure. We let $F \subset TW$ be the subbundle of TW formed by vertical vectors: $F_x = \operatorname{Ker} p_*$. Since the fibers are Riemannian we have a canonical Euclidean structure on F which is moreover canonically oriented by the orientation of the fibers. At each point y of W the tangent map p_* to the projection p gives a natural isomorphism of $T_y(W)/F_y$ with $T_{p(y)}(V)$. As by definition of the fiber $p^{-1}\{x\} = \Gamma(T_x(V))$, y is a Euclidean structure on $T_x(V)$ it follows that the quotient bundle $T(W)/F$ has a canonical Euclidean structure.

Now we choose, for instance from any affine connection on V, a subbundle N of TW, such that for every $y \in W$, N_y and F_y satisfy $N_y \cap F_y = \{0\}$, $N_y + F_y = T_y(W)$.

Let us then endow W with the unique Riemannian metric such that

$$N_y = F_y^+\,,\ \|\xi\| = \|\xi\|_F\ \xi \in F_y\,,\ \|\xi\| = \|\xi\|_{T/F}\,, \quad \forall\, \xi \in N_y\,.$$

Lemma 5.2 *Let* φ *be an orientation preserving diffeomorphism of* V *and* $\psi = \Gamma(\varphi)$ *be given by* $\psi/p^{-1}\{x\} = \Gamma(T_x(\varphi))$, $\forall\, x \in V$. *(Here* $T_x(\varphi) : T_x(V) \to T_{\varphi(x)}(V)$ *is the tangent map to* φ *at* $x \in V$.*) Then* ψ *is a diffeomorphism of* W *whose tangent map at any point has the form:* $\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix}$ *with* g_{11} *and* g_{22} *orthogonal and orientation preserving.*

Proof. By construction ψ maps the fiber $p^{-1}\{x\}$ to the fiber $p^{-1}(\varphi(x))$ by the orientation preserving isometry $\Gamma(T_x(\varphi))$, thus $g_{12}=0$ and g_{11} is orthogonal and orientation preserving. One has $p\psi(y)=\varphi p(y)$, $\forall\, y\in W$, thus for any tangent vector $X\in T_y(W)$, $y\in p^{-1}\{x\}$ one has $p_*\psi_*(X)=T_x(\varphi)\,p_*(X)$.

Let $X\in N_y$, $\psi_*(X)=Y+Z$ with $Y\in N_{\psi(y)}$, $Z\in F_{\psi(y)}$. We have to check that $\|Y\|=\|X\|$. We have:

$$\begin{aligned}\|Y\|^2 &= \|Y\|^2_{T/F}+\|Y+Z\|^2_{T/F}=\|\psi_*(X)\|^2_{T/F}\\ &= \psi(y)(p_*\psi_*(X))=\psi(y)(T_x(\varphi)\,p_*(X))\\ &= (\Gamma(T_x(\varphi))\,y)(T_x(\varphi)\,p_*(X))=y(p_*(X))=\|X\|^2\,.\end{aligned}$$

$\bullet$

Thus if we started from a discrete group Γ acting by orientation preserving diffeomorphisms of V we get a natural action of Γ on W which preserves a G_q structure. Thus Theorem 4.5 shows that the fundamental class of W/Γ yields a map of $K_i(C_0(W)\rtimes\Gamma)$ to $\mathbb{C}$ ($i=\dim W$ modulo 2). Since we can view "W/Γ" as the total space of a bundle over "V/Γ" with fiber $H_n=\mathrm{GL}(n,\mathbb{R})^+/\mathrm{SO}(n,\mathbb{R})$ we shall now proceed to construct a Thom map:

$$\beta : K_j(C_0(V)\rtimes\Gamma)\to K_{j+q}(C_0(W)\rtimes\Gamma)$$

($q=\dim H_n=\frac{n(n+1)}{2}$).

For this we shall assume that n is even (one obtains the general case by crossing V by $\mathbb{R}$ on which Γ acts trivially). Then the fibers of $W\xrightarrow{p}V$ are Spin contractible Riemannian manifolds of negative curvature, which by the ideas of Mischenko [32] and Kasparov [26], [27] yields an element of the Γ equivariant bivariant group $KK_\Gamma(C_0(V),C_0(W))$.

Let us describe carefully this element β_0.

We let E' be the hermitian complex vector bundle on W associated to the Spin representation of Spin (n) and to the Spin (n) vector bundle $F=\mathrm{Ker}\,p_*$. By construction E' is Γ-equivariant and the action of Γ is isometric. Moreover E' is canonically $\mathbb{Z}/2$ graded if $q=\frac{n(n+1)}{2}$ is even, *i.e.* if n is divisible by 4.

Let $\mathcal{E}=C_0(W,E')$ be the C^* module over $C_0(W)$ formed of continuous sections of E' vanishing at ∞. From the equivariance of E' we get an action of Γ on $\mathcal{E}$ such that, for any $g\in\Gamma$:

$$g(\xi f)=g(\xi)\,g(f)\,,\qquad \forall\,\xi\in\mathcal{E}\,,\ f\in C_0(W)$$

$$\langle g(\xi_1), g(\xi_2)\rangle = g\langle \xi_1, \xi_2\rangle \qquad \forall\, \xi_1, \xi_2 \in \mathcal{E}\,.$$

Moreover $\mathcal{E}$ is in a natural way a left $C_0(V)$ module by

$$(f\xi)(y) = f(p(y))\,\xi(y) \qquad \forall\, y \in W\,,\ f \in C_0(V)\,,\ \xi \in \mathcal{E}\,.$$

One has $g(f\xi) = g(f)\,g(\xi)$, $\forall\, g \in \Gamma$.

To get an element of $KK^i_\Gamma(C_0(V), C_0(W))$ it remains to define an endomorphism S of $\mathcal{E}$ such that: (*cf.* [27])

a) $f(S^2-1)$, $f(S-S^*)$ are compact endomorphisms for any $f \in C_0(V)$.

b) $[S,f]$ is a compact endomorphism for any $f \in C_0(V)$.

c) $f(S-S^g)$ is a compact endomorphism for any $f \in C_0(V)$, $g \in \Gamma$.

To define S we make two auxiliary choices. First we choose a smooth section s of the bundle $\Gamma(T) \xrightarrow{p} V$, or in other words we choose a Riemannian metric on V (of course not Γ invariant). Next we let, using Lemma 5.1b), σ be a continuous section of the verticle bundle F such that:

There exists a closed subset $K \subset W$ on which the projection $p : K \to V$ is proper and outside which $\sigma(y)$, $y \in p^{-1}\{x\}$, is the unit tangent vector at y defined by the geodesic joining (in the fiber) y to $s \circ p(y)$.

The equality $(S\xi)(y) = C(\sigma(y))\,\xi(y)$, where C is Clifford multiplication, then defines an endomorphism of $\mathcal{E}$, odd for the $\mathbb{Z}/2$ grading if $\frac{n(n+1)}{2}$ is even.

Lemma 5.3 *The pair $(\mathcal{E}, S)$ defines an element β_0 of $KK^i_\Gamma(C_0(V), C_0(W))$, $i = \frac{n(n+1)}{2}$ modulo 2.*

Proof. By construction $\mathcal{E}$ is the C^* module over $C_0(W)$ corresponding to the hermitian vector bundle E'. Thus endomorphisms of $\mathcal{E}$ correspond to continuous bounded sections of End E' and compact endomorphisms to those continuous sections of End E' whose norm tends to 0 at ∞ in W. Let us check conditions a) b) c).

a) To show that $f(S^2-1)$ is compact one may assume that $f \in C_c(V)$. Then as $K \cap \text{Support}\,(f \circ p)$ is compact and as $S^2 = 1$ outside K we see that $f(S^2-1)$ is a continuous section with compact support of End E'. The same applies to $f(S-S^*)$, since $S = S^*$ outside K.

b) One has $[f,S] = 0$, $\forall\, f \in C_0(V)$.

c) Let $g \in \Gamma$, then the endomorphism $S^g = gSg^{-1}$ is given by:

$$(S^g \xi)(y) = C(\sigma^g(y))\, \xi(y) \qquad \forall\, \xi \in \mathcal{E}\,,\ y \in W$$

where $\sigma^g(y) = g\sigma(yg)g^{-1}$, $\forall\, y \in W$.

Let s^g be the section of the bundle $W = \Gamma(T) \xrightarrow{p} V$ given by $s^g(x) = gs(xg)$, then we can assert that for any $y \notin Kg^{-1}$, $\sigma^g(y)$ is the unit tangent vector at y (in the fiber) defined by the geodesic joining y to $s^g(p(y))$.

Given $f \in C_c(v)$, the distance $d(s(x), s^g(x))$ (in the fiber) is bounded on the support of f. Hence by Lemma 5.1c) we get that $\|\sigma - \sigma^g\|$ as a function on W restricted to Support $(f \circ p)$ tends to 0 at ∞. Thus $f(S - S^g)$ is a compact endomorphism of $\mathcal{E}$. $\bullet$

Lemma 5.4 *Let I_V (resp. I_W) be the canonical homomorphism of $C_0(V)$ resp. $C_0(W)$) in $A = C_0(V) \times \Gamma$ (resp. $A = C_0(W) \times \Gamma$). Let $\beta \in KK(A, A')$ be associated to β_0 as in* [27]*, and β_1 be the restriction of β_0 to be the trivial subgroup $\{1\} \subset \Gamma$. Then:*

$$I_V \otimes_A \beta = \beta_1 \otimes_{C_0(W)} I_W \ \text{ in } KK(C_0(V), A')\,.$$

Here we consider I_V (and I_W) as an element of $KK(C_0(V), A)$ (resp. $KK(C_0(W), A')$).

Proof. Let us first describe the Kasparov bimodule $\mathcal{E}' = \mathcal{E} \times \Gamma$ (reduced crossed product). As a C^* module over $A = C_0(W) \times \Gamma$ it is obtained exactly as in Section 3 from the equivariant hermitian bundle $E' = S(F)$, *i.e.*:

1) $\mathcal{E}'$ is the completion of $C_c(W) \rtimes \Gamma$, $r^*(E')$ with

$$\langle \xi, \eta \rangle(x, g) = \sum_{h \in \Gamma} \langle \xi(xh, h^{-1}), \eta(xh, h^{-1}g) \rangle$$

$$(\xi f)(x, g) = \sum_{h \in \Gamma} \xi(x, h)\, f(xh, h^{-1}g)\,.$$

2) The left action of $A = C_0(V) \rtimes \Gamma$ on $\mathcal{E}'$ is given, using the action of Γ on E' by:

$$(f\xi)(x, g) = \sum_{h \in \Gamma} f(p(x), h)\, h\xi(xh, h^{-1}g)\,.$$

3) The endomorphism S' is given by

$$(f\xi)(x, g) = c(\sigma(x))\, \xi(x, g)\,.$$

Note that now the action of Γ on E' no longer enters in any of the formulae 1) 2') 3) and that the action of $C_0(V)$ on $\mathcal{E}'$ *commutes exactly* with the operator S'.

Next I_W, as an element of $KK(C_0(W), A')$ is described by the Kasparov bimodule A' with $C_0(W)$ acting by left multiplication, and $\mathcal{E} \otimes_{C_0(W)} A'$ is canonically isomorphic to $\mathcal{E}'$ as a C^* module over A'. As this isomorphism is compatible with the left actions of $C_0(V)$ (on $\mathcal{E}$ as in Lemma 5.3 and on $\mathcal{E}'$ as in 2)'), they are isomorphic as $(C_0(V), A')$ bimodules. Finally as $S \otimes 1 = S'$, where 1 is the identity endomorphism of A' we see that the Kasparov bimodule $(\mathcal{E}', S')$ over $(C_0(V), A')$ is isomorphic to $(\mathcal{E}, S) \times_{C_0(W)} A'$ hence the equality of the Lemma. •

We shall now combine Lemma 5.4 with Corollary 4.6 to give a positive answer to the problem that we posed in [11] p. 588. We take a fixed connected component of V and consider the element $[V_0]^*$ of $K^*(V) = K_*(C_0(V))$ which is the pushforward $f!(1)$ of the trivial line bundle on a one point space $\{p\}$ by the map f, $f(p) \in V_0$. More explicitly this element has support in a small disk $D \subset V$, $\dim D = \dim V$ and is the generator of $K_c^*(D)$ which fits with the orientation, *i.e.*, the integral over D of its Chern character is equal to 1. Let

$$I_V : C_0(V) \to A = C_0(V) \rtimes \Gamma \text{ and } [V_0/\Gamma]^* = I_V [V_0]^* .$$

Theorem 5.5 *Let Γ be a discrete group acting by orientation preserving diffeomorphisms of the (not necessarily compact) manifold V of dimension n. Let $i \in \{0,1\}$ be equal to n* modulo 2, *then $K_i(C_0(V) \rtimes \Gamma)$, the K theory of the reduced crossed product C^* algebra, always contains a non-trivial copy of $\mathbb{Z}$: the elements $k\,[V_0/\Gamma]^*$, $k \in \mathbb{Z}$ where V_0 is any connected component of V.*

Proof. Replacing V by $V \times \mathbb{R}$, where Γ acts trivially on $\mathbb{R}$ we may assume that n is even. Then if $s : V \to W$ is any continuous section of $W \xrightarrow{p} V$, it is K-oriented by the Spin structure of the fibers and one has $\beta_1 = s!$ in the sense of [11]. Let f be a map from the one point space $\{p\}$ to V_0, and W_0 the connected component of W above V_0. As $s \circ f$ maps p to W_0 one has (*cf.* [15]):

$$[W_0]^* = (s \circ f)!\,(1) = [V_0]^* \otimes \beta_1 .$$

Thus by Lemma 5.4:

$$I_W([W_0]^*) = [V_0]^* \otimes \beta_1 \otimes I_W = [V_0]^* \otimes I_V \otimes \beta = [V_0/\Gamma]^* \otimes \beta .$$

Hence it is enough to show that $kI_W([W_i]^*) \neq 0$, $\forall\, k \neq 0$, which follows from Corollary 4.6 and Lemma 5.2. $\bullet$

Theorem 5.6 *Let Γ be a discrete group acting by orientation preserving diffeomorphisms of the not necessarily compact manifold V. Let $P(p_i)$ be a polynomial with complex coefficients, in the Pontrjagin classes p_i of V. Then there exists an additive map of $K_i(C_0(V) \rtimes \Gamma)$ ($i = \dim V(2)$) to $\mathbb{C}$ such that:*

$$\varphi(I(x)) = \langle \mathrm{ch}\, x \cdot P(p_i), [V] \rangle \qquad \forall\, x \in K^i(V)\,.$$

Proof. By Lemma 5.2 the real vector bundle TW/F over W is naturally Γ-equivariant with a Γ-invariant Euclidean structure. By Corollary 4.7 and Lemma 5.2 there exists for any polynomial Q in the Pontrjagin classes of TW/F an additive map φ of $K_j(C_0(W) \rtimes \Gamma) = K_j(A')$ to $\mathbb{C}$ ($j = \dim W$) such that:

$$\varphi(I_W(y)) = \langle \mathrm{ch}\, y\, Q(p_i), [W] \rangle \qquad \forall\, y \in K^j(W)\,.$$

Now TW/F is the pull back by $p : W \to V$ of TV so that

$$\varphi(I_W(y)) = \langle \mathrm{ch}\, y\, p^* Q(p_i(V)), [W] \rangle\,.$$

From Lemma 5.4 there exists an additive map $\Psi : K(A) \to \mathbb{C}$ (namely $\Psi(z) = \varphi(z \otimes_A \beta)$) such that

$$\Psi(I_V(x)) = \langle \mathrm{ch}\,(x \otimes_{C_0(V)} \beta_1)\, p^* Q(p_i(V)), [W] \rangle$$

for any $x \in K^0(V)$.

Now topologically the bundle $W \xrightarrow{p} V$ is homeomorphic to the vector bundle obtained from TV (with structure group $\mathrm{SO}(n, \mathbb{R})$, n even) using the isotropy representation fo $\mathrm{SO}(n, \mathbb{R})$ in $\mathrm{GL}^+(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$. In particular since n is even this bundle is a Spin bundle and one has the Thom isomorphism: $K^0(V) \to K^j(W)$ given by $z \in K^0(V) \to z \otimes_{C_0(V)} \beta_1$.

Since this bundle is oriented one also has the Thom isomorphism in cohomology $t : H_c^*(V) \to H_c^*(W)$ and hence:

$$\Psi(I_V(x)) = \langle t^{-1}\, \mathrm{ch}\,(x \otimes_{C_0(V)} \beta_1)\, Q(p_i(V)), [V] \rangle \qquad \forall\, x \in K^0(V)\,.$$

Thus by [1] there exists a polynomial $T(p_i)$ in the Pontrjagin classes of the above bundle and hence of V, with leading term 1 such that, for any $x \in K^0(V)$:

$$t^{-1}\,\mathrm{ch}\,(x \otimes_{C_0(V)} \beta_1) = \mathrm{ch}\, x\, T(p_i)\,.$$

Since T is invertible in $H^*(V,\mathbb{C})$ we can choose Q such that $QT = P$ hence the general result.

Corollary 5.7 *Let V be a compact oriented $4k$ dimensional manifold with some non zero Pontrjagin number. Then for any discrete group Γ of diffeomorphisms of V the unit 1_A of $A = C(V) \rtimes \Gamma$ is a non torsion element of $K_0(A)$.*

(Note that Γ necessarily preserves the orientation).

The corollary is obtained from Theorem 5.6 by taking for x the class of the trivial line bundle on V.

It applies for instance to any countable group of diffeomorphisms of $P_2(\mathbb{C})$.

Note that it is not true in general that if Γ is a countable group of homeomorphisms of a compact space X the unit 1_A of $A = C(X) \rtimes \Gamma$ is non trivial in $K_0(A)$. It is true if Γ is an amenable discrete group (then to an invariant probability measure on X corresponds a trace τ on A with $\tau(1) = 1$), but fails for an obvious action of the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor space, *cf.* [17].

In Section 6 we shall prove:

Proposition. *Let Γ be a cocompact torsion free discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$. *Let it act on $P_1(\mathbb{R})$ in the obvious way, then the unit 1_A, $A = C(P_1) \rtimes \Gamma$, is a torsion element of $K_0(A)$.*

The next theorem shows that the situation is completely different for arbitrary countable subgroups of $\mathrm{PSL}(2,\mathbb{C})$. (See Corollary 5.9.)

Theorem 5.8 *Let Γ be a discrete group of orientation preserving diffeomorphisms of the not necessarily compact manifold V. Let P be an element of the subring of $H^*(V,\mathbb{C})$ generated by the Chern classes of the Γ-equivariant complex vector bundles on V (not necessarily hermitian equivariant).*
Then there exists an additive map of $K(C_0(V) \rtimes \Gamma$ to $\mathbb{C}$ such that

$$\varphi(I(x)) = \langle \mathrm{ch}\, x \cdot P, [V]\rangle \qquad \forall\, x \in K^{\dim V}(V)\,.$$

Proof. Let $E^1, \ldots, E^k$ be Γ-equivariant bundles on V, $E = \otimes E^i$ their sum. Tensoring each E^i by a suitable trivial line bundle which is Γ-equivariant one may assume that for each i the bundle $\wedge^{\dim E_i}(E_i)$ has a Γ-invariant hermitian metric $\| \, \|$. Now for each x, let $P(E_x)$ be the space of hermitian metric on the fiber E_x which are such that:

a) The subspaces E_x^i are pairwise orthogonal.

b) For any i and any orthonormal basis e_j^i of E^i one has

$$\|e_1^i \wedge \ldots \wedge e_{\dim(E^i)}^i\| = 1\,.$$

By construction $P(E_x)$ is a product of Riemannian symmetric spaces of the form $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$, thus it inherits a canonical structure of Spin Riemannian manifold simply connected with non positive sectional curvature. (Here $\mathrm{SU}(n)$ is simply connected so the existence of an invariant Spin structure is automatic.) Thus exactly as above we get a natural element β of $KK_\Gamma(C_0(V), C_0(X))$ where X is the total space of $P(E)$, whose restriction to $\{e\} \subset \Gamma$ is the Thom isomorphism : $K^*(V) \to K^*(X)$.

Now by construction the pull back to X of any of the Γ-equivariant bundles E^i has a Γ-invariant hermitian metric. Using a suitable element of $KK(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma)$ it follows (as in Corollary 4.7) that using Theorem 5.6, there exists for any product $\mathrm{ch}\,(E^{i_1}) \ldots \mathrm{ch}\,(E^{i_k}) = \mathrm{ch}\,(E^{i_1} \otimes \ldots \otimes E^{i_k})$ of Chern characters of the E^i's an additive map $\Psi : K(C_0(X)) \rtimes \Gamma \to \mathbb{C}$ with:

$$\Psi(I_X(y)) = \langle \mathrm{ch}\, y\, q^*(\Pi\, \mathrm{ch}\,(E^{i_q})), [X]\rangle \qquad \forall\, y \in K^*(X)\,.$$

As in Theorem 5.6 we get a map $\varphi : K(C_0(V) \rtimes \Gamma) \to \mathbb{C}$ such that for any $x \in K^*(V)$:

$$\varphi(I_V(x)) = \langle \mathrm{ch}\,(\mathrm{Thom}\,(x))\, q^*(\Pi\,(\mathrm{ch}\,(E^{i_q})), [X]\rangle\,.$$

Let t be the Thom isomorphism in cohomology, then $t^{-1}\mathrm{ch}\,(\mathrm{Thom}\,(x)) = \mathrm{ch}\, x\, Q$ where Q is an invertible element of the subring of $H^*(V,\mathbb{C})$ generated by the $\mathrm{ch}\,(E^i)$. Using linear combination of the maps φ's one gets the conclusion. $\bullet$

Corollary 5.9 *Let Γ be any countable subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ acting on $P_n(\mathbb{C})$. Then the canonical homomorphism of $C(P_n(\mathbb{C}))$ to $A = C(P_n(\mathbb{C})) \rtimes \Gamma$ is an injection in K-theory. In particular 1_A is a non torsion element in $K_0(A)$.*

6 Pairing of the fundamental class of V/Γ with the geometric group $K^*(V,\Gamma)$

In our joint work with P. Baum ([3]) we defined, for any discrete group Γ acting by diffeomorphisms on a manifold V, a geometric group $K^*(V,\Gamma)$ and a map:

$$\mu : K^*(V,\Gamma) \to K_*(C_0(V) \times \Gamma)\,.$$

We shall first show the usefulness of such a map in proving the vanishing of certain elements of $K_*(C_0(V)\times\Gamma)$. Then we shall put together the results of Section 5 with those of [15], [3] to compute $\langle \mu(x), [V/\Gamma]\rangle$ for any $x \in K^*(V,\Gamma)$.

For simplicity let us assume that Γ is *torsion free* and describe the features of $K^*(V,\Gamma)$ which are relevant here.

Let $E\Gamma \xrightarrow{\pi} B\Gamma$ be the universal principal Γ-bundle over the classifying space $B\Gamma$ of Γ. Let $V_\Gamma = V \times_\Gamma E\Gamma$, where Γ acts on the right on V and on the left on $E\Gamma$. This space V_Γ is known as the *homotopy quotient* of V by Γ, it is unique up to homotopy and is the total space of the bundle over $B\Gamma$ with fiber V.

Any Γ-equivariant bundle F on V is still Γ-equivariant on $V \times E\Gamma$ and hence drops down to a bundle on V_Γ. This applies in particular to the tangent bundle TV of V yielding a bundle τ on V_Γ. At a formal level the geometric group $K^*(V,\Gamma)$ is defined as follows: ([3]).

Definition 6.1 *$K^*(V,\Gamma)$ is the K homology of the pair $(B\tau, S\tau)$ of the unit ball, unit sphere bundle of τ over V_Γ.*

Since $V_\Gamma = V \times_\Gamma E\Gamma$ is not in general a finite simplicial complex we have to be precise regarding the definition of K homology for arbitrary simplicial complexes X. We take $K_*(X) = \varinjlim K_*(Y)$ where Y runs through compact subsets of X. In other words we choose K homology *with compact supports* in the sense of [38] Axiom 11 p. 203.

Let $H_*^\varepsilon(V_\Gamma,\mathbb{Q})$ be the ordinary singular homology of the pair $(B\tau, S\tau)$ over V_Γ, and with coefficients in $\mathbb{Q}$. Since it is also a theory with compact supports, the Chern character

$$\mathrm{ch} : K^*(V,\Gamma) \to H_*^\varepsilon(V_\Gamma,\mathbb{Q})$$

is a rational isomorphism.

Since we are interested mainly in the case of orientation preserving diffeomorphisms, let us assume that V is oriented and that Γ preserves this orientation. Then the bundle τ over $V_\Gamma = V \times_\Gamma E\Gamma$ is still oriented and letting U be the orientation class of τ on V_Γ, we can use the Thom isomorphism ([38] Theorem 10, p. 259):

$$\Phi : H^\tau_{q+n}(V_\Gamma, \mathbb{Q}) \to H_q(V_\Gamma, \mathbb{Q}) \qquad (n = \dim \tau = \dim V)$$

(where $\Phi(z) = p_*(U \cap z)$, $\forall\, z \in H_{q+n}((B\tau, S\tau), \mathbb{Q})$ and where p is the projection from $B\tau$ to the base V_Γ).

Thus $\Phi \circ \mathrm{ch}$ is a rational isomorphism:

$$\Phi \circ \mathrm{ch} : K^*(V, \Gamma) \to H_*(V_\Gamma, \mathbb{Q})\,.$$

To construct the map $\mu : K^*(V,\Gamma) \to K_*(C_0(V) \rtimes \Gamma)$ requires a better understanding of the K homology of an arbitrary pair (here the pair $B\tau, S\tau$ over V_Γ), this follows from:

Proposition 6.2

a) *Let M be a* Spin^c *manifold with boundary, assume that M is compact, then one has a Poincaré duality isomorphism:*

$$K^*(M) \simeq K_*(M, \partial M)\,.$$

b) *Let (X, A) be a topological pair, and $x \in K_*(X, A)$. Then there exists a compact* (Spin^c) *manifold with boundary $(M, \partial M)$, a continuous map $f : (M, \partial M) \to (X, A)$ and an element y of $K_*(M, \partial M)$ with $f_*(y) = x$.*

The Chern character is then uniquely characterized by the properties:

1) $f_*\,\mathrm{ch}(y) = \mathrm{ch}(f_*(y))$.

2) If M is a compact Spin^c manifold with boundary, and $z \in K_*(M, \partial M)$ is the image of $y \in K^*(M)$ under Poincaré duality, one has:

$$\mathrm{ch}\, z = (\mathrm{ch}\, y \cdot Td(M)) \cap [M, \partial M]\,.$$

(Here $Td(M)$ is the characteristic class associated to the Spin^c structure of M as in [2].)

Now let (N, F, g) be a triple where N is a compact manifold without boundary, $F \in K^*(N)$, g is a continuous map from N to $V_\Gamma = V \times_\Gamma E\Gamma$, which is K-oriented, *i.e.*, such that the bundle $TN \oplus g^*\tau$ is gifted with a Spin^c structure. To such a triple corresponds an element of $K_*(B\tau, S\tau)$ as follows. Let B, S be the unit ball, unit sphere bundle of $g^*\tau$ on N, then B is a Spin^c manifold with boundary so that the Poincaré duality isomorphism assigns a class $y \in K_*(B, S)$ to the pull back of F to B. Then put $[(N, F, g)] = g_*(y) \in K_*(B\tau, S\tau)$. For convenience any triple (N, F, g) as above will be called a K-cycle.

Proposition 6.3

a) *Any element of* $K^*(V, \Gamma) = K_*(B\tau, S\tau)$ *is of the form* $[(N, F, g)]$ *for some* K *cycle* (N, F, g).

b) *Let* (N, F, g) *be a* K *cycle,* N' *be a compact manifold and* $f : N' \to N$ *a continuous map which is* K*-oriented:* i.e. $TN' \oplus f^*TN$ *is gifted with a* Spin^c *structure, then, for any* $F' \in K^*(N')$ *with* $f_!(F') = F$ *one has:*

$$[(N', F', g \circ f)] = [(N, F, g)] \quad in \quad K_*(B\tau, S\tau)\,.$$

Here $f! : K^*(N') \to K^*(N)$ is the pushforward map in K theory (*cf.* [2]).

Proof. a) By Proposition 6.2b) there exist a compact Spin^c manifold with boundary $(M, \partial M)$, an element y of $K_*(M, \partial M)$ and a continuous map $f : (M, \partial M) \to (B\tau, S\tau)$ with $x = f_*(y)$. By transversality one may assume that the inverse image in M of the 0 section of τ is a sub-manifold N of M with normal bundle ν the restriction of $f^*(\tau)$ to N. Since the boundary of M maps to $S\tau$, the manifold N is closed without boundary. Let g be the restriction of f to N then g is a continuous map from N to V_Γ and the bundle $TN \oplus g^*\tau = TN \oplus \nu$ is gifted with a Spin^c structure. Let B be the unit ball bundle $B \xrightarrow{p} N$ of the bundle $\nu = g^*\tau$. Then since $p^* : K^*(N) \to K^*(B)$ is an isomorphism the answer follows from Proposition 6.2a).

b) Let (B, S), (B', S') be the unit ball, unit sphere bundle of $g^*\tau$ and $f^*g^*\tau$ over N and N', and $\tilde{f} : (B', S') \to (B, S)$ the natural extension of f. One has $\tilde{f}!(p'^*(F')) = p^*(F)$, thus the conclusion follows since $\tilde{f}!$ is Poincaré dual to $\tilde{f}_* : K_*(B', S') \to K_*(B, S)$. •

If we translate the Chern character $\Phi \circ \mathrm{ch}$ in terms of K-cycles we get:

Proposition 6.4 *Let* (N, F, g) *be a* K *cycle, then* $\Phi \circ \mathrm{ch}([N, F, g]) = g_*(\mathrm{ch}\, F \cdot Td(TN \oplus g^*\tau) \cap [N]) \in H_*(V_\Gamma, \mathbb{Q})$.

Note that we assumed that τ was oriented, hence N is oriented since the bundle $TN \oplus g^*\tau$ is Spin^c hence oriented.

The proof of the proposition is straightforward.

Let us now give another description of K cycles using the equality $V_\Gamma = V \times_\Gamma E\Gamma$. Thus on V_Γ one has a principal Γ bundle: $V \times E\Gamma$ with projection p. For $x \in V$, $t \in E\Gamma$ one has $(x,t)g = (xg, g^{-1}t)$, $\forall g \in \Gamma$. Given a K cycle (N, F, g) one can pull back to N the principal Γ-bundle of V_Γ. One gets in this way a Γ principal bundle $\tilde{N} \xrightarrow{q} N$ over N and a continuous map $\tilde{g} : \tilde{N} \to V \times E\Gamma$ which is Γ-equivariant. Since $E\Gamma$ is contractible we need only retain the continuous Γ-equivariant map:

$$h = \mathrm{pr}_V \circ \tilde{g} : \tilde{N} \to V\,.$$

Thus we see that we can equivalently describe a K-cycle by a Γ-principal bundle $\tilde{N} \xrightarrow{q} N$, an element F of $K^*(N)$, a Γ-invariant Spin^c structure on the bundle $T\tilde{N} \oplus h^*\,TV$. We do not have to assume that N is compact, but we do assume that F is an element of K theory with compact support. Let us describe $\mu(N, F, h)$ for such a K-cycle.

By construction $h : \tilde{N} \to V$ is a K-oriented map, thus by [15] one can associate to h a Kasparov bimodule $(\mathcal{E}, F)$ over $(C_0(\tilde{N}), C_0(V))$. Since h is Γ equivariant *and the action of* Γ *on* $\tilde{N}$ *is proper* it is not difficult to turn this Kasparov bimodule in a Γ-equivariant one.

Let us describe this element $h!$ of $KK_\Gamma(C_0(\tilde{N}), C_0(V))$. The action of Γ on $\tilde{N} \times V$, given by $(x,y)g = (xg, yg)$, $\forall\, x \in \tilde{N}$, $y \in V$, $g \in \Gamma$, is proper. Moreover one can assume that h is smooth. As Graph $(h) \subset \tilde{N} \times V$ is a Γ-invariant submanifold there exist a Γ-invariant neighbourhood M of Graph (h) in $\tilde{N} \times V$ and a Γ equivariant isomorphism X of the bundle (M, p), $p(x,y) = x$, over $\tilde{N}$ with the bundle $h^*(TV)$.

Thus $X(x,y) \in T_{h(x)}(V)$, $\forall (x,y) \in M$ and one has:

$$X(xg, yg) = g^{-1} X(x,y) \qquad \forall (x,y) \in M\,,\ g \in \Gamma\,.$$

Now let us endow the bundles $T\tilde{N}$ and $h^*\,TV$ over $\tilde{N}$ with Γ invariant Euclidean structures. Let then S be the Γ-equivariant hermitian bundle of Spinors associated to the Γ-invariant Spin^c structure of $T\tilde{N} \oplus h^*\,TV$.

For each $y \in V$, let $\Omega_y = \{x \in \tilde{N}, (x,y) \in M\}$. One has:

$$\Omega_{yg} = (\Omega_y)\, g \qquad \forall\, y \in V\,,\ g \in \Gamma\,.$$

Let $H_y = L^2(\Omega_y, S)$ be the Hilbert space of L^2 sections of the restriction of S to Ω_y. Since S is Γ-equivariant, each $g \in \Gamma$ defines an isometry $\xi \to g(\xi)$ of H_{yg} on H_y given by:

$$g(\xi)(x) = g(\xi(x,g)) \qquad \forall\, x \in \Omega_y\,.$$

Finally, as in [11], [15], we can define on each Ω_y a symbol of order 0, $\sigma_y(x,\xi)$, where $\xi \in T_x(\tilde{N})$, $\|\xi\| = 1$, trivial at ∞ in each Ω_y, by the equality:

$$\sigma_y(x,\xi) = \mathrm{Cliff}\,(m(x,y)\,\xi + n(x,y)\,X(x,y))$$

where:

1) Cliff means Clifford multiplication: $S_x \to S_x$ by the vector in $T_x\,\tilde{N} \oplus T_{h(x)}(V)$.

2) $m(x,y)$ and $n(x,y)$ are positive scalars depending only upon $\|X(x,y)\| = \rho$ and such that:

a) $m^2 + n^2\rho^2 = 1$

b) If $\rho \leq 1$ then $m = 1$, $n = 0$

c) If $\rho \geq 2$ then $m = 0$.

It follows that $m(xg,yg) = m(x,y)$, $n(xg,yg) = n(x,y)$, and that the transformed $(\sigma_y)^g$ of σ_y by $g \in \Gamma$ is equal to σ_{yg}. Using the Γ-invariant Riemannian metric on N we have a Γ equivariant manner to replace each σ_y by a bounded pseudodifferential operator of order 0, F_y, on $L^2(\Omega_y, S)$, with σ_y as principal symbol.

Thus $g^{-1} F_y\, g = F_{yg}$ with obvious notations.

Let $C_0(\tilde{N})$ act in each H_y by left multiplication. Then the C^* module $\mathcal{E}$ over V associated to the continuous field of Hilbert spaces $(H_y)_{y \in V}$ is a Γ-equivariant bimodule, and with the endomorphism F defined by the family $(F_y)_{y \in V}$ it yields the desired element of $KK_\Gamma(C_0(\tilde{N}), C_0(V))$, noted $h!$.

We did not worry about the $\mathbb{Z}/2$ gradings but it clearly depends on the mod 2 dimension of $T\tilde{N} \oplus h^*(TV)$.

Note also that F is exactly Γ invariant.

Before we state the definition and first properties of the map μ let us recall that for any Γ-principal bundle such as $\tilde{N}$ on N one has a natural Morita equivalence $C_0(\tilde{N}) \rtimes \Gamma \simeq C_0(N)$ ([36]).

Lemma 6.5

a) *There exists an additive map μ of $K^*(V,\Gamma)$ to $K_*(C_0(V) \rtimes \Gamma)$ such that for any K-cycle $(N, \tilde{N}, F, h)$ as above $\mu(N, F, h) = F \otimes h!$, where $F \in K_*(C_0(N))$ is viewed as an element of $K(C_0(\tilde{N} \rtimes \Gamma))$ through the Morita equivalence and $h!$ as an element of $KK(C_0(\tilde{N}) \times \Gamma, C_0(V) \rtimes \Gamma)$.*

b) *If $h : \tilde{N} \to V$ is a submersion, then $h! \in KK_\Gamma(C_0(\tilde{N}), C_0(V))$ coincides with the equivariant family of Dirac operators along the fibers of h.*

We refer to [15], [3]. The proof is just an equivariant form of [15]. We shall now work out the simplest possible case, *i.e.* we take $N = V$, $\tilde{N} = V \times \Gamma$ and the Γ equivariant map $h : \tilde{N} \to V$ is given by: $h(x, g) = xg \in V$, $(x, g) \in V \times \Gamma$. (Here $\tilde{N}$ is the trivial Γ principal bundle, the right action of Γ is given by $(x, g)g' = (x, gg')$.)

Lemma 6.6 *Let $F \in K_c^*(V)$, and consider the K-cycle (N, F, h) where $h : \tilde{N} = V \times \Gamma \to V$ is the above map. Then $\mu(N, F, h)$ is equal to $I_*(F)$ where I is the inclusion $C_0(V) \to C_0(V) \rtimes \Gamma$.*

Proof. Here we have a natural cross section $V \xrightarrow{j} \tilde{N} = V \times \Gamma$ given by $x \to (x, e)$, e being the unit of Γ, so that the Morita equivalence: $K_*(C_0(\tilde{N} \rtimes \Gamma)) \simeq K_*(C_0(V))$ is simply given by the homomorphism $\rho : C_0(V) \to C_0(\tilde{N}) \rtimes \Gamma$ equal to $j_* : C_0(V) \to C_0(\tilde{N})$ composed with $I_{\tilde{N}} : C_0(\tilde{N}) \to C_0(\tilde{N}) \rtimes \Gamma$. Thus it is enough to check that $I_V = \rho \otimes_{C_0(\tilde{N}) \rtimes \Gamma} h!$ in $KK(C_0(V), C_0(V) \rtimes \Gamma)$.

Let $\mathcal{E}$ be the Kasparov bimodule given by the continuous field of Hilbert spaces $H_x = \ell^2(h^{-1}\{x\})$, $x \in V$ over V on which $C_0(\tilde{N})$ acts in the obvious way. Then Γ acts naturally on $\mathcal{E}$ so that $\mathcal{E}$ is an element of $KK_\Gamma(C_0(\tilde{N}), C_0(V))$ and $h!$ as an element of $KK(C_0(\tilde{N}) \rtimes \Gamma, C_0(V) \rtimes \Gamma)$ is just $\mathcal{E} \rtimes \Gamma$. It is obvious that the composition with $\rho : C_0(V) \to C_0(\tilde{N}) \rtimes \Gamma$ yields the Kasparov bimodule I_V (*i.e.* $A = C_0(V) \rtimes \Gamma$ as a C^* module over A with $C_0(V) \subset A$ acting by left multiplication). •

With this we are ready to prove:

Theorem 6.7 *Let V be a compact oriented manifold on which Γ acts by orientation preserving diffeomorphisms. Assume that in the induced fibration over $B\Gamma$ with fiber $V : V \times_\Gamma E_\Gamma \to B\Gamma$ the fundamental class $[V]$ of the fiber becomes 0 in $H_n(V \times_\Gamma E\Gamma, \mathbb{Q})$. Then the unit of the C^* algebra $A = C(V) \rtimes \Gamma$ is a torsion element of $K_0(A)$.*

Proof. By [15], [3] we know that the map $\mu : K^*(V, \Gamma) \to K_*(C(V) \rtimes \Gamma)$ is well defined. Thus it is enough to find $x \in K^*(V, \Gamma)$ with $\mu(x) = 1_A$ and $\operatorname{ch} x = 0$ (since the Chern character is a rational isomorphism (see above)). Now Lemma 6.6 shows that $\mu(x) = 1_A$ where x is the following K cycle $(V, 1_V, g)$ where 1_V stands for the trivial line bundle on V, g is the map from V to V_Γ given by $g(x) = (s \times t_0)/\Gamma$ for some $t_0 \in E\Gamma$, and where the K orientation of the bundle $TV \oplus TV$ comes from its natural complex structure. By Proposition 6.4 the Chern character of this K-cycle is equal to $g_*(Td(T_{\mathbb{C}}(V) \cap [V]))$. Let τ be the bundle on V_Γ associated to TV, since the restriction of τ to any fiber coincides with TV, we see that the Chern character of x is given by $Td(\tau_{\mathbb{C}})\, g_*[V]$ and hence is equal to 0. •

Corollary 6.8 *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a torsion free cocompact discrete subgroup. Let Γ acts on $V = P_1(\mathbb{R})$ in the obvious way, then in the C^* algebra $A = C(V) \times \Gamma$ the unit 1_A is a torsion element of $K_0(A)$.*

Proof. Let $H = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ be the Poincaré space, and $M = H/\Gamma$ be the quotient Riemann surface. Let us identify H to $E\Gamma$ since it is a contractible space on which Γ acts freely and properly. Then as in [31] p. 313 we can identify the induced bundle $V_\Gamma = P_1(\mathbb{R}) \times_\Gamma E\Gamma = P_1(\mathbb{R}) \times_\Gamma H$ over $B\Gamma = M$ to the unit sphere bundle of M, called η. It follows that the Euler class $e(\eta)$ of η is equal to $2 - 2g$ times the generator of $H^2(M, \mathbb{Z})$. Applying this with the Gysin sequence of the tangent vector bundle TM: (*cf.* [31] p. 143)

$$H^0(M, \mathbb{Z}) \xrightarrow{Ue} H^2(M, \mathbb{Z}) \xrightarrow{\pi^*} H^2(\eta, \mathbb{Z}) \to \dots$$

shows that $\pi^*([M]^*)$ is a torsion element of $H^2(\eta, \mathbb{Z})$, where $\pi : \eta \to M$ is the projection and $[M]^*$ is the generator of $H^2(M, \mathbb{Z})$. But η is an oriented manifold and the homology class of the fiber: $[P_1(\mathbb{R})]$ is Poincaré dual to $\pi^*([M])$ and hence is also a torsion element so that $\operatorname{ch} x = 0$. •

We shall now state the main result of this paper, whose proof will occupy the end of this section.

Theorem 6.9 *Let Γ be a discrete group acting by orientation preserving diffeomorphisms of the not necessarily compact manifold V. Let $\mathcal{R}$ be the $\mathbb{C}$-subalgebra of $H^*(V_\Gamma, \mathbb{C})$ generated by the Chern classes of equivariant bundles on V. Then for any $P \in \mathcal{R}$ there exists an additive map φ of $K(A)$ to $\mathbb{C}$, $A = C_0(V) \rtimes \Gamma$, such that*

$$\varphi(\mu(x)) = \langle \Phi \circ \mathrm{ch}(x)), P \rangle \qquad \forall\, x \in K^*(V, \Gamma)\,.$$

Here $\Phi \circ \mathrm{ch}$ is the Chern character: $K^*(V,\Gamma) \to H_*(V_\Gamma, \mathbb{Q})$ and P being an element of $H^*(V_\Gamma, \mathbb{C})$ the pairing is well defined.

We shall first show that we may assume, to prove Theorem 6.9, that the action of Γ on V is almost isometric. With the notations of Section 5, the Γ equivariant map $p : W \to V$ turns $W_\Gamma = W \times_\Gamma E\Gamma$ into a bundle over $V_\Gamma = V \times_\Gamma E\Gamma$, with fiber $H_n = \mathrm{GL}(n,\mathbb{R})^+/\mathrm{SO}(n,\mathbb{R})$. Let $f : V_\Gamma \to W_\Gamma$ be a continuous cross section of this bundle and identify it topologically with the real vector bundle $\rho(\tau)$ where ρ is the isotropy representation of $\mathrm{SO}(n,\mathbb{R})$ in H_n. Assume n even (otherwise replace V by $V \times \mathbb{R}$ with trivial action of Γ on $\mathbb{R}$). Then $\rho(\tau)$ is a Spin real vector bundle.

Lemma 6.10

a) *The map θ which to each K-cycle (N, F, g) for V assigns the K cycle $(N, F, f \circ g)$ for W is an isomorphism of $K^*(V,\Gamma)$ with $K^*(W,\Gamma)$.*

b) *With the notations of* a) *one has*

$$\Phi_W \circ \mathrm{ch}(\theta(x)) = f_*((\Phi_V\, \mathrm{ch}(x))\, Td\, \rho(\tau_V)) \qquad \forall\, x \in K^*(V,\Gamma)\,.$$

c) *Let $\beta \in KK(C_0(V) \rtimes \Gamma, C_0(W) \rtimes \Gamma)$ be as in Section* 5*, then*

$$\mu(\theta(x)) = \mu(x) \otimes_A \beta \qquad \forall\, x \in K^*(V,\Gamma)\,.$$

Proof. a) Let τ_W be the real vector bundle on W/Γ associated to the (Γ-equivariant) tangent bundle TW. By construction one has $f^*(\tau_W) = \tau_V \oplus \rho(\tau_V)$ and $f : V_\Gamma \to W_\Gamma$ is a homotopy equivalence. As $\rho(\tau_V)$ is a Spin vector bundle, one has a Thom isomorphism:

$$K_{*,\tau_V \oplus \rho(\tau_V)}(V_\Gamma) \simeq K_{*,\tau_V}(V_\Gamma)$$

(where for any real vector bundle E on a topological space X we put $K_{*,E}(X) = K_*(BE, SE)$, the relative theory of the pair: unit ball, unit sphere bundle of E).

Thus one has a natural isomorphism: $K_{*,\tau_W}(W_\Gamma) \simeq K_{*,\tau_V}(V_\Gamma)$. To each K cycle (N, F, g) for V corresponds the K cycle $(N, F, f \circ g)$ where the Spin^c structure on $TN \oplus (f \circ g)^* \tau_W = TN \oplus g^* \tau_V \oplus g^* \rho(\tau_V)$ comes from the Spin^c structure on $TN \oplus g^* \tau_V$ and the Spin structure on $g^* \rho(\tau_V)$.

b) With the notations of a) using Proposition 6.4 one has:

$$\begin{aligned}
\Phi_W \operatorname{ch}(\theta(x)) &= (f \circ g)_* (\operatorname{ch} F \cdot Td(TN \oplus g^* \tau_V \oplus g^* \rho(\tau_V))) \\
&= (f \circ g)_* ((\operatorname{ch} F \cdot Td(TN \oplus g^* \tau_V))\, g^*\, Td\, \rho(\tau_V)) \\
&= f_* ((\Phi_V \operatorname{ch}(x))\, Td\, \rho(\tau_V))\,.
\end{aligned}$$

c) Let us take the notations of Lemma 6.5. Thus (N, F, g) is a K cycle $\tilde N \xrightarrow{q} N$ is a Γ principal bundle and $h : \tilde N \to V$ is a Γ-equivariant smooth map. Now h lifts to an equivariant map $h_W : \tilde N \to W$ such that $p \circ h_W = h$. This can be seen using $f : V_\Gamma \to W_\Gamma$ or be proven by a direct argument. The new triple $(\tilde N, F, h_W)$ now describes the K cycle $\theta(\tilde N, F, h)$. Now $p : W \to V$ is a Γ equivariant submersion, with Γ invariant metric and Spin structure along the fibers, so that the family of Dirac operators along the fibers defines an element $p!$ of $KK_\Gamma(C_0(W), C_0(V))$. The argument of [15] shows that, with the notations of Lemma 6.5:

$$h! = h_W! \otimes_{C_0(W)} p! \quad \text{in} \quad KK_\Gamma(C_0(\tilde N), C_0(V))\,.$$

Thus the conclusion: $h! \otimes \beta_0 = h_W!$ follows from the equality

$$p! \otimes \beta_0 = \mathrm{id} \quad \text{in} \quad KK_\Gamma(C_0(W), C_0(W))\,.$$

This last equality follows from [27].

Now given an action of Γ on a manifold V, we let $\mathcal{R}_0 \subset H^*(V, \mathbb{C})$ be the subalgebra of $\mathcal{R}$ generated by the Chern characters of those Γ-equivariant bundles on V which can be endowed with a Γ-invariant hermitian metric.

Lemma 6.11 *To prove Theorem* 6.9 *one can assume:*

a) *that the action of* Γ *on* V *is almost isometric,*

b) *that* $P \in \mathcal{R}_0 \subset \mathcal{R}$.

Proof. Let us assume that Theorem 6.9 is proven under the hypothesis a) b) and prove it when P belongs to the subalgebra of $\mathcal{R}$ generated by $\mathcal{R}_0$ and the Pontrjagin classes of τ. Let $W \xrightarrow{p} V$ be as above, and $p_\Gamma : W_\Gamma \to V_\Gamma$ be the corresponding fibration. Then since $p^*(TV)$ can be endowed with a Γ invariant Euclidean metric, it follows that $p_\Gamma^*(P) \in \mathcal{R}_0^W \subset H^*(W_\mathcal{R}, \mathbb{C})$. Let $Q \in \mathcal{R}_0^W$ be such that $p_\Gamma^*(P) = (Td\, p_\Gamma^*\, \rho(\tau_V))\, Q$. Then by hypothesis there exists an additive map Ψ of $K_*(C_p(W) \rtimes \Gamma)$ to $\mathbb{C}$ such that:

$$\Psi(\mu(y)) = \langle \Phi_W \circ \mathrm{ch}(y), Q \rangle \qquad \forall\, y \in K^*(W, \Gamma)\,.$$

Define $\varphi : K_*(C_0(V) \rtimes \Gamma) \to \mathbb{C}$ by $\varphi(z) = \Psi(z \otimes_A \beta)$. Then by Lemma 6.10 one has, for any $x \in K^*(V, \Gamma)$:

$$\begin{aligned} \varphi(\mu(x)) &= \Psi(\mu(x) = \Psi(\mu(\theta(x))) \\ &= \langle \Phi_W\, \mathrm{ch}(\theta(x)), Q \rangle = \langle f_*((\Phi_V \circ \mathrm{ch}\, x)\, Td\, \rho(\tau_V), Q \rangle \\ &= \langle \Phi_V \circ \mathrm{ch}\, x, P \rangle\,. \end{aligned}$$

Thus Theorem 6.9 is now known for any action of Γ on V and any $P \in \mathcal{R}_0$. To treat the general case one applies the same idea as in Theorem 5.8 with an appropriate variant of Lemma 6.10. •

To prove Theorem 6.9 under hypothesis a) b) we shall need another variation on Lemma 6.10. We take the trivial bundle over V with fiber $\mathbb{R}^m$, m even, and make it equivariant in the obvious way. Thus the total space U of this bundle is $U = V \times \mathbb{R}^m$, the projection $U \to V$ is $\mathrm{pr}_V : (x, \xi) \to x$ and the action of Γ is given by $(x, \xi)g = (xg, \xi)$, $\forall\, (x, \xi) \in U$, $g \in \Gamma$. The map $f : V \to U$, $f(x) = (x, 0)$ is Γ equivariant, with pr_V as a Γ equivariant left inverse, and both are equivariantly K oriented. Also $U_\Gamma = V_\Gamma \times \mathbb{R}^m$ and $C_0(U) \rtimes \Gamma = (C_0(V) \rtimes \Gamma) \otimes C_0(\mathbb{R}^m)$. We let $b \in K_0(C_0(\mathbb{R}^m))$ be the Bott element, then:

Lemma 6.12

a) *The map ρ which to each K-cycle (N, F, g) for V assigns the K-cycle $(N, F, f \circ g)$ for U is an isomorphism of $K^*(V, \Gamma)$ with $K^*(U, \Gamma)$.*

b) *With the notations of* a) *one has:*

$$\Phi_U\, \mathrm{ch}(\rho(x)) = f_*(\Phi_V\, \mathrm{ch}\, x) \qquad \forall\, x \in K^*(V, \Gamma)\,.$$

c) *Let $x \in K^*(V,\Gamma)$, then $\mu(\rho(x)) = \mu(x) \otimes b$.*

The proof is much easier than that of Lemma 6.10 and is omitted. For each even integer m we let $U_m = V \times \mathbb{R}^m$, $b_m \in K_0(C_0(\mathbb{R}^m))$ be as above, and $\varphi_m : K_0(C_0(U_m) \rtimes \Gamma) \to \mathbb{C}$ be the unique additive map (we take $n = \dim V$ even) determined by Theorem 4.5.

Lemma 6.13

a) *For any even m and $z \in K_0(C_0(V) \rtimes \Gamma)$ one has:*

$$\varphi_m(z \otimes b_m) = \varphi_0(z)\,.$$

b) *Let $x = (N, F, g)$ be a K-cycle for U_m with $g : \tilde{N} \to U_m$ etale, and $T\tilde{N} \oplus g^* TV$ K-oriented in the obvious way. Then*

$$\varphi_m(\mu(x)) = \langle \operatorname{ch} F, [N] \rangle\,.$$

Proof. a) Let S^m be the m sphere and Γ act on $V \times S^m$ by $(x,s)g = (xg, s)$, $\forall\, x \in V$, $s \in S^m$. Let $\Psi_m : K_0(C_0(V \times S^m) \rtimes \Gamma) \to \mathbb{C}$ be given by Theorem 4.5. It is enough to show that for any complex vector bundle E on S^m and any $z \in K_0(C_0(V) \rtimes \Gamma)$ one has $\Psi_m(z \otimes E) = \varphi_0(z) \langle \operatorname{ch} E, S^m \rangle$. We can represent E by an idempotent $e \in M_q(C^\infty(S^m))$ for some $q < \infty$. Let $\mathcal{C} \subset C_0(V) \rtimes \Gamma$ be the smallest subalgebra containing $C_c(V \rtimes \Gamma)$ and stable under holomorphic functional calculus, one can assume that z is represented by the difference $z = [f] - [f_0]$ where $f \in M_k(\tilde{\mathcal{C}})$ is an idempotent and $f_0 \in M_k(\mathbb{C}) \subset M_k(\tilde{\mathcal{C}})$ is equal to f modulo $M_k(\mathcal{C})$.

Then $z \otimes E$ is presented by the pair of idempotents $(f \otimes e + f_0 \otimes (1 - e), f_0 \otimes 1)$ of $M_{kq}(\widetilde{\mathcal{C} \otimes C^\infty}(S^m))$. Now let $\mathcal{C}' \subset C_0(V \times S^m) \rtimes \Gamma$ be the smallest subalgebra containing $C_c(V \times S^m \rtimes \Gamma)$ and stable under holomorphic functional calculus. One has $\mathcal{C} \otimes 1 \subset \mathcal{C}'$ hence the algebraic tensor product $\mathcal{C} \otimes C^\infty(S^m)$ is contained in $\mathcal{C}'$. Moreover the canonical cyclic cocycle on $\mathcal{C}'$, given by Theorem 4.5, τ' when restricted to this algebraic tensor product is the cup product $\tau \,\#\, \tau''$ of the canonical cyclic cocycle τ on $\mathcal{C}$ by the fundamental class τ'' of S^m. (See [] for the cup product.) This is clear on $C_c^\infty(V \rtimes \Gamma) \otimes C^\infty(S^m)$ and follows in general from the construction of τ and τ' of Section 2.
Now $\Psi_m(z \otimes E) = \tau \,\#\, \tau''((f - f_0) \otimes e, \ldots, (f - f_0) \otimes e)$ is easy to compute using the same argument as in [7] and yields:

$$\tau(f - f_0, \ldots, f - f_0)\, \tau''(e, \ldots, e)\,.$$

b) Replacing V by U_m one may as well assume that $m = 0$.

By Lemma 6.5 one has $\mu(N, F, \pi) = [F] \otimes [M] \otimes [\pi']$, where $[M] \in KK(C_0(N), C_0(\tilde{N}) \rtimes \Gamma)$ is the class of the $C_0(N), C_0(\tilde{N}) \rtimes \Gamma$ bimodule of Morita equivalence: M and $\pi!$ is as in Lemma 6.5b). As the fibers of π are 0-dimensional both $\pi!$ and $[M] \otimes [\pi!]$ are easy to describe. $[M] \otimes [\pi!] \in KK(C_0(N), C_0(V) \rtimes \Gamma)$ is described by the following $C_0(N)$, $C_0(V) \rtimes \Gamma$ bimodule $\mathcal{E}$.

First note that $C_c^\infty(\tilde{N})$ is in a natural way a $C_c^\infty(N)$, $C_c^\infty(V \rtimes \Gamma)$ bimodule with

$$(\xi f)(x) = \sum_\Gamma \xi(xg)\, f(\pi(xg), g^{-1}) \qquad \forall\, \xi \in C_c^\infty(N),\ f \in C_c^\infty(V \rtimes \Gamma)$$

$$(h\xi)(x) = h(x/\Gamma)\, \xi(x) \qquad \forall\, h \in C_c^\infty(N),\ \xi \in C_c^\infty(\tilde{N})\,.$$

Moreover one has a $C_c^\infty(V \rtimes \Gamma)$ valued sesquilinear form given by:

$$\langle \xi, \eta \rangle(y, g) = \sum_{\pi(x) = y} \bar{\xi}(x)\, \eta(x, g)\,.$$

Completion with respect to the norm $\|\xi\|^2 = \|\langle \xi, \xi \rangle\|_{C_0(V) \rtimes \Gamma}$ yields the desired bimodule $\mathcal{E}$.

Note that, as a C^* module over $C_0(V) \rtimes \Gamma$, $\mathcal{E}$ corresponds, in the sense of [11] Section 7, to the Γ-equivariant field of Hilbert spaces $H_x = \ell^2(\pi^{-1}\{x\})$, $x \in V$. Since the map π is etale, the fibers are 0-dimensional and there is an obvious Γ-invariant connection on this field of Hilbert spaces. Using it we shall now compute the cyclic cocycle τ_1 on $C_c^\infty(V \rtimes \Gamma)$ of Section 4 and the bimodule $C_c^\infty(\tilde{N})$ as in [7].

With the notations of Section 4, let $\Omega \rtimes \Gamma$ be the crossed product of the de Rham complex: $C_c^\infty(V) \to \Omega^1 \to \ldots \to \Omega^n$ by Γ and $\int : \Omega^n \times \Gamma \to \mathbb{C}$ the canonical trace. A connection ∇ on $C_c^\infty(\tilde{N})$ is given a linear map $\nabla : C_c^\infty(\tilde{N}) \to C_c^\infty(\tilde{N}) \otimes_{C_c^\infty(V \rtimes \Gamma)} (\Omega^1 \rtimes \Gamma)$ which satisfies the Leibnitz rule:

$$\nabla(\xi f) = (\nabla \xi) f + \xi \otimes df \qquad \forall\, \xi \in C_c^\infty(N),\ f \in C_c^\infty(V \rtimes \Gamma)\,.$$

Let $C_c^\infty(\tilde{N}) \to \tilde{\Omega}^1 \to \ldots \to \tilde{\Omega}^n$ be the de Rham complex on $\tilde{N}$. It is a right module over $\Omega \rtimes \Gamma$ with, for $\omega \in \tilde{\Omega}$, $\alpha \sum \alpha_g\, U_g \in \Omega \rtimes \Gamma$:

$$\omega\alpha = \sum_\Gamma g(\omega)\, \rho_g^*(\alpha_{g^{-1}})$$

where $\rho_g(x) = \pi(xg)$, $\forall\, x \in \tilde{N}$, $g \in \Gamma$.

Moreover there is a natural isomorphism of the induced module: $C_c^\infty(\tilde{N}) \otimes_{C_c^\infty(V \rtimes \Gamma)} (\Omega \rtimes \Gamma)$ with the above module $\tilde{\Omega}$, given by

$$I(\xi \otimes \alpha) = \xi\alpha \qquad \forall\, \xi \in C_c^\infty(\tilde{N}),\ \alpha \in \Omega \rtimes \Gamma\,.$$

The equality $\nabla\omega = d\omega$, $\forall\, \omega \in \tilde{\Omega}$ defines a connection, whose curvature ∇^2 is equal to 0. It follows that the cycle on $C_c^\infty(N)$ yielding τ_1 is very simple to describe: 1) any $f \in C_c^\infty(N)$ defines an endomorphism $\tilde{f}$ of degree 0 of $\tilde{\Omega}$, the multiplication by $f \circ q$ where $q : \tilde{N} \to N$ is the projection. 2) For any f, $\nabla\tilde{f} - \tilde{f}\nabla$ is the multiplication by $df \circ q$. To determine $\tau_1(f^0, \ldots, f^n)$, $f^i \in C_c^\infty(N)$ one has to compute:

$$\int \tilde{f}^0(\nabla f^1 - f^1\nabla) \ldots (\nabla\tilde{f}^n - \tilde{f}^n\nabla)\,.$$

As the computation is local over N, and is easily done in the case $N = V$, $\tilde{N} = V \times \Gamma$ (*cf.* [7]) one gets

$$\tau_1(f_0, \ldots, f^n) = \int_N f^0 df^1 \wedge \ldots \wedge df^n\,.$$

Finally one has $\varphi_0(\mu(N, F, \pi)) = \varphi_0([F] \otimes [\mathcal{E}]) = \langle \tau_1, [F] \rangle = \langle \mathrm{ch}\, F, [N] \rangle$. •

Putting together Lemmas 6.12 and 6.13 we get:

Lemma 6.14 *With the notations of Lemma* 6.13 *one has:*

$$\varphi_0(\mu(x)) = \langle \Phi\, \mathrm{ch}\, x, Td^{-1}(\tau_V \oplus \tau_V) \rangle \qquad \forall\, x \in K^*(V, \Gamma)\,.$$

Proof. Let $(M, \tilde{M}, E, h)$ be a K-cycle in the class x, where $\tilde{M}$ is a Γ covering of the compact manifold M, $E \in K(M)$, and h a γ-equivariant smooth map of $\tilde{M}$ to V, as above. Let then $j : M \to \mathbb{R}^m$ be an auxiliary imbedding of M in an even dimensional Euclidean space. Then with the notations of Lemma 6.12, the class $\rho(x)$ contains the K cycle $(M, \tilde{M}, E, h')$ where $h'(x) = (h(x), j(x/\Gamma))$, $\forall\, x \in \tilde{M}$. This map h' is an immersion of $\tilde{M}$ to $U = V \times \mathbb{R}^m$ so that its graph is transverse to the projection $p_U : \tilde{M} \times U \to U$. As the action of Γ on $\tilde{M} \times U$ is proper there exists an equivariant smooth map k from the total space of the normal bundle of h' to $\tilde{M} \times U$ such that $p_U \circ k$ is etale. Since the normal bundle of h' is K oriented by hypothesis, one can find in the class $\rho(x)$ a K-cycle $(N, \tilde{N}, F, \pi)$ with π etale, using Proposition 6.3b). Now by Lemma 6.13b):

$$\varphi_m(\mu(\rho(x))) = \langle \mathrm{ch}\, F, [N] \rangle = \langle \Phi_U \, \mathrm{ch}\, \rho(x), Td^{-1}(\tau_U \oplus \tau_U) \rangle$$

since by Proposition 6.4 one has $\Phi \, \mathrm{ch}(\rho(x)) = (\pi_\Gamma)_* (\mathrm{ch}\, F \, Td(T_{\mathbb{C}}\, N) \cap [N])$. By Lemma 6.12b) one gets:

$$\varphi_m(\mu(\rho(x))) = \langle \Phi_V \, \mathrm{ch}\, x, Td^{-1}(\tau_V \oplus \tau_V) \rangle \, .$$

As $\mu(\rho(x)) = \mu(x) \otimes b$ (Lemma 6.12c)) and $\varphi_m(\mu(x) \otimes b) = \varphi_0(\mu(x))$ one has $\varphi_0(\mu(x)) = \langle \Phi_V \, \mathrm{ch}\, x, Td^{-1}(\tau_v \oplus \tau_v) \rangle$. •

Lemma 6.15 *Let E be a Γ-invariant bundle on V which can be endowed with a Γ-invariant hermitian metric.*

a) *There exists a map $m_E : K^*(V, \Gamma) \to K^*(V, \Gamma)$ such that $m_E((N, F, g)) = (N, F \otimes g^* E, g)$ for any K-cycle (N, F, g).*

b) *One has $\Phi \circ \mathrm{ch}(m_E(x)) = \Phi \circ \mathrm{ch}(x) \cdot \mathrm{ch}\, E$, $\forall\, x \in K^*(V, \Gamma)$.*

c) *Let $\varepsilon_E \in KK(C_0(V) \rtimes \Gamma, C_0(V) \rtimes \Gamma)$ be as in Corollary 4.7, then:*

$$\mu(m_E(x)) = \mu(x) \otimes \varepsilon_E \qquad \forall\, x \in K^*(V, \Gamma) \, .$$

Proof. a) and b) are immediate. Note that E is viewed as a bundle on V_Γ in both cases. The proof of c) is the same as in Corollary 4.7. •

Proof of Theorem 6.9. By Lemma 6.11 we can assume that the action of Γ on V is almost isometric and that $P \in \mathcal{R}_0$. Putting together Lemmas 6.14 and 6.15, we can find for any $Q \in \mathcal{R}_0$ an additive map from $K(C_0(V) \rtimes \Gamma)$ to $\mathbb{C}$ such that

$$\Psi(\mu(x)) = \langle \Phi \, \mathrm{ch}\, x, Td^{-1}(\tau_V) \, Q \rangle \qquad \forall\, x \in K^*(V, \Gamma) \, .$$

Thus taking $Q = Td\,(\tau_V \oplus \tau_V)\, P$ yields the answer. •

7 Cyclic cohomology and Gel'fand-Fuchs cohomology

Our aim in this section is to improve the result of Section 6, namely Theorem 6.9, by enlarging the ring $\mathcal{R} \subset H^*(V_\Gamma, \mathbb{C})$. The new ring $\tilde{\mathcal{R}}$ will now contain besides the Pontrjagin classes of τ, all the higher characteristic classes of Gel'fand Fuchs (*cf.* [18]). More precisely, let $n = \dim V$, then to the action of the discrete group Γ on V corresponds a groupoid homomorphism π from $V \rtimes \Gamma$ to the groupoid Γ_V (*cf.* [21] Chapter III, Example 2) of germs of diffeomorphisms of V. To the pair $(x,g) \in V \times \Gamma$ it assigns the germ at $xg \in V$ of the diffeomorphism $y \to yg^{-1}$ of V. Let then $B\pi$ be the corresponding map of classifying spaces; $B\pi : V_\Gamma \to B\Gamma_V$. By [21] Theorem 4.4 the differentiable cohomology $H_d^*(\Gamma_V, \mathbb{R})$ is isomorphic to the Gel'fand Fuchs cohomology $H_d^*(\mathcal{A}_n, O_n) = H^*(\mathrm{WO}_n)$ (*cf.* [18]). Thus $(B\pi)^*$ yields a map from $H^*(\mathrm{WO}_n)$ to $H^*(V_\Gamma, \mathbb{R})$ and our aim is to show that we can enlarge $\mathcal{R}$ by the range of this map. Using the technique of Section 6 together with the existence of Γ-invariant differential forms on higher frame bundles over V the general result is easy to obtain. Since we want to have explicit formulae on $C_c^\infty(V \rtimes \Gamma)$ we shall spend most of this section with a specific example, namely the Godbillon-Vey class $\mathrm{GV} \in H^3(\mathrm{WO}_1)$. We shall associate to it a 2-trace on the C^* algebra $A = C(S^1) \rtimes \Gamma$ where Γ is a discrete group acting by diffeomorphisms of S^1.

The following lemma has useful generalizations in different directions, we shall discuss this point in a remark at the end of this section.

Lemma 7.1 *Let V be an n-dimensional oriented manifold, G a Lie group acting smoothly on V by orientation preserving diffeomorphisms. Let ω be a k-cocycle on G with coefficients in the G-module of n-forms on V : $\omega \in Z^k(G, \Omega_V^n)$, such that $\omega(g^1, \dots, g^k) = 0$ if one of the g^q is 1 or if $g^1, \dots, g^k = 1$. Let $\nu = dg$ be a right Haar measure on G. The following equality defines a cyclic k-cocycle on the algebra $C_c^\infty(V \rtimes G)$:*

$$\tau(f^0, \dots, f^k) = \int f^0(\gamma_0) \dots f^k(\gamma_k)\, \omega(g_1, \dots, g_k)(x)\, dg_1 \dots dg_k \, .$$

We have used the following notations: the elements $\gamma_0, \dots, \gamma_k$ of the groupoid $V \rtimes G$ are uniquely determined for $x \in V$, $g_1, \dots, g_k \in G$ by: $\gamma_0 \gamma_1 \dots \gamma_k = x$,

$h(\gamma_j) = g_j$ where $h : V \rtimes G \to G$ is the homomorphism given by the second projection.

Proof. Let us first check that τ is a Hochschild cocycle: $b\tau = 0$. With obvious notations one has for $j = 0, 1, \ldots, k$:

$$
\begin{aligned}
& \tau(f^0, \ldots, f^j j^{j+1}, \ldots, f^{k+1}) \\
= & \int f^0(\gamma_0) \ldots f^{k+1}(\gamma_{k+1})\, \omega(g_1, \ldots, g_j\, g_{j+1}, \ldots, g_{k+1})(x) dg_1 \ldots dg_{k+1}\,.
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
& \tau(f^{k+1} f^0, f^1, \ldots, f^k) \\
= & \int f^{k+1}(\gamma_{k+1})\, f^0(\gamma_0) \ldots f^k(\gamma_k)\, \omega(g_1, \ldots, g_k)(x g_{k+1}^{-1})\, dg_1 \ldots dg_{k+1}\,.
\end{aligned}
$$

Thus since ω is a k-cocycle we get $b\tau = 0$. (Note that G acts on the right on Ω_V^n by $\omega g = \varphi_g^* \omega$ where $\varphi_g(x) = x g^{-1}$, $\forall\, x \in V$, $g \in G$.) Next, since $\omega(g_1, \ldots, g_k) = 0$ if $g_1 \ldots g_k = 1$ it follows using the cocycle property that $\omega(g_1, \ldots, g_k) = (-1)^k\, \omega(g_0, \ldots, g_{k-1})\, g_k$ for any $g_0, \ldots, g_k \in G$ with $g_0 g_1 \ldots g_k = 1$. The same computation yields:

$$
\tau(f^1, \ldots, f^k, f^0) = (-1)^k\, \tau(f^0, \ldots, f^k)\,.
$$

•

We shall now apply Lemma 7.1 when $V = S^1$ and ω is the 2-cocycle on Γ with values in Ω_V^1 given by the following formula:

$$
\omega(g_1, g_2) = d\ell(g_1 g_2)\, \ell(g_2) - \ell(g_1 g_2)\, d\ell(g_2) \qquad \forall\, g_1, g_2 \in \Gamma
$$

where for $g \in \Gamma$, $\ell(g) \in C^\infty(S^1)$ is the logarithm of the Jacobian of the diffeomorphism associated to g:

$$
\ell(g) = \mathrm{Log}\left(\frac{d(x g^{-1})}{dx}\right)\,.
$$

This formula is simply a normalized form of the 2-cocycle of Thurston ([44]) as shown by the next lemma.

Lemma 7.2

a) $\ell(g_1 g_2) = \ell(g_1)\, g_2 + \ell(g_2)$, $\forall\, g_1, g_2 \in \Gamma$.

b) $\omega(g_2, g_3) - \omega(g_1 g_2, g_3) + \omega(g_1, g_2 g_3) - \omega(g_1, g_2)\, g_3 = 0$, $\forall\, g_1, g_2, g_3 \in \Gamma$.

c) $\omega(g_1, g_2) = 0$ *if* $g_1 = 1$ *or* $g_2 = 1$ *or* $g_1 g_2 = 1$.

d) ω *is cohomologous to* $2c$ *where* $c(g_1, g_2) = ((d\ell(g_1)\, g_2)\, \ell(g_2)$, $\forall\, g_1, g_2 \in \Gamma$.

Proof. a) Follows from the chain rule.
b) By a) both ℓ and $d\ell$ are 1-cocycles with values in Ω_V^0 and Ω_V^1 respectively, their cup product is given by c and hence is a Hochschild cocycle, thus b) will follow from d).
c) One has $\ell(1) = 0$.
d) Let $\rho(g) = c(g^{-1}, g) \in \Omega_V^1$, $\forall\, g \in \Gamma$. Let us compute $(b\rho)(g_1, g_2) = \rho(g_2) - \rho(g_1 g_2) + \rho(g_1)\, g_2$. One has $\rho(g) = -\ell(g)\, d\ell(g)$ thus one gets: $-\ell(g_2)\, d\ell(g_2) + \ell(g_1 g_2)\, d\ell(g_1 g_2) - (\ell(g_1)\, g_2)(d\ell(g_1)\, g_2) = \ell(g_1)\, g_2\, d\ell(g_2) + \ell(g_2)(d\ell(g_1)\, g_2)$. Thus since $\omega(g_1, g_2) = -\ell(g_1)\, g_2\, d\ell(g_2) + \ell(g_2)(d\ell(g_1)\, g_2)$ we get $\omega + b\rho = 2c$. •

By Lemma 7.1 we can associate to the cocycle ω a cyclic cocycle τ on $C_c^\infty(S^1 \rtimes \Gamma)$:

$$\tau(f^0, f^1, f^2) = \sum \int f^0(\gamma_0)\, f^1(\gamma_1)\, f^2(\gamma_2)\, \omega(g_1, g_2)\,.$$

More explicitly, with $f^j = \sum f_g^j\, U_g$ this gives:

$$\tau(f^0, f^1, f^2) = \sum_{g_0 g_1 g_2 = 1} \int_{S^1} f_{g_0}^0(x)\, f_{g_1}^1(x g_0)\, f_{g_2}^2(x g_0 g_1)\, \omega(g_1, g_2)(x)\,.$$

Theorem 7.3

1) *The densely defined cyclic cocycle* τ *is a* 2*-trace on the* C^* *algebra* $A = C(S^1) \rtimes \Gamma$ *(reduced crossed product).*

2) *For any* $x \in K^*(S^1, \Gamma)$ *(the geometric group of Section* 6*) one has* $\langle \mu(x), \tau \rangle = \langle \Phi\, \mathrm{ch}\, x, (B\pi)^* \mathrm{GV} \rangle$, *where* $\mathrm{GV} \in H^3(\mathrm{WO}_1)$ *is the Godbillon-Vey class, and* $\pi : S^1 \rtimes \Gamma \to \Gamma_{S^1}$ *the natural homomorphism.*

It follows from 2) and [19] that the map: $K_0(A) \to \mathbb{C}$ determined by τ is non trivial when Γ is the fundamental group $\Gamma = \pi_1(M)$ of a Riemann surface, acting on $P_1(\mathbb{R})$ as a discrete cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We shall now begin the proof of this theorem.

For the definition of $\ell(g)$ we used a 1-form on S^1, we call it dx, and note that it is of course not canonically given by the smooth manifold S^1 and in particular not Γ invariant in general. We let φ be the state on A which is associated to this 1-form, thus,

$$\varphi\left(\sum f_g\, U_g\right) = \int_{S^1} f_1(x)\, dx\,.$$

The modular automorphism group σ_t^φ of φ leaves A invariant and the corresponding unbounded derivation D_φ of A is given by:

$$\text{a)}\quad D_\varphi(f) = 0 \quad \forall\, f \in C^\infty(S^1) \qquad \text{b)}\quad D_\varphi(U_g) = U_g\,\ell(g) \quad \forall\, g \in \Gamma\,.$$

Let τ_1 be the 1-trace on A constructed in Section 1, *i.e.*, one has

$$\tau_1(f^0,f^1) = \sum_{g_0g_1=1} \int_{S^1} f^0_{g_0}(xg_0^{-1})\, df^1_{g_1}(x)$$

where

$$f^j = \sum f^j_g\, U_g \in C_c^\infty(S^1 \rtimes \Gamma)\,.$$

In general τ_1 is not invariant under the modular automorphism group σ_t^φ, more precisely:

Lemma 7.4

a) *There exists a one trace $\dot\tau_1$ on A such that, on $C_c^\infty(S^1 \rtimes \Gamma)$,*

$$\dot\tau_1(f^0,f^1) = \lim_{\varepsilon\to 0} \frac{1}{i\varepsilon}\,(\tau_1(\sigma_\varepsilon^\varphi(f^0),\sigma_\varepsilon^\varphi(f^1)) - \tau_1(f^0,f^1))\,.$$

b *$\dot\tau_1$ is the one trace on A associated to the* 1*-cocycle of* Γ *with coefficients in $\Omega^1_{S^1}$, given by $g \to d\ell(g)$.*

Proof. For any $f^0,f^1 \in C_c^\infty(S^1 \rtimes \Gamma)$ one has

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{i\varepsilon}\,(\tau_1(\sigma_\varepsilon^\varphi(f^0),\sigma_\varepsilon^\varphi(f^1)) - \tau_1(f^0,f^1))\\
=\ & \tau_1(D_\varphi\, f^0,f^1) + \tau_1(f^0,D_\varphi\, f^1)\\
=\ & \sum_{g_0g_1=1} \int_{S^1} (f^0_{g_0}(xg_0^{-1})\,\ell_{g_0}(x)\, df^1_{g_1}(x) + f^0_{g_0}(xg_0^{-1})\, df^1_{g_1}(x)\,\ell_{g_1}(xg_1)\\
+\ & f^0_{g_0}(xg_0^{-1})\, f^1_{g_1}(x)\, d\ell_{g_1}(xg_1))\,.
\end{aligned}$$

As $\ell_{g_0}(x)+\ell_{g_1}(xg_1) = (\ell_{g_0}+(\ell_{g_1})\, g_0)(x) = 0$, the first two terms in the integral cancel and we get:

$$\sum_{g_0 g_1 = 1} \int_{S^1} f^0_{g_0}(xg_0^{-1})\, f^1_{g_1}(x)\, d\ell_{g_1}(xg_1) = \dot\tau_1(f^0, f^1)\,.$$

Changing x to xg_0, yields the following equality:

$$\dot\tau_1(f^0, f^1) = \sum_{g_0 g_1 = 1} \int_{S_1} f^0_{g_0}(x)\, f^1_{g_1}(xg_0)\, d\ell_{g_1}(x) = \sum \int f^0(\gamma_0)\, f^1(\gamma_1)\, d\ell(g_1)\,.$$

Using Lemma 7.1 this defines a cyclic 1-cocycle on $C_c^\infty(S^1 \rtimes \Gamma)$ and one checks that for $f^1 \in C_c^\infty(S^1 \rtimes \Gamma)$, the linear functional L, $f^0 \to \dot\tau_1(f^0, f^1)$ is continuous on A (it is of the form $L(f^0) = \int f^0 d\mu(x)$ where μ is a finite Radon measure on $S^1 \rtimes \Gamma$). Thus $\dot\tau_1$ is a 1-trace. •

What is surprising is that the second derivative $\ddot\tau_1$ of τ_1 with respect to σ_t^φ, at $t = 0$, does vanish.

Lemma 7.5 *One has* $\ddot\tau_1 = 0$.

Proof. As above, for any $f^0, f^1 \in C_c^\infty(S^1 \rtimes \Gamma)$ one has:

$$\begin{aligned}
\ddot\tau_1(f^0, f^1) &= \dot\tau_1(D_\varphi\, f^0, f^1) + \dot\tau_1(f^0, D_\varphi\, f^1) \\
&= \sum_{g_0 g_1 = 1} \int_{S_1} f^0_{g_0}(x)\, \ell_{g_0}(xg_0)\, f^1_{g_1}(xg_0)\, d\ell_{g_1}(x) \\
&+ \sum_{g_0 g_1 = 1} \int_{S_1} f^0_{g_0}(x)\, f^1_{g_1}(xg_0)\, \ell_{g_1}(x)\, d\ell_{g_1}(x)\,.
\end{aligned}$$

As $\ell_{g_0}(xg_0) + \ell_{g_1}(x) = 0$ when $g_0 g_1 = 1$ we get the conclusion. •

Thus $\dot\tau_1$ is a 1-trace on A which is invariant under the one parameter group of automorphisms σ_t^φ, this allows us to define a 2-trace on A by a formula which (*cf.* [7]) is an extension to the non commutative framework of the contraction $i_X C$ of a closed current by a vector field X such that $\partial_X C = 0$.

Lemma 7.6

a) *Let τ_0 be a 1-trace on a C^* algebra A, invariant under a one parameter group of automorphisms with generator D, with* $\mathrm{Dom}\, D \cap \mathrm{Dom}\, \tau_0$ *dense in A. Then the following equality defines a 2-trace $\tau = i_D\, \tau_0$ on A:*

$$\tau(x^0, x^1, x^2) = \tau_0(D(x^2)\, x^0, x^1) - \tau_0(x^0 D(x^1), x^2)\,.$$

b) *With the notations of Theorem* 7.3*, one has:*

$$\tau = i_{D_\varphi}(\dot{\tau}_1)\,.$$

Proof. a) Let us check that on $\mathrm{Dom}\, D \cap \mathrm{Dom}\, \tau_0$, τ is a cyclic 2-cocycle. One has:

$$\begin{aligned}
\tau(x^1, x^2, x^0) &= \tau_0(D(x^0)\, x^1, x^2) - \tau_0(x^1 D(x^2), x^0) \\
&= \tau_0(D(x^0 x^1), x^2) - \tau_0(x^0 D(x^1), x^2) \\
&\quad + \tau_0(D(x^2)\, x^0, x^1) + \tau_0(x^0 x^1, D(x^2)) \\
&= \tau(x^0, x^1, x^2)
\end{aligned}$$

since $\tau_0(D(a), b) + \tau_0(a, D(b)) = 0$. Next, τ is a Hochschild cocycle, in fact as a map from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}^*, \mathcal{A} = \mathrm{Dom}\, D \cap \mathrm{Dom}\, \tau$ it is given by:

$$c(x^1, x^2) = \delta(x^1)\, D(x^2) - D(x^1)\, \delta(x^2)$$

where $\delta : \mathcal{A} \to \mathcal{A}^*$ is the derivation given by τ_0.

Now let us show that for any $a^1, a^2 \in \mathcal{A}$ there exists $C < \infty$ such that $|\hat{\tau}(x^1 da^1\, x^2 da^2)| \leq C\, \|x^1\|_A\, \|x^2\|_A$, $\forall\, x^j \in \mathcal{A}$. One has

$$\begin{aligned}
\hat{\tau}(x^1 da^1\, x^2 da^2) &= \tau(x^1, a^1 x^2, a^2) - \tau(x^1 a^1, x^2, a^2) \\
&= \tau_0(D(a^2)\, x^1, a^1 x^2) - \tau_0(x^1 D(a^1 x^2), a^2) \\
&\quad - \tau_0(D(a^2)\, x^1 a^1, x^2) + \tau_0(x^1 a^1 D(x^2), a^2) \\
&= \tau_0(x^2 D(a^2)\, x^1, a^1) - \tau_0(x^1 D(a^1)\, x^2, a^2)
\end{aligned}$$

which can easily be estimated since τ_0 is a 1-trace.

b) Let $f^0, f^1, f^2 \in C_c^\infty(S^1 \rtimes \Gamma)$, one has, with $\tau' = i_{D_\varphi}(\dot\tau_1)$;

$$\begin{aligned}
\tau'(f^0,f^1,f^2) &= \dot\tau_1(D_\varphi(f^2)\,f^0,f^1) - \dot\tau_1(f^0 D_\varphi(f^1),f^2)\\
&= \sum_{g_0g_1g_2=1} \int_{S^1} (D_\varphi(f^2))_{g_2}(x)\, f^0_{g_0}(xg_2)\, f^1_{g_1}(xg_1^{-1})\, d\ell_{g_1}(x)\\
&- \sum_{g_0g_1g_2=1} \int_{S^1} f^0_{g_0}(x)(D_\varphi f^1_{g_1})(xg_0)\, f^2_{g_2}(xg_2^{-1})\, d\ell_{g_2}(x)\\
&= \sum_{g_0g_1g_2=1} \int_{S^1} f^0_{g_0}(x)\, f^1_{g_1}(xg_0)\, f^2_{g_2}(xg_0g_1)(\ell(g_2)(x)\, d\ell(g_1)(xg_2^{-1})\\
&- \ell(g_1)(xg_0g_1)\, d\ell_{g_2}(x))\\
&= \sum \int f^0(\gamma_0)\, f^1(\gamma_1)\, f^2(\gamma_2)\, \omega'(g_1,g_2)
\end{aligned}$$

where $\omega'(g_1,g_2) = \ell(g_2)(d\ell(g_1)\,g_2) - (\ell(g_1)\,g_2)\,d\ell(g_2) = 2\omega(g_1,g_2)$. •

Thus we have proven that τ is a 2-trace on A and also that it is obtained in a canonical manner from the 1-trace τ_1 (given by the fundamental class of S^1/Γ) and the modular automorphism group σ_t^φ. Let us now pass to the proof of b) in Theorem 7.3.

Let J_k^+ be the bundle of positive frames of order k on the oriented manifold S^1 (*cf.* [21] I.8). By definition a positive frame of order k at $y \in S^1$ is the jet or order k, $j_0^k(f)$, at $0 \in \mathbb{R}$ of a local orientation preserving diffeomorphism f of a neighborhood of 0 in $\mathbb{R}$ to a neighborhood of $y = f(0)$. Let us write $S^1 = \mathbb{R}/\mathbb{Z}$ with y the corresponding coordinate. Then natural coordinates in J_k^+ are $(y, y_1, \ldots, y_k)$, $y \in \mathbb{R}/\mathbb{Z}$, $y_1 > 0$, $y_{k'} \in \mathbb{R}$, $k' = 2, \ldots, k$. To such a point one associates the jet:

$$f(t) = y + ty_1 + t^2y_2 + \ldots + t^k y_k\,, \quad t \in \mathbb{R}\,.$$

By construction J_k^+ is a principal G^k bundle over S^1, where G^k is the Lie group of k jets of orientation preserving diffeomorphisms of $\mathbb{R}$ which fix $0 \in \mathbb{R}$. For any $f \in J_k^+$, $g \in G^k$, the product fg is just the composition of the jets.

We shall deal only with $k = 1, 2$ in which case G^1 is the multiplicative group $\mathbb{R}_+^*$ and G^2 the group of lower triangular unimodular two by two real matrices with positive diagonal elements. We let $\rho : G^2 \to G^1$ be the canonical homomorphism (any 2-jet yields a 1-jet) and H its kernel. Any $g \in H$ has the form

$$g(t) = t + bt^2 \qquad \forall\, t \in \mathbb{R}\,.$$

By construction $\mathrm{Diff}^+(S^1)$ acts on J_k^+ by composition: $f \to \varphi \circ f$ and this action commutes with the right action of G^k. Our starting point will be the following fact:

Lemma 7.7 ([18] [5]) *The* 3*-form* $\alpha = \frac{2}{3}\, dy \wedge dy_1 \wedge dy_2$ *on* $J_2^*(S^1)$ *is invariant under the action of* Diff^+*.*

Let A_k be the C^* algebra crossed product of $C_1(J_k^+)$ by Γ. Then α is a Γ-invariant Radon measure on J_2^+ and hence yields a trace τ_2 on A_2, which is semi finite and lower semi continuous. One has $C_c^\infty(J_2^+ \rtimes \Gamma) \subset \mathrm{Dom}\, \tau_2$ and

$$\tau_2 \left(\sum f_g\, U_g \right) = \int f_1\, \alpha\, .$$

We let $\varphi_2 : K_0(A_2) \to \mathbb{C}$ be the map given by the trace τ_2. Let $m \in KK_\Gamma(C_0(J_2^+) \rtimes G^2, C(S^1))$ be the canonical Morita equivalence between the C^* algebra $C_0(J_2^+) \rtimes G^2$ and $C(S^1)$ (*cf.* [36]). It yields an isomorphism:

$$m_0 : K_0(A_2 \rtimes G^2) \to K_0(A)\, .$$

Let $\Phi_2 : K_0(A_2) \to K_0(A_2 \rtimes G^2)$ be the Thom isomorphism ([9]) and $\varphi : K_0(A) \to \mathbb{C}$ the additive map:

$$\varphi(x) = \varphi_2\, \Phi_2^{-1}\, m_0^{-1}(x) \qquad \forall\, x \in K_0(A)\, .$$

(We have chosen an orientation on G^2 and hence J_2^+.)

Lemma 7.8 *For any element* $x \in K^*(S^1, \Gamma)$ *one has:*

$$\varphi(\nu(x)) = \langle \Phi\, \mathrm{ch}\,(x), (B\pi)^*\, \mathrm{GV} \rangle\, .$$

Proof. Let $J_{2_\Gamma}^+ = J_2^+ \times_\Gamma E\Gamma$, p_2 the projection of J_2^+ on S^1, $(p_2)_\Gamma$ the corresponding projection of $J_{2,\Gamma}^+$, on S_Γ^1. One obtains in this way a principal G^2 bundle over S_Γ^1 and since G^2 is contractible one can choose a continuous section $f : S_\Gamma^1 \to J_{2,\Gamma}^+$. As in Lemma 6.10 there exists a map θ from $K^*(S^1, \Gamma)$ to $K^*(J_2^+, \Gamma)$ such that, for any $x \in K^*(S^1, \Gamma)$ one has:

a) $\Phi_J\, \mathrm{ch}(\theta(x)) = f_*\, \Phi_{S^1}(\mathrm{ch}(x))$.

b) $\mu(\theta(x)) = \Phi_2^{-1}\, m_0^{-1}(\mu(x))$.

Now the Γ-invariant 3-form α on J_2^+ determines an element $[\alpha]$ of the cohomology group: $H^3(J_{2,\Gamma}^+, \mathbb{R})$, as is easily seen using singular cohomology, *i.e.*

defining a singular 3-cocycle on $J_{2,\Gamma}^+$ by integration of α over any singular 3-chain. This is possible using the Γ invariance of α.

By remark 7.16 we get:

$$\varphi_2(\mu(y)) = \langle \Phi_J \operatorname{ch}(y), [\alpha] \rangle \qquad \forall\, y \in K^*(J_2^+, \Gamma)\,.$$

Thus using a) b) we just have to check now that $f^*[\alpha] = (B\pi)^* \,\mathrm{GV}$, or equivalently that $p_\Gamma^*(B\pi)^* \,\mathrm{GV} = [\alpha]$. This is a simple geometric fact which could be used as a definition of GV, let us only sketch its proof, assuming to simplify that $E\Gamma$ is a manifold. Then the pull back by p_Γ of the foliation of $S_\Gamma^1 = S^1 \times_\Gamma E\Gamma$ whose leaves are the $\{x\} \times_\Gamma E\Gamma$, $x \in S^1$, is the foliation of $J_{2,\Gamma}^+\, E\Gamma$. The result then follows using [19] from the equalities

$$d\omega = \beta \wedge \omega\,, \quad \alpha = d\beta \wedge \beta\,,$$

where $\beta = 2\,\frac{y_2}{y_1^2}\, dy - \frac{dy_1}{y_1}$. •

To end the proof of Theorem 7.3 we shall show that

$$(*) \qquad \varphi(x) = \langle x, \tau \rangle \qquad \forall\, x \in K_0(A)\,.$$

The solvable group G^2 is the semi-direct product $H \rtimes G^1$ of $H = \operatorname{Ker}\rho$ by the group $G^1 = \mathbb{R}_+^*$, so that the Thom isomorphism $\Phi_2 : K_0(A_2) \to K_0(A_2 \rtimes G^2)$ is the composition of the two Thom isomorphisms $\Phi'' : K_0(A_2) \to K_1(A_2 \rtimes H)$ and $\Phi' : K_1(A_2 \rtimes H) \to K_0((A_2 \rtimes H) \rtimes G^1)$.

Since J_2^+ is a Γ-equivariant principal G bundle over J_1^+, the Morita equivalence $(C_0(J_2^+) \rtimes \Gamma) \rtimes H \simeq C_0(J_1^+) \rtimes \Gamma$ yields an isomorphism $m_1 : K_1(A_2 \rtimes H) \to K_1(A_1)$. Let Φ_1 be the Thom isomorphism $\Phi_1 : K_1(A_1) \to K_0(A_1 \rtimes G^1)$ and $m_2 : K_0(A_1 \rtimes G^1) \to K_0(A)$ the isomorphism obtained by Morita equivalence.

Lemma 7.9 $m_0\, \Phi_2 = m_2\, \Phi_1\, m_1\, \Phi''$.

Proof. One checks that $m_0\, \Phi' = m_2\, \Phi_1\, m_1$, the result then follows from the equality $\Phi_2 = \Phi' \circ \Phi''$. •

To prove $(*)$ we shall construct a 1-trace τ_3 on A_1 such that
(α) $\langle \tau_3, m_1 \Phi''(x) \rangle = \langle \tau_2, x \rangle$, $\forall\, x \in K_0(A_2)$.
(β) $\langle \tau, m_2\, \Phi_1(y) \rangle = \langle \tau_3, y \rangle$, $\forall\, y \in K_1(A_1)$.

Since the 3-form α is invariant under the action of H on J_2^+ (one has $(y,y_1,y_2)\,g_b = (y,y_1,y_2+by_1)$, for $g_b\in H$, $g_b(t)=t+bt^2$) the dual weight $\hat\tau_2$ of τ_2 on $A_2\rtimes H$ is a trace and the equality $\tau_3'(x^0,x^1)=\hat\tau_2(x^0D_H(x^1))$, $x^j\in C_c^\infty(H,\mathrm{Dom}\,\tau_2)$ where D_H is the derivation of $A_2\rtimes H$ which generates the dual action, determines a 1-trace on $A_2\rtimes H$. By [9] one has: $\langle\tau_3',\Phi''(x)\rangle=\langle\tau_2,x\rangle$, $\forall\,x\in K_0(A_2)$. We define τ_3 as the 1-trace on A_1 associated by Lemma 7.1 to the 1-cocycle $\omega\in Z^1(\Gamma,\Omega^2_{J_1^+})$ given by:

$$\omega(g)=d\ell(g)\wedge\frac{dy_1}{y_1}\,.$$

Lemma 7.10 $\langle\tau_3,m_1(z)\rangle=\langle\tau_3',z\rangle$, $\forall\,z\in K_1(A_1\rtimes H)$.

Proof. Let G be the Lie group $\Gamma\times H$ and consider the following two G spaces:

1) J_2^+ with the natural right action of $\Gamma\times H$.

2) $J_1^+\times\mathbb{R}$ with the right action of $\Gamma\times H$ given by the product of the natural right action of Γ on J_1^+ and the action by translation of H on $\mathbb{R}$.

One has $A_2\rtimes H=C_0(J_2^+)\rtimes G$ and the 1-trace τ_3' is given, using Lemma 7.1, by the following 1-cocycle of G with values in $\Omega^3_{J_2^+}$:

$$\omega'(g,b)=b\alpha\in\Omega^3_{J_2^+}\qquad\forall\,(g,b)\in\Gamma\times H\,.$$

For any $(y,y_1,y_2)\in J_2^+$, $(g,b)\in G$ one has:

$$(y,y_1,y_2)(g,b)=\left(yg,y_1v_1,(y_2+by_1)\,v_1+\frac{1}{\tau}\,y_1^2\,v_2\right)$$

where $v_1=\frac{dy\,g}{dy}$, $v_2=\frac{d^2yg}{dy^2}$.

For any $(y,y_1)\in J_1^+$, $t\in\mathbb{R}$, and $(g,b)\in G$ one has:

$$((y,y_1),t)(g,b)=((yg,y_1v_1),t+b)\,.$$

For any $(y,y_1)\in J_1^+$, $g\in\Gamma$, put $v((y,y_1)\,g)=\frac{1}{2}\,y_1\,\frac{v_2}{v_1}$ where $v_1=\frac{dy\,g}{dy}$, $v_2=\frac{d^2yg}{(dy)^2}$. Then the following formula defines a groupoid isomorphism θ of $(J_1^+\times\mathbb{R})\times G$ on $J_2^+\times G$:

$$\theta(((y,y_1),t),(g,b))=((y,y_1,ty_1),(g,b-v(y,y_1,g)))\,.$$

Since the transverse function ν on $(J_1^+ \times \mathbb{R}) \rtimes G$ given by the Haar measure on G is mapped under θ to the transverse function ν' on $J_2^+ \rtimes G$ given by the same Haar measure on G, the trace τ_3'' on $C_0(J_1^+ \times \mathbb{R}) \times G$ which corresponds to τ_3' under θ is associated by Lemma 7.1 to the following 1-cocycle $\omega'' \in Z^1(G, \Omega^3_{J_1^+ \times \mathbb{R}})$:

$$\omega''(g,b) = (b + v(y, y_1, g^{-1}))\, \frac{2}{y_1^2}\, dy \wedge dy_1 \wedge dt\,.$$

Of course $C_0(J_1^+ \times \mathbb{R}) \rtimes G = (C_0(J_1^+) \rtimes \Gamma) \otimes (C_0(\mathbb{R}) \rtimes H) = A_1 \otimes k$, where k is the elementary C^* algebra of compact operators. The Morita equivalence isomorphism: $K_1(A) \simeq K_1(A_1 \otimes k)$ is given by any homomorphism $h : A_1 \to A_1 \otimes k$ of the form: $h(x) = x \otimes e$, with e a minimal idempotent of k. Choosing $e \in C_c^\infty(\mathbb{R} \times H)$, let us check that $\tau_3'' \circ h = \tau_3$, which will complete the proof of Lemma 7.10.

Since $\int e(\gamma^0)\, e(\gamma^1)\, bdb\, dt = 0$, where $\gamma^0, \gamma^1 \in \mathbb{R} \rtimes H$ the groupoid coming from the action of H on $\mathbb{R}$, and $\gamma^0 = (\gamma^1)^{-1}$, one gets

$$\begin{aligned} \tau_3'' \circ h(f^0, f^1) &= \sum \int (f^0 \otimes e)(\gamma^0)(f^1 \otimes e)(\gamma^1)\, \omega''(g_1, b_1) \\ &= \sum \int (f^0 \otimes e)(\gamma^0)(f^1 \otimes e)(\gamma^1)\, v(y, y_1, g_1^{-1}) \\ &\quad \frac{2}{y_1^2}\, dy \wedge dy_1 \wedge dt \wedge db_1 \end{aligned}$$

where $\gamma^0, \gamma^1 \in (J_1^+ \times \mathbb{R}) \times G$, $\gamma^1 = (\gamma^0)^{-1}$ and $(y, y_1, t) = r(\gamma^0) = s(\gamma^1)$. One has $v(y, y_1, g_1^{-1})\, \frac{2}{y_1^2}\, dy \wedge dy_1 \wedge dt \wedge db_1 = d\ell(g_1) \wedge \frac{dy_1}{y_1} \wedge dt \wedge db_1$ and writing $(J_1^+ \times \mathbb{R}) \rtimes G$ as a product: $(j_1^+ \rtimes \Gamma) \times (\mathbb{R} \rtimes H)$ the last integral gives:

$$\left(\sum \int f^0(\gamma^0)\, f^1(\gamma^1)\, d\ell(g_1) \wedge \frac{dy_1}{y_1} \right) \times \mathrm{Trace}\,(e^2)\,.$$

As $\mathrm{Trace}\,(e^2) = \mathrm{Trace}\,(e) = 1$ we get the desired result. $\bullet$

We have proven α) it remains to prove β). The action of $G^1 = \mathbb{R}_+^*$ on J_1^+ is given by $(y, y_1)\, g_a = (y, ay_1)$ where $g_a(t) = at$. Thus it preserves the 1-cocycle $\omega \in Z^1(\Gamma, \Omega^2)$ given by $\omega(g) = \frac{1}{2}\, d\ell(g) \wedge \frac{dy_1}{y_1}$. It follows that the 1-trace τ_3 on $A_1 = C_0(J_1^+) \rtimes \Gamma$ is invariant under the action of G^1. Let $\hat\tau_3$ be the dual 1-trace on the crossed product $A_1 \rtimes G^1$. It is defined thanks to the following lemma:

Lemma 7.11

a) *Let $(A_1, \theta, \mathbb{R})$ be a C^* dynamical system and τ_3 a θ-invariant 1-trace on A_1. Then the following formula defines a 1-trace $\hat{\tau}_3$ on $\hat{A}_1 = A_1 \rtimes_\theta \mathbb{R}$, invariant under the dual action:*

$$\hat{\tau}_3(y^0, y^1) = \int \tau_3(y^0(t), \theta_t(y^1(-t))\, dt$$

where $y^0, y^1 \in C_c^\infty(\mathbb{R}, \mathrm{Dom}\,\tau_3)$.

b) *Let Φ_1 be the Thom isomorphism: $K_1(A_1) \to K_0(A_1 \rtimes_\theta \mathbb{R})$ and D the generator of the dual action, then*

$$\langle \tau_3, y \rangle = \langle i_D\, \hat{\tau}_3, \Phi_1(y) \rangle \qquad \forall\, y \in K_1(A_1)\,.$$

The notation i_D has been defined in Lemma 7.6 above.

Proof. a) Let us check that $\hat{\tau}_3$ is a cyclic 1-cocycle on its domain. One has:

$$\begin{aligned}
\hat{\tau}_3(y^1, y^0) &= \int \tau_3(y^1(t), \theta_t(y^0(-t)))\, dt \\
&= -\int \tau_3(\theta_t(y^0(-t)), y^1(t))\, dt \\
&= -\int \tau_3(y^0(-t), \theta_t(y^1(t)))\, dt \\
&= -\,\hat{\tau}_3(y^0, y^1)
\end{aligned}$$

using the θ invariance of τ_3.

$$\begin{aligned}
& \hat{\tau}_3(y^0y^1, y^2) - \hat{\tau}_3(y^0, y^1y^2) + \hat{\tau}_3(y^2y^0, y^1) \\
= & \int [\tau_3(y^0(t_0)\, \theta_{t_0}(y^1(t_1)), \theta_{t_0+t_1}(y^2(t_2)) \\
- & \ \tau_3(y^0(t_0), \theta_{t_0}(y^1(t_1))\, \theta_{t_0+t_1}(y^2(t_2))) \\
+ & \ \tau_3(y^2(t_2)\, \theta_{t_2}(y^0(t_0)), \theta_{t_0+t_1}(y^1(t_1)))]\, dt_1\, dt_2
\end{aligned}$$

where $t_0 + t_1 + t_2 = 0$.

Using the invariance of τ_3 and $b\tau_3 = 0$ one gets $b\hat{\tau}_3 = 0$. Let us now prove that for $y^1 \in \mathrm{Dom}\,\hat{\tau}_3 = C_c^\infty(\mathbb{R}, \mathrm{Dom}\,\tau_3)$ the linear functional $y^0 \to \hat{\tau}_3(y^0, y^1)$ is continuous on $\hat{A}_1$. It is of the form $L(y_0) = \int \langle \varphi_t, \theta_{-t}(y^0(t)) \rangle\, dt$ where

$t \to \varphi_t$ is a C^∞ map with compact support from $\mathbb{R}$ to the dual A_1^* with the weak topology $\sigma(A_1^*, A_1)$. Let $t_0 \in \mathbb{R}$ be such that $\operatorname{Supp}\varphi \subset [-t_0, t_0]$ and let $z(t) = \int_{-t_0}^{t} \int_{-t_0}^{s'} \theta_{-s}(y^0(s))\, ds\, ds' = \langle (y^0)^* \xi, \eta\rangle$ with $\xi, \eta \in L^2(\mathbb{R})$. Thus $\|z(t)\| \leq C\,\|y^0\|$, $\forall\, t \in [-t_0, t_0]$ and hence integrating by parts in $\int \langle \varphi_t, \theta_{-t}(y^0(t))\rangle\, dt$ one checks that $|L(y^0)| \leq C'\|y^0\|$, $\forall\, y^0$.

b) We shall first reduce to the case when the action θ is trivial:

Let $B = A_1 \otimes C[0,1]$ and α the action of $\mathbb{R}$ on B given by:

$$(\alpha_t(b))_\lambda = \theta_{\lambda t}(b_\lambda) \qquad \forall\, \lambda \in [0,1]\,,\ t \in \mathbb{R}\,,\ b \in B\,.$$

For $\lambda \in [0,1]$, let τ^λ be the 1-trace on B defined by:

$$\tau^\lambda(b^0, b^1) = \tau_3(b^0_\lambda, b^1_\lambda)\,,\ b^i \in C([0,1], \operatorname{Dom}\tau_3)\,.$$

As τ^λ is α-invariant, $\hat\tau^\lambda$ is a 1-trace on $\hat B = B \rtimes_\alpha \mathbb{R}$ and $i_D\,\hat\tau_\lambda$ is a 2-trace on $\hat B$, where D is the generator of the dual action.

Let $\varphi_\lambda = i_D\,\hat\tau_\lambda$. For any $x \in K_0(\hat B)$ the value of $\langle x, \varphi_\lambda\rangle$ depends continuously upon $\lambda \in [0,1]$.

This follows from the density and stability under homomorphic functional calculus of the following subalgebra of $\hat B$:

$$\mathcal{C} = \{y \in \operatorname{Dom} D \subset \hat B\,,\ \lambda \to \delta_\lambda(y) \text{ is continuous from } [0,1] \text{ to } (\hat B)^* \\ \text{endowed with } \tau((\hat B)^*, \hat B)\}\,.$$

Here δ_λ is the unbounded derivation associated to $\hat\tau_\lambda$.

Thus to show that $\langle x, \varphi_0\rangle = \langle x, \varphi_1\rangle$, $\forall\, x \in K_0(\hat B)$, it is enough to show that $\langle x, \varphi_\lambda\rangle = \langle x, \varphi_1\rangle$ for $\lambda \neq 0$. Let then $\hat A_\lambda$ be the crossed product of A_1 by the action θ^λ, $\theta^\lambda_t = \theta_{\lambda t}$, $\forall\, t \in \mathbb{R}$, and ρ_λ be the isomorphism of $\hat A_1$ on $\hat A_\lambda$ given by

$$(\rho_\lambda(f))(s) = \lambda f(\lambda s)\,, \quad \forall\, s \in \mathbb{R}\,, \quad \forall\, f \in C_c^\infty(\mathbb{R}, A_1) \subset \hat A_1\,.$$

With obvious notations one has $\rho_\lambda^*(i_D\,\hat\tau_3) = i_D\,\hat\tau_3$, and hence $\langle x, \varphi_\lambda\rangle = \langle x, \varphi_1\rangle$, $\forall\, x \in K_0(\hat B)$ (see [9]). We have now reduced to the case of the trivial action. Then $\hat A_1 = A_1 \otimes C_0(\hat{\mathbb{R}})$ and $i_D\,\hat\tau_3 = \tau_3 \,\#\, \tau'$ where τ' is the fundamental class. Restricting τ_3 to C^∞ of any unitary $u \in \operatorname{Dom}\tau_3$ one easily concludes. •

The proof of Theorem 7.3 is now ended by the following:

Lemma 7.12 *For any $x \in K_0(A_1 \rtimes G^1)$ one has:*

$$\langle \tau, m_2(x) \rangle = \langle i_D \, \hat{\tau}_3, x \rangle \, .$$

Proof. We proceed as in lemma 7.10. Thus let $G = \Gamma \times G^1$ and consider the following two G-spaces:

1) J_1^+, 2) $S^1 \times \mathbb{R}$ with the product of the action of Γ on S^1 by the action of $G^1 = \mathbb{R}_+^*$ given by $(t,a) \to t + \mathrm{Log}\, a$, $\forall \, t \in \mathbb{R}$, $a \in \mathbb{R}_+^*$.

One has $A_1 \rtimes G^1 = C_0(J_1^+) \rtimes G$ and the 2-trace $i_D \, \hat{\tau}_3$ is given using Lemma 7.1 by the following 2-cocycle of G with values in $\Omega^2_{J_1^+}$; $\mu(g_1,a_1,g_2,a_2) = -(\mathrm{Log}\, a_1)\, \omega(g_2) + (\mathrm{Log}\, a_2)\, \omega(g_1)$. For any $(y,y_1) \in J_1^+$; $(g^1,a) \in G$ one has:

$$(y,y_1)(g,a) = (yg, y_1 \, a v_1) \text{ where } v_1 = \frac{dyg}{dy} \, .$$

For any $y \in S^1$, $t \in \mathbb{R}$ and $(g,a) \in G$ one has:

$$(y,t)(g,a) = (yg, t + \mathrm{Log}\, a) \, .$$

This shows that the following formula defines a groupoid isomorphism θ' of $(S^1 \times \mathbb{R}) \rtimes G$ on $J_1^+ \rtimes G$:

$$\theta'((y,t),(g,a)) = ((y, \exp(t)), (g, a v_1^{-1}))$$

where $v_1 = \frac{dyg}{dy}$. As in Lemma 7.10 it follows that the 2-trace τ' on $C_0(S^1 \times \mathbb{R}) \rtimes G$ corresponding to $i_D \, \hat{\tau}_3$ by θ' is given using Lemma 7.1 by the following 2-cocycle $\mu' \in Z^2(G, \Omega^2_{S^1 \times \mathbb{R}})$:

$$\begin{aligned} \mu'(g_1,a_1,g_2,a_2) \;&=\; \left(\mathrm{Log}\, a_2 + \mathrm{Log}\, \frac{dyg_2^{-1}}{dy} \right) d\ell(g_1)\, g_2 \wedge dt \\ &-\; \left(\mathrm{Log}\, a_1 + \mathrm{Log}\, \frac{d(y(g_1g_2)^{-1})}{d(yg_2^{-1})} \right) d\ell(g_2) \wedge dt \, . \end{aligned}$$

Exactly as in Lemma 7.10 one considers a homomorphism h of A in $(C(S^1) \rtimes \Gamma) \otimes (C_0(\mathbb{R}) \rtimes G^1)$ of the form $h(f) = f \otimes e$ and the 2-trace $\tau' \circ h$ is obtained using Lemma 7.1 from the following 2-cocycle $\mu'' \in Z^2(\Gamma, \Omega^1_{S^1})$:

$$\begin{aligned} \mu''(g_1,g_2) \;&=\; \mathrm{Log} \left(\frac{dyg_2^{-1}}{dy} \right) d\ell(g_1) - \left(\mathrm{Log}\, \frac{d(y(g_1g_2)^{-1})}{d(yg_2^{-1})} \right) d\ell(g_2) \\ & (\ell(g_2)(d\ell(g_1)g_2) - (\ell(g_1)g_2)\, d\ell(g_2)) = \omega(g_1,g_2) \, . \end{aligned}$$

$\bullet$

We have now ended the proof of Theorem 7.3. As an immediate corollary we get:

Corollary 7.13 *Let* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *be a torsion free discrete cocompact subgroup. Let it act in the obvious way on* $P_1(\mathbb{R}) = S^1$ *and let* τ *be the* 2*-trace on* $A = C(S^1) \rtimes \Gamma$ *given by Theorem* 7.3*. Let* $e \in K_0(C_r^*(\Gamma))$ *be the* Γ*-index of the* $\bar\partial$ *operator on* $U = S^1 \backslash \mathrm{PSL}(2,\mathbb{R})$*, then:*

$$\langle \tau, e\rangle \neq 0\,.$$

This shows that the K theory map associated to τ is non trivial in many cases (for any group Γ containing one of the groups appearing in Corollary 7.13). This map is however trivial when $\Gamma = \mathbb{Z}$ acts by rotation on S^1, since the $d\ell(g) = 0$.

Proof. By construction one has $e = \mu(\varepsilon)$ where $\varepsilon \in K^*(pt,\Gamma)$ is the K cocycle described by the triple (M,E,f) where M is the Riemann surface U/Γ, E is the trivial line bundle on M and f is the classifying map: $M \to B\Gamma$ which is K-oriented by the complex structure of M. Let $i : C_r^*(\Gamma) \to A = C(S^1) \rtimes \Gamma$ be the canonical homomorphism. Then $i_*(e) \in K_0(A)$ is equal to $\mu(\varepsilon')$ where $\varepsilon' \in K^*(S^1,\Gamma)$ is the K cocycle described by the triple (M',E',f') where: $M' = S^1 \times_\Gamma \tilde M$, E' is the trivial line bundle, and f' is the tautological map to $S^1 \times_\Gamma E\Gamma$. Note that $TM' \oplus f'^*\tau$ is K oriented by the complex structure of $\tilde M$.

Let us compute $\langle \Phi\,\mathrm{ch}\,(\varepsilon'), (B\pi)^*\,\mathrm{GV}\rangle$. Let us identify M' with $S^1_\Gamma = S^1 \times_\Gamma E\Gamma$. By construction, $(B\pi)^*\,\mathrm{GV}$ is the Godbillon-Vey class for the foliation of $M' = S^1_\Gamma$ obtained from the foliation of $S^1 \times \tilde M$ by $pts \times \tilde M$. Thus by [44], $(B\pi)^*\,\mathrm{GV}$ is equal to a nonzero multiple of the generator of $H^3(M',\mathbb{Z}) \subset H^3(M',\mathbb{R})$. Since E' is the trivial line bundle one has $\mathrm{ch}\,E' = 1$ and the component of $\Phi\,\mathrm{ch}(\varepsilon')$ in $H_3(M',\mathbb{Q})$ is equal to $[M']$, hence the answer. $\bullet$

Our next result improves the vanishing theorem of S. Hurder ([23]).

Theorem 7.14 *Let* Γ *be a discrete group acting by orientation preserving diffeomorphisms on* S^1*. Let* M *be the von Neumann algebra crossed product* $M = L^\infty(S^1) \rtimes \Gamma$ *and assume that the flow of weights* $W(M)$ (cf. [16], [42]) *has no invariant probability measure. Then* $\langle \tau, e\rangle = 0$, $\forall\, e \in K_0(C(S^1) \rtimes \Gamma)$ *where* τ *is the* 2*-trace given by Theorem* 7.3*.*

In particular if M is semi-finite one has $\langle \tau, e\rangle = 0$ for any $e \in K_0(C(S^1)\times\Gamma)$.

Proof. Let τ_3 be the 1-trace on $A_1 = C_0(J_1^+) \rtimes \Gamma$ defined above and δ the corresponding unbounded derivation from A_1 to A_1^*. Let us prove that any element $x \in Z(A_1^{**})$ of the center of the bidual of A_1, belongs to the domain of δ^* and satisfies $\delta^*(x) = 0$. We have to show that $\langle \delta(y), x\rangle = 0$, $\forall\, y \in C_c^\infty(J_1^+ \rtimes \Gamma)$. With $\omega(g) = d\ell(g) \wedge \frac{dy_1}{y_1}$, $\forall\, g \in \Gamma$, one has $\delta\left(\sum y_g\, U_g\right) = \sum y_g\, U_g\, \omega_g = \sum U_g(y_g\, g)\, \omega_g$ where for $f \in C_c^\infty(J_1^+)$, $f\omega_g$ is the element of A_1^* given by $\langle h, f\omega_g\rangle = \int_{J_1^+} h_1\, f\omega_g$, $\forall\, h = \sum h_g\, U_g \in C_0(J_1^+) \rtimes \Gamma$. One has $f\omega_g \in N_* \subset A_1^*$, where N_* is the predual of $N = L^\infty(J_1^*) \rtimes \Gamma$. Thus $\delta(y) \in N_*$, $\forall\, y \in C_c^\infty(J_1^+ \rtimes \Gamma)$ so that we can assume that $x \in N \subset A_1^{**}$. Let $x = \sum x_g\, U_g$, with $x_g \in L^\infty(J_1^+)$. Since $x \in \mathrm{Center}(N) = Z(N)$ the diffeomorphism φ of J_1^+ associated to g is equal to identity almost everywhere where $x_g \neq 0$. One has $\varphi(u,v) = \left(ug, v\, \frac{dug}{du}\right) = (ug, v \exp \ell(g)(u))$. Thus we can assume that $x_g(u,v) \neq 0 \Rightarrow ug = u$ and $\ell(g)(u) = 0$. Since $\ell'(g)$ is equal to zero almost everywhere in $\{u : \ell(g)(u) = 0\}$ we see that the differential form $x_g\, \omega(g^{-1})$ is equal to 0 almost everywhere. Hence

$$\langle \delta(y), x\rangle = \sum \int_{J_1^+} (y_{g^{-1}}\, g^{-1})\, x_g\, \omega(g^{-1}) = 0\,.$$

This shows that for any $x \in Z(N)$ the derivation $\delta^x : \delta^x(y) = x\delta(y) \in N_* \subset A_1^*$, $y \in \mathrm{Dom}\,\delta = C_c^\infty(J_1^+ \rtimes \Gamma)$, is a 1-trace on A_1 which extends to the domain $\mathcal{A}$ of the closure of δ. Let $u \in \mathrm{GL}_q(\tilde{\mathcal{A}})$ and δ_q^x the derivation $\delta^x \otimes \mathrm{id}_q$ from $M_q(\tilde{\mathcal{A}})$ to $M_q(A_1^*)$. The equality

$$L(x) = \langle u^{-1}, \delta_q^x(u)\rangle \qquad \forall\, x \in Z(N)$$

defines a normal linear functional on $Z(N)$ which, since δ^x is a 1-trace, depends only upon the class of u in $K_1(A_1)$. As the group $G^1 = \mathbb{R}_+^*$ is connected, it acts trivially on $K_1(A_1)$ and as τ_3 is invariant under G^1 we get $L(\theta_\lambda(x)) = L(x)$, $\forall\, \lambda \in \mathbb{R}_+^*$, $\forall\, x \in Z(N)$. Since the action of $G^1 = \mathbb{R}_+^*$ on $Z(N)$ is by definition the flow of weights of M we conclude that $L = 0$, hence:

$$\langle \tau_3, z\rangle = 0 \qquad \forall\, z \in K_1(A_1)\,.$$

The conclusion now follows from the equality b) above. $\bullet$

Let α be the 3-form of Lemma 7.7, B a measurable Γ invariant subset of J_1^+ and $J_1^+ \to J_1^+$ be the projection. Then the product α^B of α by the characteristic function of $p^{-1}(B)$ is closed and Γ invariant so that it determines, as in

Lemma 7.8, a singular 3-cocycle $[\alpha^B] \in H^3(J_2^+ \times_\Gamma E\Gamma, \mathbb{R}) = H^3(S^1_\Gamma, \mathbb{R})$. Moreover $[\alpha^{\theta_\lambda(B)}] = [\alpha^B]$. With this one gets a direct geometric proof of Theorem 7.14 when a) the element $e \in K_0(C(S^1) \rtimes \Gamma)$ is in the image $\mu(K^*(S^1, \Gamma))$ and b) the fixed point set of any $g \in \Gamma$, $g \neq 1$ is negligible.

Theorem 7.15 *Let Γ be a discrete group acting by orientation preserving diffeomorphisms of a manifold V of dimension n. Let $B\pi$ be the classifying map $V_\Gamma \to B\Gamma_n$ and γ a Gel'fand-Fuchs class $\gamma \in H^*(\mathrm{WO}_n) = H_d^*(\Gamma_n, \mathbb{R})$. Then there exists an additive map φ of $K_*(A)$, $A = C_0(V) \rtimes \Gamma$, to $\mathbb{C}$ such that, for any $x \in K^*(V, \Gamma)$, one has:*

$$\varphi(\mu(x)) = \langle \Phi \operatorname{ch}(x), (B\pi)^* \gamma \rangle \,.$$

Proof. Let $J_k^+(V)$ be the positive higher frame bundle over V, *i.e.* an element of $J_k^+(V)$ is the k-jet $j_0^k(f)$ at 0 of a germ of orientation preserving local diffeomorphism of a neighborhood of 0 in $\mathbb{R}^n$ with an open subset of V. As above (*cf.* [21]) $J_k^+(V)$ is both a Γ-manifold and a principal G^k bundle, over V, where G^k is the Lie group of k jets of orientation preserving local diffeomorphisms of $\mathbb{R}^n$ fixing 0.

The group $\mathrm{SO}(n)$ sits naturally in G^k as a maximal compact subgroup. Let $V_k = J_k^+(V)/\mathrm{SO}(n)$, and (*cf.* [18], [5]) $\gamma \to \alpha(\gamma)$ be the natural map of the complex $\mathrm{WO}(n)$ to the complex of $\mathrm{Diff}^+(V)$ invariant differential forms on V_k. Given $\gamma \in H^q(\mathrm{WO}_n)$ let k be large enough so that $\alpha(\gamma)$ is already defined on V_k. The pull back to V_k of the tangent bundle TV is a Γ-equivariant bundle on V_k, thus by Remark 7.16, given any element P of the Pontrjagin ring of this bundle on $V_{k,\Gamma} = V_k \times_\Gamma E\Gamma$, there exists an additive map Ψ_P of $K_*(C_0(V_k) \rtimes \Gamma)$ to $\mathbb{C}$ such that:

$$\Psi_P(\mu(x)) = \langle \Phi \operatorname{ch} x, [\alpha(\gamma)] \cdot P \rangle \qquad \forall\, x \in K^*(V_k, \Gamma) \,.$$

Here the class $[\alpha(\gamma)] \in H^q(V_k \times_\Gamma E\Gamma)$ is defined as in Lemma 7.8. As in Section 6 we may as well assume that n is even, in which case the Spin condition is satisfied by the action of G^k on $G^k/\mathrm{SO}(n)$ and one gets by [27] a natural element β of $KK_\Gamma(C_0(V), C_0(V_k))$. It yields a map $\beta' : K_*(C_0(V) \rtimes \Gamma) \to K_*(C_0(V_k) \rtimes \Gamma)$ which when composed with Ψ_P for a suitable P gives the correct answer: $\Psi_P \circ \beta' \circ \mu$ when applied to any $x \in K^*(V, \Gamma)$, exactly as in Theorem 6.7. $\bullet$

Remark 7.16

a) Theorem 7.15 is much less precise than Theorem 7.3, in as much as it does not describe the map φ by a cyclic cocycle on $C_c^\infty(V \rtimes \Gamma)$. Note that $V_1 = J_1^+(V)/\mathrm{SO}(n)$ is naturally isomorphic to the Γ-bundle W of Section 6, and in particular the action of Γ on V_1 is almost isometric. This fact together with the solvability of the simply connected group $H_k = \mathrm{Ker}(G^k \to G^1)$ allows to describe φ by an m-trace on the Banach algebra $B \subset C_0(V_1) \rtimes \Gamma$ of Section 6.

b) The construction of the 2-cocycle τ of Theorem 7.3 starting from the 1-trace associated to the fundamental class of V/Γ can be formulated purely in C^* algebraic terms. The 1-trace δ and the modular automorphism group σ_t^φ satisfy the auxiliary condition $\frac{d^2}{dt^2}\,\sigma_t^\varphi(\delta) = 0$, then the formula $i_D \frac{d}{dt}(\delta)$ makes sense and yields τ.

c) Lemma 7.1 could be formulated in the framework of [12] but in our differentiable context the relevant generalization would be the construction of map from the bicomplex of group cochains of Γ with coefficients in the de Rham currents on V to the bicomplex (b, B) defined in [7] associated to the algebra $C_c^\infty(V \rtimes \Gamma)$.

8 Geometric corollaries

In this section we shall state the analogues of Theorems 6.9 and 7.14 in the context of foliations. Choosing an open disconnected manifold T transverse to the foliation (V, F), with $\dim T = \mathrm{Codim}\, F$ which intersects every leaf of (V, F) one may replace the C^* algebra $C^*(V, F)$ of the foliation by the Morita equivalent C^* algebra of the groupoid $(\mathrm{Graph}(V, F))_T = \{\gamma \in \mathrm{Graph}(V, F), s(\gamma) \in T, r(\gamma) \in T\}$. This differentiable groupoid G_T is now "discrete" in as much as the range and source maps are etale maps. The point is that in the results of Sections 3, 4, 5, 6, 7 we could formulate everything in terms of the differentiable "discrete" groupoid $V \rtimes \Gamma$, so that these results still hold in this new category.

Theorem 8.1 *Let (V, F) be a (not necessarily compact) foliated manifold which is transversally oriented. Let $G = \mathrm{Graph}(V, F)$ be its holonomy groupoid and $\pi : BG \to B\Gamma_q$ be the map classifying the natural Haefliger stucture ($q = \mathrm{Codim}\, V$), τ the bundle on BG given by the transverse bundle of (V, F). Let $\mathcal{R} \subset H^*(BG, \mathbb{C})$ be the ring generated by the Pontrjagin classes of τ, the Chern classes of holonomy equivariant bundles on V and $\pi^*(H^*(\mathrm{WO}_q))$.*

For any $P \in \mathcal{R}$ there exists an additive map φ of $K_*(C^*(V,F))$ to $\mathbb{C}$ such that:

$$\varphi(\mu(x)) = \langle \Phi \,\mathrm{ch}(x), P\rangle \qquad \forall\, x \in K^*(V,F)\,.$$

Here $K^*(V,F)$ is the geometric group as defined in [3] § 8, ch is the Chern character: $K^*(V,F) \to H_*(B\tau, S\tau)$ where τ is the bundle on BG corresponding to the transverse bundle of (V,F), and $\Phi : H_*(B\tau,S\tau) \to H_*(BG)$ is the Thom isomorphism.

An important step in the proof of this theorem is the longitudinal index theorem for foliations [15].

We shall now state several corollaries.

Corollary 8.2 *Let (V,F) be a transversally oriented foliation of codimension q, then the class $[V/F]^*$ of* [11] *is a non torsion element of $K_q(C^*(V,F))$.*

The proof is the same as in Theorem 5.5. Note that V need not be compact. When (V,F) is not transversally orientable, with V connected $[V/F]^*$ is a 2-torsion element (see [43] for relevant computations). We now pass to two other corollaries which are purely geometric, *i.e.* the C^* algebra $C^*(V,F)$ disappears in the statement. Its role is to allow to integrate in K theory *in two steps*: 1) Along the leaves of the foliation (this provides under suitable K orientation hypothesis a map from $K^*(V)$ to $K_*(C^*(V,F))$. 2) Over the space of leaves (this provides a map from $K_*(C^*(V,F))$ to $\mathbb{C}$).

That the composition of these two steps is the same as integration (in K theory) over V is a corollary of Theorem 8.1.

It follows from [28], [37] that if the bundle F is a Spin bundle which can be endowed with a Euclidean metric with strictly positive (lower bounded by $\varepsilon > 0$) scalar curvature (it makes sense since F is integrable) then the longitudinal integral of the trivial bundle does vanish (*i.e.* the K theory index of the longitudinal Dirac operator is equal to 0). Thus the integral over V of the trivial bundle does vanish or in other words $\hat{A}(V) = 0$ (*cf.* [2] for the definition of $\hat{A}$, note that here only F is assumed to be Spin so that $\hat{A}(V)$ is not a priori an integer). More precisely:

Corollary 8.3 *Let V be a compact foliated oriented manifold. Assume that the integrable bundle $F \subset TV$ is a Spin bundle and is endowed with a metric of strictly positive ($\geq \varepsilon > 0$) scalar curvature. Let $\mathcal{R}$ be the subring* *of $H^*(V,\mathbb{C})$ generated by the Pontrjagin classes of $\tau = TV/F$, the Chern classes of holonomy equivariant bundles and the range of the natural map: $H^*(\mathrm{WO}_q) \to H^*(V,\mathbb{C})$. Then $\langle \hat{A}(F)\,\omega, [V]\rangle = 0$, $A\omega \in \mathcal{R}$; where $\hat{A}(F)$ is the $\hat{A}$ class of this Spin bundle.*

When $F = TV$ *i.e.* when the foliation has just one leaf, this is exactly the content of the well known vanishing theorem of A. Lichnerowicz ([28]).

As an immediate application we see that no spin foliation of a compact manifold V, with non-zero $\hat{A}$ genus: $\hat{A}(V) \neq 0$, does admit a metric of strictly positive scalar curvature.

Proof. The projection $V \to V/F$ is K oriented by the Spin structure on F and hence defines a K-cocycle $x \in K^*(V,F)$ (*cf.* [3]). The argument of [37] shows that the analytical index of the Dirac operator along the leaves of (V,F) is equal to 0 in $K_*(C^*(V,F))$, so that one has $\mu(x) = 0$.

Let $f : V \to BG$ be the map associated as in [11] to the projection $V \to V/F$. There exists a polynomial P in the Pontrjagin classes of τ, with leading coefficient 1, such that

$$\Phi \circ \mathrm{ch}(x) = f_*(\hat{A}(F) \cup [V]) \cup P \in H_*(BG)\,.$$

Thus, since $\mu(x) = 0$, the result follows from Theorem 8.1. •

Corollary 8.4 *Let (V,F), (V',F') be oriented and transversally oriented compact foliated manifolds. Let $f : V \to V'$ be a smooth, orientation preserving, leafwise homotopy equivalence* (cf. [4])*. Then for any element P of the ring $\mathcal{R} \subset H^*(V,\mathbb{C})$ of Corollary* 8.3 *one has:*

$$\langle (f^*\mathcal{L}(V') - \mathcal{L}(V)), P \cup [V]\rangle = 0$$

where $\mathcal{L}(V)$ (resp. (V')) is the L-class of V (resp. V') cf. [31].

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