

Hopf algebras, cyclic cohomology and the transverse index theorem

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Introduction

In this paper we present the solution of a longstanding internal problem of noncommutative geometry, namely the computation of the index of transversally elliptic operators on foliations.

The spaces of leaves of foliations are basic examples of noncommutative spaces and already exhibit most of the features of the general theory. The index problem for longitudinal elliptic operators is easy to formulate in the presence of a transverse measure, cf. [Co] [M-S], and in general it leads to the construction (cf. [C-S]) of a natural map from the geometric group to the K -theory of the leaf space, i.e. the K -theory of the associated C^* -algebra. This “assembly map” μ is known in many cases to exhaust the K -theory of the C^* -algebra but property T in the group context and its analogue for foliations provide conceptual obstructions to tentative proofs of its surjectivity in general. One way to test the K -group, $K(C^*(V, F)) = K(V/F)$ for short, is to use its natural pairing with the K -homology group of $C^*(V, F)$. Cycles in the latter represent “abstract elliptic operators” on V/F and the explicit construction for *general* foliations of such cycles is already quite an elaborate problem. The delicate point is that we do not want to assume any special property of the foliation such as, for instance, the existence of a holonomy invariant transverse metric as in Riemannian foliations. Equivalently, we do not want to restrict in anyway the holonomy pseudogroup of the foliation.

In [Co1] [H-S] [C-M] a general solution was given to the construction of transversal elliptic operators for foliations. The first step ([Co1]) consists in passing by a Thom isomorphism to the total space of the bundle of transversal metrics. This first step is a geometric adaptation of the reduction of an arbitrary factor of type III to a crossed product of a factor of type II by a one-parameter group of automorphisms. Instead of only taking care of the volume distortion (as in the factor case) of the involved elements of the pseudogroup, it takes care of their full Jacobian. The second step ([H-S]) consisted in realizing that while the standard theory of *elliptic* pseudodifferential operators is too restrictive to allow the construction of the desired K -homology cycle, it suffices to replace it by its refinement to *hyppoelliptic* operators. This was used in [C-M] in order to construct a *differential* (hyppoelliptic) operator Q , solving the general construction of the K -cycle.

One then arrives at a well posed general index problem. The index defines a

map: $K(V/F) \rightarrow \mathbb{Z}$ which is simple to compute for those elements of $K(V/F)$ in the range of the assembly map. The problem is to provide a general formula for the cyclic cocycle $\text{ch}_*(D)$, which computes the index by the equality

$$(1) \quad \langle \text{ch}_*(D), \text{ch}^*(E) \rangle = \text{Index } D_E \quad \forall E \in K(V/F),$$

where the Chern character $\text{ch}^*(E)$ belongs to the cyclic homology of V/F . We showed in [C-M] that the spectral triple given by the algebra \mathcal{A} of the foliation, together with the operator D in Hilbert space \mathcal{H} actually fulfills the hypothesis of a general abstract index theorem, holding at the operator theoretic level. It gives a “local” formula for the cyclic cocycle $\text{ch}_*(D)$ in terms of certain residues that extend the ideas of the Wodzicki-Guillemin-Manin residue as well as of the Dixmier trace. Adopting the notation \int for such a residue, the general formula gives the components φ_n of the cyclic cocycle $\varphi = \text{ch}_*(D)$ as universal *finite* linear combinations of expressions which have the following general form

$$(2) \quad \int a^0 [D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} |D|^{-n-2|k|}, \quad \forall a^j \in \mathcal{A},$$

where for an operator T in \mathcal{H} the symbol $T^{(k)}$ means the k^{th} iterated commutator of D^2 with T .

It was soon realized that, although the general index formula easily reduces to the local form of the Atiyah-Singer index theorem when D is say a Dirac operator on a manifold, the actual explicit computation of all the terms (2) involved in the cocycle $\text{ch}_*(D)$ is a rather formidable task. As an instance of this let us mention that even in the case of codimension one foliations, the printed form of the explicit computation of the cocycle takes around one hundred pages. Each step in the computation is straightforward but the explicit computation for higher values of n is clearly impossible without a new organizing principle which allows to bypass them.

In this paper we shall adapt and develop the theory of cyclic cohomology to Hopf algebras and show that this provides exactly the missing organizing principle, thus allowing to perform the computation for arbitrary values of n . We shall construct for each value of n a specific Hopf algebra $\mathcal{H}(n)$, show that it acts on the C^* -algebra of the transverse frame bundle of any codimension n foliation (V, F) and that the index computation takes place within the cyclic cohomology of $\mathcal{H}(n)$. We compute this cyclic cohomology explicitly as Gelfand-Fuchs cohomology. While the link between cyclic cohomology and Gelfand-Fuchs cohomology was already known ([Co]), the novelty consists in the fact that the entire differentiable transverse structure is now captured by the action of the Hopf algebra $\mathcal{H}(n)$, thus reconciling our approach to noncommutative geometry to a more group theoretical one, in the spirit of

the Klein program.

I. Notations

We let M be an n -dimensional smooth manifold (not necessarily connected or compact but assumed to be oriented). Let us first fix the notations for the frame bundle of M , $F(M)$, in local coordinates

$$(1) \quad x^\mu \quad \mu = 1, \dots, n \quad \text{for } x \in U \subset M.$$

We view a frame with coordinates x^μ, y_j^μ as the 1-jet of the map

$$(2) \quad j : \mathbb{R}^n \rightarrow M, \quad j(t) = x + yt \quad \forall t \in \mathbb{R}^n$$

where $(yt)^\mu = y_i^\mu t^i \quad \forall t = (t^i) \in \mathbb{R}^n$.

Let φ be a local (orientation preserving) diffeomorphism of M , it acts on $F(M)$ by

$$(3) \quad \varphi, j \rightarrow \varphi \circ j = \tilde{\varphi}(j)$$

which replaces x by $\varphi(x)$ and y by $\varphi'(x)y$ where

$$(4) \quad \varphi'(x)_\beta^\alpha = \partial_\beta \varphi(x)^\alpha \quad \text{where } \varphi(x) = (\varphi(x)^\alpha).$$

We restrict our attention to orientation preserving frames $F^+(M)$, and in the one dimensional case ($n = 1$) we take the notation

$$(5) \quad y = e^{-s}, \quad s \in \mathbb{R}.$$

In terms of the coordinates x, s one has,

$$(6) \quad \tilde{\varphi}(s, x) = (s - \log \varphi'(x), \varphi(x))$$

and the invariant measure on F is ($n = 1$)

$$(7) \quad \frac{dx dy}{y^2} = e^s ds dx.$$

One has a canonical right action of $GL^+(n, \mathbb{R})$ on F^+ which is given by

$$(8) \quad (g, j) \rightarrow j \circ g, \quad g \in GL^+(n, \mathbb{R}), \quad j \in F^+$$

it replaces y by yg , $(yg)_j^\mu = y_i^\mu g_j^i \quad \forall g \in GL^+(n, \mathbb{R})$ and F^+ is a $GL^+(n, \mathbb{R})$ principal bundle over M .

We let Y_i^j be the vector fields on F^+ generating the action of $GL^+(n, \mathbb{R})$,

$$(9) \quad Y_i^j = y_i^\mu \frac{\partial}{\partial y_j^\mu} = y_i^\mu \partial_\mu^j.$$

In the one dimensional case one gets a single vector field,

$$(10) \quad Y = -\partial_s .$$

The action of Diff^+ on F^+ preserves the \mathbb{R}^n valued 1-form on F^+ ,

$$(11) \quad \alpha^j = (y^{-1})^j_\beta dx^\beta .$$

One has $y_j^\mu \alpha^j = dx^\mu$ and $dy_j^\mu \wedge \alpha^j + y_j^\mu d\alpha^j = 0$.

Given an affine torsion free connection Γ , the associated one form ω ,

$$(12) \quad \omega_j^\ell = (y^{-1})^\ell_\mu (dy_j^\mu + \Gamma_{\alpha,\beta}^\mu y_j^\alpha dx^\beta)$$

is a 1-form on F^+ with values in $\mathbf{GL}(n)$ the Lie algebra of $GL^+(n, \mathbb{R})$. The $\Gamma_{\alpha,\beta}^\mu$ only depend on x but not on y , moreover one has,

$$(13) \quad d\alpha^j = \alpha^k \wedge \omega_k^j = -\omega_k^j \wedge \alpha^k$$

since Γ is *torsion free*, i.e. $\Gamma_{\alpha,\beta}^\mu = \Gamma_{\beta,\alpha}^\mu$.

The natural horizontal vector fields X_i on F^+ associated to the connection Γ are,

$$(14) \quad X_i = y_i^\mu (\partial_\mu - \Gamma_{\alpha,\mu}^\beta y_j^\alpha \partial_\beta^j),$$

they are characterized by

$$(15) \quad \langle \alpha^j, X_i \rangle = \delta_i^j \quad \text{and} \quad \langle \omega_k^\ell, X_i \rangle = 0 .$$

For $\psi \in \text{Diff}^+$, the one form $\tilde{\psi}^* \omega$ is still a connection 1-form for a new affine torsion free connection Γ' . The new horizontal vector fields X'_i are related to the old ones by

$$(16) \quad X'_i = \tilde{\varphi}_* X_i \circ \tilde{\psi}, \quad \varphi = \psi^{-1} .$$

When ω is the trivial flat connection $\Gamma = 0$ one gets

$$(17) \quad \Gamma' = \psi'(x)^{-1} d\psi'(x), \quad \Gamma'_{\alpha,\beta}^\mu = (\psi'(x)^{-1})^\mu_\rho \partial_\beta \partial_\alpha \psi^\rho(x) .$$

II. Crossed product of $F(M)$ by Γ and action of $\mathcal{H}(n)$

We let M be gifted with a *flat* affine connection ∇ and let Γ be a pseudogroup of local diffeomorphisms, preserving the orientation,

$$(1) \quad \psi : \text{Dom } \psi \rightarrow \text{Range } \psi$$

where both the domain, $\text{Dom } \psi$ and range, $\text{Range } \psi$ are open sets of M . By the functoriality of the construction of $F(M)^+$, $\forall \psi \in \Gamma$ we let $\tilde{\psi}$ be the corresponding local diffeomorphism of $F^+(M)$.

We let $\mathcal{A} = C_c^\infty(F^+) \rtimes \Gamma$ be the crossed product of F^+ by the action of Γ on F^+ . It can be described directly as $C_c^\infty(G)$ where G is the etale smooth groupoid,

$$(2) \quad G = F^+ \rtimes \Gamma,$$

an element γ of G being given by a pair (x, φ) , $x \in \text{Range } \varphi$, while the composition is,

$$(3) \quad (x, \varphi) \circ (y, \psi) = (x, \varphi \circ \psi) \quad \text{if } y \in \text{Dom } \varphi \quad \text{and} \quad \varphi(y) = x.$$

In practice we shall generate the crossed product \mathcal{A} as the linear span of monomials,

$$(4) \quad f U_\psi^*, \quad f \in C_c^\infty(\text{Dom } \psi)$$

where the star indicates a contravariant notation. The multiplication rule is

$$(5) \quad f_1 U_{\psi_1}^* f_2 U_{\psi_2}^* = f_1(f_2 \circ \tilde{\psi}_1) U_{\psi_2 \psi_1}^*$$

where by hypothesis the support of $f_1(f_2 \circ \tilde{\psi}_1)$ is a compact subset of

$$(6) \quad \text{Dom } \psi_1 \cap \psi_1^{-1} \text{Dom } \psi_2 \subset \text{Dom } \psi_2 \psi_1.$$

The canonical action of $GL^+(n, \mathbb{R})$ on $F(M)^+$ commutes with the action of Γ and thus extends canonically to the crossed product \mathcal{A} . At the Lie algebra level, this yields the following derivations of \mathcal{A} ,

$$(7) \quad Y_\ell^j (f U_\psi^*) = (Y_j^\ell f) U_\psi^*.$$

Now the flat connection ∇ also provides us with associated horizontal vector fields X_i on $F^+(M)$ (cf. section I) which we extend to the crossed product \mathcal{A} by the rule,

$$(8) \quad X_i (f U_\psi^*) = X_i(f) U_\psi^*.$$

Now, of course, unless the ψ 's are *affine*, the X_i do not commute with the action of ψ , but using (16) and (17) of section I we can compute the corresponding commutator and get,

$$(9) \quad X_i - U_\psi X_i U_\psi^* = -\gamma_{ij}^k Y_k^j$$

where the functions γ_{ij}^k are,

$$(10) \quad \gamma_{ij}^k = y_i^\mu y_j^\alpha (y^{-1})_\beta^k \Gamma_{\alpha\mu}^\beta,$$

$$\Gamma_{\alpha,\beta}^\mu = (\psi'(x)^{-1})_\rho^\mu \partial_\beta \partial_\alpha \psi^\rho(x).$$

It follows that, for any $a, b \in \mathcal{A}$ one has

$$(11) \quad X_i(ab) = X_i(a)b + aX_i(b) + \delta_{ij}^k(a)Y_k^j(b)$$

where the linear operators δ_{ij}^k in \mathcal{A} are defined by,

$$(12) \quad \delta_{ij}^k(fU_\psi^*) = \gamma_{ij}^k fU_\psi^*.$$

To prove (11) one takes $a = f_1 U_{\psi_1}^*$, $b = f_2 U_{\psi_2}^*$ and one computes $X_i(ab) = X_i(f_1 U_{\psi_1}^* f_2 U_{\psi_2}^*) = X_i(f_1 U_{\psi_1}^* f_2 U_{\psi_1}^*) U_{\psi_2 \psi_1}^* = X_i(f_1) U_{\psi_1}^* f_2 U_{\psi_2}^* + f_1 (X_i U_{\psi_1}^* - U_{\psi_1}^* X_i) f_2 U_{\psi_2}^* + f_1 U_{\psi_1}^* X_i(f_2 U_{\psi_2}^*)$.

One then uses (9) to get the result.

Next the γ_{ij}^k are characterized by the equality

$$(13) \quad \widetilde{\psi}^* \omega - \omega = \gamma_{j\ell}^k \alpha^k = \gamma \alpha$$

where α is the canonical \mathbb{R}^n -valued one form on $F^+(M)$ (cf. I).

The equality $\widetilde{\psi_2 \psi_1}^* \omega - \omega = \widetilde{\psi_1}^* (\widetilde{\psi_2}^* \omega - \omega) + (\widetilde{\psi_1}^* \omega - \omega)$ together with the invariance of α thus show that the γ_{ij}^k form a 1-cocycle, so that each δ_{ij}^k is a derivation of the algebra \mathcal{A} ,

$$(14) \quad \delta_{ij}^k(ab) = \delta_{ij}^k(a)b + a\delta_{ij}^k(b).$$

Since the connection ∇ is flat the commutation relations between the Y_j^ℓ and the X_i are those of the affine group,

$$(15) \quad \mathbb{R}^n \rtimes GL^+(n, \mathbb{R}).$$

The commutation of the Y_j^ℓ with δ_{ab}^c are easy to compute since they correspond to the tensorial nature of the δ_{ab}^c . The X_i however do not have simple commutation relations with the δ_{ab}^c , and one lets

$$(16) \quad \delta_{ab, i_1 \dots i_n}^c = [X_{i_1}, \dots, [X_{i_n}, \delta_{ab}^c] \dots].$$

All these operators acting on \mathcal{A} are of the form,

$$(17) \quad T(fU_\psi^*) = hfU_\psi^*$$

where $h = h^\psi$ is a function depending on ψ .

In particular they all commute pairwise,

$$(18) \quad [\delta_{ab, i_1 \dots i_n}^c, \delta_{a'b', i'_1, \dots, i'_m}^{c'}] = 0.$$

It follows that the linear space generated by the Y_j^ℓ , X_i , $\delta_{ab, i_1 \dots i_n}^c$ forms a Lie algebra and we let \mathcal{H} be the corresponding envelopping algebra. We endow \mathcal{H} with a coproduct in such a way that its action on \mathcal{A} ,

$$(19) \quad h, a \rightarrow h(a), \quad h \in \mathcal{H}, \quad a \in \mathcal{A}$$

satisfies the following rule,

$$(20) \quad h(ab) = \sum h_{(0)}(a) h_{(1)}(b) \quad \forall a, b \in \mathcal{A} \quad \text{where} \quad \Delta h = \sum h_{(0)} \otimes h_{(1)}.$$

One gets from the above discussion the equalities

$$(21) \quad \Delta Y_i^j = Y_i^j \otimes 1 + 1 \otimes Y_i^j$$

$$(22) \quad \Delta X_i = X_i \otimes 1 + 1 \otimes X_i + \delta_{ij}^k \otimes Y_k^j$$

$$(23) \quad \Delta \delta_{ij}^k = \delta_{ij}^k \otimes 1 + 1 \otimes \delta_{ij}^k.$$

These rules, together with the equality

$$(24) \quad \Delta(h_1 h_2) = \Delta h_1 \Delta h_2 \quad \forall h_j \in \mathcal{H}$$

suffice to determine completely the coproduct in \mathcal{H} . As we shall see \mathcal{H} has an antipode S , we thus get a Hopf algebra $\mathcal{H}(n)$ which only depends upon the integer n and which acts on any crossed product,

$$(25) \quad \mathcal{A} = C_c^\infty(F) \rtimes \Gamma$$

of the frame bundle of a *flat* manifold M by a pseudogroup Γ of local diffeomorphisms.

We shall devote a large portion of this paper to the understanding of the structure of the Hopf algebra $\mathcal{H}(n)$ as well as of its cyclic cohomology. For notational simplicity we shall concentrate on the case $n = 1$ but all the results are proved in such a way as to extend in a straightforward manner to the general case.

To end this section we shall show that provided we replace \mathcal{A} by a Morita equivalent algebra we can bypass the flatness condition of the manifold M .

To do this we start with an arbitrary manifold M (oriented) and we consider a locally finite open cover (U_α) of M by domains of local coordinates. On $N = \coprod U_\alpha$, the disjoint union of the open sets U_α , one has a natural pseudogroup Γ_0 of diffeomorphisms which satisfy

$$(26) \quad \pi \psi(x) = \pi(x) \quad \forall x \in \text{Dom } \psi$$

where $\pi : N \rightarrow M$ is the natural projection.

Equivalently one can consider the smooth etale groupoid which is the graph of the equivalence relation $\pi(x) = \pi(y)$ in N ,

$$(27) \quad G_0 = \{(x, y) \in N \times N ; \pi(x) = \pi(y)\}.$$

One has a natural Morita equivalence,

$$(28) \quad C_c^\infty(M) \simeq C_c^\infty(G_0) = C_c^\infty(N) \rtimes \Gamma_0$$

which can be concretely realized as the reduction of $C_c^\infty(G_0)$ by the idempotent,

$$(29) \quad e \in C_c^\infty(G_0) = C_c^\infty(N) \rtimes \Gamma_0, \quad e^2 = e,$$

associated to a partition of unity in M subordinate to the cover (U_α) ,

$$(30) \quad \sum \varphi_\alpha(x)^2 = 1, \quad \varphi_\alpha \in C_c^\infty(U_\alpha)$$

by the formula,

$$(31) \quad e(u, \alpha, \beta) = \varphi_\alpha(u) \varphi_\beta(u).$$

We have labelled the pair $(x, y) \in G_0$ by $u = \pi(x) = \pi(y)$ and the indices α, β so that $x \in U_\alpha, y \in U_\beta$.

This construction also works in the presence of a pseudogroup Γ of local diffeomorphisms of M since there is a corresponding pseudogroup Γ' on N containing Γ_0 and such that, with the above projection e ,

$$(32) \quad (C_c^\infty(N) \rtimes \Gamma')_e \simeq C_c^\infty(M) \rtimes \Gamma.$$

Now the manifold N is obviously flat and the above construction of the action of the Hopf algebra $\mathcal{H}(n)$ gives an action on $\mathcal{A}' = C_c^\infty(N) \rtimes \Gamma', (\mathcal{A}')_e = \mathcal{A} = C_c^\infty(M) \rtimes \Gamma$.

III. One dimensional case, the Hopf algebras \mathcal{H}_n

We first define a bialgebra by generators and relations. As an algebra we view \mathcal{H} as the envelopping algebra of the Lie algebra which is the linear span of $Y, X, \delta_n, n \geq 1$ with the relations,

$$(1) \quad [Y, X] = X, [Y, \delta_n] = n \delta_n, [\delta_n, \delta_m] = 0 \quad \forall n, m \geq 1, [X, \delta_n] = \delta_{n+1} \quad \forall n \geq 1.$$

We define the coproduct Δ by

$$(2) \quad \Delta Y = Y \otimes 1 + 1 \otimes Y, \quad \Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \quad \Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1$$

and with $\Delta \delta_n$ defined by induction using (1).

One checks that the presentation (1) is preserved by Δ , so that Δ extends to an algebra homomorphism,

$$(3) \quad \Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

and one also checks the coassociativity.

For each n we let \mathcal{H}_n be the algebra generated by $\delta_1, \dots, \delta_n$,

$$(4) \quad \mathcal{H}_n = \{P(\delta_1, \dots, \delta_n) ; P \text{ polynomial in } n \text{ variables}\}.$$

We let $\mathcal{H}_{n,0}$ be the ideal,

$$(5) \quad \mathcal{H}_{n,0} = \{P; P(0) = 0\}.$$

By induction on n one proves the following

Lemma 1. *For each n there exists $R_{n-1} \in \mathcal{H}_{n-1,0} \otimes \mathcal{H}_{n-1,0}$ such that $\Delta \delta_n = \delta_n \otimes 1 + 1 \otimes \delta_n + R_{n-1}$.*

Proof. It holds for $n = 1, n = 2$. Assuming that it holds for n one has $\Delta \delta_{n+1} = \Delta [X, \delta_n] = [\Delta X, \Delta \delta_n] = [X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \delta_n \otimes 1 + 1 \otimes \delta_n + R_{n-1}] = \delta_{n+1} \otimes 1 + 1 \otimes \delta_{n+1} + [X \otimes 1 + 1 \otimes X, R_{n-1}] + \delta_1 \otimes [Y, \delta_n] + [\delta_1 \otimes Y, R_{n-1}] = \delta_{n+1} \otimes 1 + 1 \otimes \delta_{n+1} + R_n$ where

$$(6) \quad R_n = [X \otimes 1 + 1 \otimes X, R_{n-1}] + n \delta_1 \otimes \delta_n + [\delta_1 \otimes Y, R_{n-1}].$$

Since $[X, \mathcal{H}_{n-1,0}] \subset \mathcal{H}_{n,0}$ and $[Y, \mathcal{H}_{n-1,0}] \subset \mathcal{H}_{n-1,0} \subset \mathcal{H}_{n,0}$, one gets that $R_n \in \mathcal{H}_{n,0}$. ■

For each $k \leq n$ we introduce a linear form $Z_{k,n}$ on \mathcal{H}_n

$$(7) \quad \langle Z_{k,n}, P \rangle = \left(\frac{\partial}{\partial \delta_k} P \right) (0).$$

One has by construction,

$$(8) \quad \langle Z_{k,n}, PQ \rangle = \langle Z_{k,n}, P \rangle Q(0) + P(0) \langle Z_{k,n}, Q \rangle$$

and moreover $\varepsilon, \langle \varepsilon, P \rangle = P(0)$ is the counit in \mathcal{H}_n ,

$$(9) \quad \langle L \otimes \varepsilon, \Delta P \rangle = \langle \varepsilon \otimes L, \Delta P \rangle = \langle L, P \rangle \quad \forall P \in \mathcal{H}_n.$$

(Check both sides on a monomial $P = \delta_1^{a_1} \dots \delta_n^{a_n}$.)

Thus in the dual algebra \mathcal{H}_n^* one can write (8) as

$$(10) \quad \Delta Z_{k,n} = Z_{k,n} \otimes 1 + 1 \otimes Z_{k,n}.$$

Moreover the $Z_{k,n}$ form a basis of the linear space of solutions of (10) and we need to determine the Lie algebra structure determined by the bracket.

We let for a better normalization,

$$(11) \quad Z'_{k,n} = (k+1)! Z_{k,n}.$$

Lemma 2. *One has $[Z'_{k,n}, Z'_{\ell,n}] = (\ell-k) Z'_{k+\ell,n}$ if $k+\ell \leq n$ and 0 if $k+\ell > n$.*

Proof. Let $P = \delta_1^{a_1} \dots \delta_n^{a_n}$ be a monomial. We need to compute $\langle \Delta P, Z_{k,n} \otimes Z_{\ell,n} - Z_{\ell,n} \otimes Z_{k,n} \rangle$. One has

$$\Delta P = (\delta_1 \otimes 1 + 1 \otimes \delta_1)^{a_1} (\delta_2 \otimes 1 + 1 \otimes \delta_2 + R_1)^{a_2} \dots (\delta_n \otimes 1 + 1 \otimes \delta_n + R_{n-1})^{a_n}.$$

We look for the terms in $\delta_k \otimes \delta_\ell$ or $\delta_\ell \otimes \delta_k$ and take the difference. The latter is non zero only if all $a_j = 0$ except $a_q = 1$. Moreover since R_m is homogeneous of degree $m+1$ one gets $q = k+\ell$ and in particular $[Z'_{k,n}, Z'_{\ell,n}] = 0$ if $k+\ell > n$. One then computes by induction using (6) the bilinear part of R_m . One has $R_1^{(1)} = \delta_1 \otimes \delta_1$, and from (6)

$$(12) \quad R_n^{(1)} = [(X \otimes 1 + 1 \otimes X), R_{n-1}^{(1)}] + n \delta_1 \otimes \delta_n.$$

This gives

$$(13) \quad R_{n-1}^{(1)} = \delta_{n-1} \otimes \delta_1 + C_n^1 \delta_{n-2} \otimes \delta_2 + \dots + C_n^{n-2} \delta_1 \otimes \delta_{n-1}.$$

Thus the coefficient of $\delta_k \otimes \delta_\ell$ is $C_{k+\ell}^{\ell-1}$ and we get

$$(14) \quad [Z_{k,n}, Z_{\ell,n}] = (C_{k+\ell}^{\ell-1} - C_{k+\ell}^{k-1}) Z_{k+\ell,n}.$$

One has $\frac{(k+1)!(\ell+1)!}{(k+\ell+1)!} (C_{k+\ell}^{\ell-1} - C_{k+\ell}^{k-1}) = \frac{\ell(\ell+1)-k(k+1)}{k+\ell+1} = \ell - k$ thus using (11) one gets the result. ■

For each n we let \mathcal{A}_n^1 be the Lie algebra of vector fields

$$(15) \quad f(x) \partial / \partial x \quad , \quad f(0) = f'(0) = 0$$

modulo $x^{n+2} \partial$.

The elements $Z_{k,n} = \frac{x^{k+1}}{(k+1)!} \partial / \partial x$ are related by (11) to $Z'_{k,n} = x^{k+1} \partial / \partial x$ which satisfy the Lie algebra of lemma 2.

Thus \mathcal{A}_n^1 is the Lie algebra of jets of order $(n+1)$ of vector fields which vanish to order 2 at 0.

Proposition 3. *The Hopf algebra \mathcal{H}_n is the dual of the envelopping algebra $\mathcal{U}(\mathcal{A}_n^1)$, $\mathcal{H}_n = \mathcal{U}(\mathcal{A}_n^1)^*$.*

Proof. This follows from the Milnor-Moore theorem. ■

Since the \mathcal{A}_n^1 form a projective system of Lie algebras, with limit the Lie algebra \mathcal{A}^1 of formal vector fields which vanish to order 2 at 0, the inductive limit \mathcal{H}^1 of the Hopf algebras \mathcal{H}_n is,

$$(16) \quad \mathcal{H}^1 = \mathcal{U}(\mathcal{A}^1)^*.$$

The Lie algebra \mathcal{A}^1 is a *graded* Lie algebra, with one parameter group of automorphisms,

$$(17) \quad \alpha_t(Z_n) = e^{nt} Z_n$$

which extends to $\mathcal{U}(\mathcal{A}^1)$ and transposes to $\mathcal{U}(\mathcal{A}^1)^*$ as

$$(18) \quad \langle [Y, P], a \rangle = \left\langle P, \frac{\partial}{\partial t} \alpha_t(a)_{t=0} \right\rangle \quad \forall P \in \mathcal{H}^1, a \in \mathcal{U}(\mathcal{A}^1).$$

Indeed (α_t^*) is a one parameter group of automorphisms of \mathcal{H}^1 such that

$$(19) \quad \alpha_t^*(\delta_n) = e^{nt} \delta_n.$$

One checks directly that α_t^* is compatible with the coproduct on \mathcal{H}^1 and that the corresponding Lie algebra automorphism is (17).

Now (cf. [Dix] 2.1.11) we take the basis of $\mathcal{U}(\mathcal{A}^1)$ given by the monomials,

$$(20) \quad Z_n^{a_n} Z_{n-1}^{a_{n-1}} \dots Z_2^{a_2} Z_1^{a_1}, \quad a_j \geq 0.$$

To each $L \in \mathcal{U}(\mathcal{A}^1)^*$ one associates (cf. [Dix] 2.7.5) the formal power series

$$(21) \quad \sum \frac{L(Z_n^{a_n} \dots Z_1^{a_1})}{a_n! \dots a_1!} x_1^{a_1} \dots x_n^{a_n},$$

in the commuting variables $x_j, j \in \mathbb{N}$.

It follows from [Dix] 2.7.5 that we obtain in this way an isomorphism of the algebra of polynomials $P(\delta_1, \dots, \delta_n)$ on the algebra of polynomials in the x_j 's. To determine the formula for δ_n in terms of the x_j 's, we just need to compute

$$(22) \quad \langle \delta_n, Z_n^{a_n} \dots Z_1^{a_1} \rangle.$$

Note that (22) vanishes unless $\sum j a_j = n$.

In particular, for $n = 1$, we get

$$(23) \quad \rho(\delta_1) = x_1$$

where ρ is the above isomorphism.

We determine $\rho(\delta_n)$ by induction, using the derivation

$$(24) \quad D(P) = \sum \delta_{n+1} \frac{\partial}{\partial \delta_n} (P)$$

(which corresponds to $P \rightarrow [X, P]$).

One has by construction,

$$(25) \quad \langle \delta_n, a \rangle = \langle \delta_{n-1}, D^t(a) \rangle \quad \forall a \in \mathcal{U}(\mathcal{A}^1)$$

where D^t is the transpose of D .

By definition of Z_n as a linear form (7) one has,

$$(26) \quad D^t Z_n = Z_{n-1}, \quad n \geq 2, \quad D^t Z_1 = 0.$$

Moreover the compatibility of D^t with the coproduct of \mathcal{H}^1 is

$$(27) \quad D^t(ab) = D^t(a)b + aD^t(b) + (\delta_1 a)\partial_t b \quad \forall a, b \in \mathcal{U}(\mathcal{A}^1)$$

where $a \rightarrow \delta_1 a$ is the natural action of the algebra \mathcal{H}^1 on its dual

$$(28) \quad \langle P, \delta_1 a \rangle = \langle P\delta_1, a \rangle \quad \forall P \in \mathcal{H}^1, a \in \mathcal{U}(\mathcal{A}^1).$$

To prove (27) one pairs both side with $P \in \mathcal{H}^1$. The *l.h.s.* gives $\langle P, D^t(ab) \rangle = \langle \Delta[X, P], a \otimes b \rangle = \langle [X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \Delta P], a \otimes b \rangle$. The terms in $[X \otimes 1, \Delta P]$ yield $\langle \Delta P, D^t a \otimes b \rangle$ and similarly for $[1 \otimes X, \Delta P]$. The term $[\delta_1 \otimes Y, \Delta P]$ yield $\langle \Delta P, \delta_1 a \otimes \partial_t b \rangle$ thus one gets (27).

Lemma 4. *When restricted to $\mathcal{U}(\mathcal{A}^2)$, D^t is the unique derivation, with values in $\mathcal{U}(\mathcal{A}^1)$ satisfying (26), moreover*

$$D^t(Z_n^{a_n} \dots Z_2^{a_2} Z_1^{a_1}) = D^t(Z_n^{a_n} \dots Z_2^{a_2}) Z_1^{a_1} + Z_n^{a_n} \dots Z_2^{a_2} \frac{a_1(a_1-1)}{2} Z_1^{a_1-1}.$$

Proof. The equality $\Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1$ shows that $a \rightarrow \delta_1 a$ is a derivation of $\mathcal{U}(\mathcal{A}^1)$. One has $\delta_1 Z_n = 0$ for $n \neq 1$ so that $\delta_1 = 0$ on $\mathcal{U}(\mathcal{A}^2)$ and the first statement follows from (27) and (26). The second statement follows from,

$$(29) \quad D^t(Z_1^m) = \frac{m(m-1)}{2} Z_1^{m-1}$$

which one proves by induction on m using (27). ■

Motivated by the first part of the lemma, we enlarge the Lie algebra \mathcal{A}^1 by adjoining an element Z_{-1} such that,

$$(30) \quad [Z_{-1}, Z_n] = Z_{n-1} \quad \forall n \geq 2,$$

we then define Z_0 by

$$(31) \quad [Z_{-1}, Z_1] = Z_0, [Z_0, Z_k] = k Z_k.$$

The obtained Lie algebra \mathcal{A} , is the Lie algebra of formal vector fields with $Z_0 = x \frac{\partial}{\partial x}$, $Z_{-1} = \frac{\partial}{\partial x}$ and as above $Z_n = \frac{x^{n+1}}{(n+1)!} \frac{\partial}{\partial x}$.

We can now compare D^t with the bracket with Z_{-1} . They agree on $\mathcal{U}(\mathcal{A}^2)$ and we need to compute $[Z_{-1}, Z_1^m]$. One has

$$(32) \quad [Z_{-1}, Z_1^m] = \frac{m(m-1)}{2} Z_1^{m-1} + m Z_1^{m-1} Z_0$$

$$([Z_{-1}, Z_1^m Z_1] = \left(\frac{m(m-1)}{2} Z_1^{m-1} + m Z_1^{m-1} Z_0 \right) Z_1 + Z_1^m Z_0 = \left(\frac{m(m-1)}{2} + m \right) Z_1^m + (m+1) Z_1^m Z_0).$$

Thus, if one lets \mathcal{L} be the left ideal in $\mathcal{U}(\mathcal{A})$ generated by Z_{-1}, Z_0 we get,

Proposition 5. *The linear map $D^t : \mathcal{U}(\mathcal{A}^1) \rightarrow \mathcal{U}(\mathcal{A}^1)$ is uniquely determined by the equality $D^t(a) = [Z_{-1}, a] \bmod \mathcal{L}$.*

Proof. For each monomial $Z_n^{a_n} \dots Z_1^{a_1}$ one has $D^t(a) - [Z_{-1}, a] \in \mathcal{L}$, so that this holds for any $a \in \mathcal{U}(\mathcal{A}^1)$. Moreover, using the basis of $\mathcal{U}(\mathcal{A})$ given by the $Z_n^{a_n} \dots Z_1^{a_1} Z_0^{a_0} Z_{-1}^{a_{-1}}$ we see that $\mathcal{U}(\mathcal{A})$ is the direct sum $\mathcal{L} \oplus \mathcal{U}(\mathcal{A}^1)$. ■

(The linear span of the $Z_n^{a_n} \dots Z_0^{a_0} Z_{-1}^{a_{-1}}$ with $a_0 + a_{-1} > 0$ is a left ideal in $\mathcal{U}(\mathcal{A})$ since the product $(Z_m^{b_m} \dots Z_{-1}^{b_{-1}}) (Z_n^{a_n} \dots Z_0^{a_0} Z_{-1}^{a_{-1}})$ can be expressed by decomposing $(Z_m^{b_m} \dots Z_{-1}^{b_{-1}} Z_n^{a_n} \dots Z_1^{a_1})$ as a sum of monomials $Z_q^{c_q} \dots Z_1^{c_1} Z_0^{c_0} Z_{-1}^{c_{-1}}$ which are then multiplied by $Z_0^{a_0} Z_{-1}^{a_{-1}}$ which belongs to the augmentation ideal of \mathcal{U} (Lie algebra of Z_0, Z_1).

We now define a linear form L_0 on $\mathcal{U}(\mathcal{A})$ by

$$(33) \quad L_0(Z_n^{a_n} \dots Z_1^{a_1} Z_0^{a_0} Z_{-1}^{a_{-1}}) = 0 \text{ unless } a_0 = 1, a_j = 0 \quad \forall j,$$

and $L_0(Z_0) = 1$.

Proposition 6. *For any $n \geq 1$ one has*

$$\langle \delta_n, a \rangle = L_0(\underbrace{[Z_{-1}, a] \dots}_{n \text{ times}}) \quad \forall a \in \mathcal{U}(\mathcal{A}^1).$$

Proof. Let us first check it for $n = 1$. We let $a = Z_n^{a_n} \dots Z_1^{a_1}$. Then the degree of a is $\sum j a_j$ and $L_0([Z_{-1}, a]) \neq 0$ requires $\sum j a_j = 1$ so that the only possibility is $a_1 = 1, a_j = 0 \quad \forall j$. In this case one gets $L_0([Z_{-1}, Z_1]) = L_0(Z_0) = 1$. Thus by (23) we get the equality of Proposition 6 for $n = 1$.

For the general case note first that \mathcal{L} is stable under right multiplication by Z_{-1} and hence by the derivation $[Z_{-1}, \cdot]$. Thus one has

$$(34) \quad (D^t)^n(a) = [Z_{-1}, \dots [Z_{-1}, a] \dots] \bmod \mathcal{L} \quad \forall a \in \mathcal{U}(\mathcal{A}^1).$$

Now for $a \in \mathcal{L}$ one has $L_0([Z_{-1}, a]) = 0$. Indeed writing $a = (Z_n^{a_n} \dots Z_1^{a_1}) (Z_0^{a_0} Z_{-1}^{a_{-1}}) = bc$ with $b \in \mathcal{U}(\mathcal{A}^1), c = Z_0^{a_0} Z_{-1}^{a_{-1}}$, one has $[Z_{-1}, a] = [Z_{-1}, b]c + b[Z_{-1}, c]$. Since $b \in \mathcal{U}(\mathcal{A}^1)$ and $[Z_{-1}, c]$ has strictly negative degree one has $L_0(b[Z_{-1}, c]) = 0$. Let $Z_n^{b_n} \dots Z_1^{b_1} Z_0^{b_0}$ be a non zero component of $[Z_{-1}, b]$, then unless all b_i are 0 it contributes by 0 to $L_0([Z_{-1}, b]c)$. But $[Z_{-1}, b] \in \mathcal{U}(\mathcal{A}^0)_0$ has no constant term. Thus one has

$$(35) \quad L_0([Z_{-1}, a]) = 0 \quad \forall a = Z_n^{a_n} \dots Z_1^{a_1} Z_0^{a_0} Z_{-1}^{a_{-1}}$$

except if all $a_j = 0, j \neq 1$ and $a_1 = 1$. $L_0([Z_{-1}, Z_1]) = 1$.

Using (25) one has $\langle \delta_n, a \rangle = \langle \delta_1, (D^t)^{n-1}(a) \rangle$ and the lemma follows. ■

One can now easily compute the first values of $\rho(\delta_n)$, $\rho(\delta_1) = x_1, \rho(\delta_2) = x_2 + \frac{x_1^2}{2}, \rho(\delta_3) = x_3 + x_2 x_1 + \frac{x_1^3}{2}, \rho(\delta_4) = x_4 + x_3 x_1 + 2x_2^2 + 2x_2 x_1^2 + \frac{3}{4} x_1^4$.

The affine structure provided by the δ_n has the following compatibility with left multiplication in $\mathcal{U}(\mathcal{A}^1)$,

Proposition 7. a) One has $R_{n-1} = \sum R_{n-1}^k \otimes \delta_k$, $R_{n-1}^k \in \mathcal{H}_{n-1,0}$.
b) For fixed $a_0 \in \mathcal{U}(\mathcal{A}^1)$ there are $\lambda_n^k \in \mathbb{C}$ such that

$$\langle \delta_n, (a_0 a) \rangle = \langle \delta_n, a_0 \rangle \varepsilon(a) + \sum \lambda_n^k \langle \delta_k, a \rangle.$$

Proof. a) By induction using (6). b) Follows, using $\lambda_n^k = \langle R_{n-1}^k, a_0 \rangle$. ■

The antipode S in $\mathcal{U}(\mathcal{A}^1)$ is the unique antiautomorphism such that

$$(36) \quad S Z_n = -Z_n \quad \forall n.$$

It is non trivial to express in terms of the coordinates δ_n .

In fact if we use the basis Z_j of \mathcal{A}^1 but in reverse order to construct the map ρ we obtain a map $\tilde{\rho}$ whose first values are $\tilde{\rho}(\delta_1) = z_1$, $\tilde{\rho}(\delta_2) = z_2 + \frac{z_1^2}{2}$, $\tilde{\rho}(\delta_3) = z_3 + 3 z_1 z_2 + \frac{1}{2} z_1^3$, $\tilde{\rho}(\delta_4) = z_4 + 2 z_2^2 + 6 z_1 z_3 + 9 z_1^2 z_2 + \frac{3}{4} z_1^4$.

One has $\langle \delta_n, S(Z_m^{a_m} \dots Z_1^{a_1}) \rangle = (-1)^{\sum a_j} \langle \delta_n, Z_1^{a_1} \dots Z_m^{a_m} \rangle$ so that $\rho(S^t \delta_n) = \sum \langle \delta_n, S(Z_m^{a_m} \dots Z_1^{a_1}) \rangle x_1^{a_1} \dots x_m^{a_m} = \sum (-1)^{\sum a_j} \langle \delta_n, Z_1^{a_1} \dots Z_m^{a_m} \rangle x_1^{a_1} \dots x_m^{a_m} = \tilde{\rho}(\delta_n)$ with $z_j = -x_j$ in the latter expression.

Thus $\rho(S^t \delta_1) = -x_1$, $\rho(S^t \delta_2) = -x_2 + \frac{x_1^2}{2}$, $\rho(S^t \delta_3) = -x_3 + 3 x_1 x_2 - \frac{x_1^3}{2}$, $\rho(S^t \delta_4) = -x_4 + 2 x_2^2 + 6 x_1 x_3 - 9 x_1^2 x_2 + \frac{3}{4} x_1^4$. We thus get

$$(37) \quad S^t \delta_1 = -\delta_1, \quad S^t \delta_2 = -\delta_2 + \delta_1^2, \quad S^t \delta_3 = -\delta_3 + 4 \delta_1 \delta_2 - 2 \delta_1^3, \dots$$

The antipode S is characterized abstractly as the inverse of the element $L(a) = a$ in the algebra of linear maps L from $\mathcal{U}(\mathcal{A}^1)$ to $\mathcal{U}(\mathcal{A}^1)$ with the product

$$(38) \quad (L_1 * L_2)(a) = \sum L_1(a_{(1)}) L_2(a_{(2)}) \quad \Delta a = \sum a_{(1)} \otimes a_{(2)}, \quad a \in \mathcal{U}.$$

Thus one has

$$(39) \quad \sum (S^t \delta_{n,(1)}) \delta_{n,(2)} = 0 \quad \forall n, \quad \Delta \delta_n = \sum \delta_{n,(1)} \otimes \delta_{n,(2)}$$

writing $S^t \delta_n = -\delta_n + P_n$ where $P_n(\delta_1, \dots, \delta_{n-1})$ is homogeneous of degree n , this allows to compute $S^t \delta_n$ by induction on n .

Remark. Note that the Schwartzian expression $\sigma = \delta_2 - \frac{1}{2} \delta_1^2$ is uniquely characterized by

$$(40) \quad \rho(\sigma) = x_2;$$

thus, $\rho^{-1}(x_n)$ can be regarded as higher analogues of the Schwartzian.

Let us now describe in a conceptual manner the action of the Hopf algebra \mathcal{H}_∞ on the crossed product,

$$(41) \quad \mathcal{A} = C_c^\infty(F) \rtimes \Gamma$$

of the frame bundle of a 1-manifold M by the pseudogroup Γ associated to $\text{Diff}^+ M$. We are given a flat connection ∇ on M , which we view as a $GL(1)$ -equivariant section,

$$(42) \quad \gamma : F \rightarrow J(M)$$

from F to the space of jets $J(M) = \{\mathbb{R}^n \xrightarrow{j} M\}$. For $\alpha \in F$, $\gamma(\alpha)$ is the jet

$$(43) \quad \gamma(\alpha) = (\exp \nabla) \circ \alpha$$

where $\exp \nabla$ is the exponential map associated to the connection ∇ . We let G be the groupoid $F \rtimes \Gamma$ and let $(\varphi, \alpha) \in G$ with

$$(44) \quad s(\varphi, \alpha) = \alpha \in F, \quad r(\varphi, \alpha) = \tilde{\varphi} \alpha \in F, \quad \varphi \in \Gamma.$$

We let $G(\mathcal{A}^1) \subset \mathcal{U}(\mathcal{A}^1)$ (completed I -adically, $I =$ augmentation ideal) be the group like elements,

$$(45) \quad \psi, \Delta \psi = \psi \otimes \psi.$$

We then have a canonical homomorphism γ from G to $G(\mathcal{A}^1)$ given by

$$(46) \quad \gamma(\varphi, \alpha) = \gamma(\tilde{\varphi} \alpha)^{-1} \circ \varphi \circ \gamma(\alpha)$$

where we identify $G(\mathcal{A}^1)$ with the group of germs of diffeomorphisms by the equality

$$A f = f \circ \varphi^{-1} \quad \forall A \in G(\mathcal{A}^1), \quad f \text{ function on } \mathbb{R}.$$

Theorem 8. For any $f \in C_c^\infty(G)$ and $n \in \mathbb{N}$ one has,

$$(\delta_n f)(g) = \delta_n(\gamma(g)^{-1}) f(g) \quad \forall g \in G.$$

Proof. We first define a representation π of \mathcal{A}^1 in the Lie algebra of vector fields on $F(\mathbb{R})$ preserving the differential form $e^s ds dx$, $y = e^{-s}$,

$$(47) \quad \pi(Z_n) = -\frac{x^n}{n!} \partial_s + \frac{x^{n+1}}{(n+1)!} \partial_x.$$

(One has $i_{Z_n}(e^s ds dx) = -\frac{x^n}{n!} e^s dx - \frac{x^{n+1}}{(n+1)!} e^s ds$ which is closed.) Let then H be the function of $F(\mathbb{R})$ given by

$$(48) \quad H(s, x) = s.$$

By construction the representation π is in fact representing \mathcal{A} , and moreover for any $a \in \mathcal{U}(\mathcal{A})$ one has,

$$(49) \quad L_0(a) = -(\pi(a) H)(0).$$

Indeed, for $a = Z_0$ the r.h.s. is 1 and given a monomial $Z_n^{a_n} \dots Z_1^{a_1} Z_0^{a_0} Z_{-1}^{a_{-1}}$, it vanishes if $a_{-1} > 0$ or if $a_0 > 1$ and if $a_{-1} = 0$, $a_0 = 0$. If $a_{-1} = 0$, $a_0 = 1$ the only case in which it does not vanish is $a_j = 0 \quad \forall j > 0$.

One has $\pi(Z_{-1}) = \partial_x$ and it follows from Proposition 6 that,

$$(50) \quad \langle \delta_n, a \rangle = -(\partial_x^n \pi(a) H)(0).$$

Now if $a = A \in G(\mathcal{A}^1)$ we have, with $\psi = \varphi^{-1}$, that

$$(51) \quad (\pi(a) f)(s, x) = f(s - \log \psi'(x), \psi(x)) \quad \forall f \text{ function on } F(\mathbb{R})$$

and we thus have,

$$(52) \quad \langle \delta_n, A \rangle = (\partial_x^n \log \psi'(x))_{x=0}.$$

We now consider $F(M)$ with the same notations in local coordinates, i.e. (y, x) with $y = e^{-s}$. In crossed product terms we have,

$$(53) \quad \delta_n(f U_\varphi) = f \gamma_n U_\varphi, \quad \gamma_n(y, x) = y^n \partial_x^n (\log \psi'(x)), \quad \psi = \varphi^{-1}.$$

Now U_φ , as a function on G is the characteristic function of the set $\{(\varphi, \alpha); \alpha \in F\}$ and one has $\gamma(\varphi, \alpha)$, for $\alpha = (y, x)$, given by

$$(54) \quad t \rightarrow (\varphi(x + yt) - \varphi(x)) / y \varphi'(x) = \gamma(\varphi, \alpha)(t).$$

IV. The dual algebra \mathcal{H}^*

To understand the dual algebra \mathcal{H}^* we associate to $L \in \mathcal{H}^*$, viewed as a linear form on \mathcal{H} , assumed to be continuous in the I -adic topology, the function with values in $\mathcal{U}(\mathcal{A}^1)$,

$$(1) \quad f(s, t), \quad \langle f(s, t), P \rangle = \langle L, P e^{tX} e^{sY} \rangle \quad \forall P \in \mathcal{H}^1.$$

We shall now write the product in \mathcal{H}^* in terms of the functions $f(s, t)$. We first recall the expansional formula,

$$(2) \quad e^{A+B} = \sum_{n=0}^{\infty} \int_{\sum_{j=1}^n u_j \geq 0} e^{u_0 A} B^{u_1 A} B \dots e^{u_n A} \Pi du_j.$$

We use this formula to compute Δe^{tX} , say with $t > 0$,

$$(3) \quad \Delta e^{tX} = \sum_{n=0}^{\infty} \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \Pi ds_i \delta_1(s_1) \dots \delta_1(s_n) e^{tX} \otimes Y(s_1) \dots Y(s_n) e^{tX}$$

where $\delta_1(s) = e^{sX} \delta_1 e^{-sX}$, $Y(s) = e^{sX} Y e^{-sX} = Y - sX$. One has,

$$(4) \quad (Y - s_1 X) e^{tX} e^{sY} = (\partial_s + (t - s_1) \partial_t) e^{tX} e^{sY}.$$

(Since $e^{tX} Y e^{sY} = (Y - tX) e^{tX} e^{sY}$.)

We thus get the following formula for the product, ($t > 0$),

$$(5) \quad (f_1 f_2)(s, t) = \sum_{n=0}^{\infty} \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \prod ds_i \delta_1(s_1) \dots \delta_n(s_n) f_1(s, t) (\partial_s + (t - s_n) \partial_t) \dots (\partial_s + (t - s_1) \partial_t) f_2(s, t).$$

We apply this by taking for f_1 the constant function,

$$(6) \quad f_1(s, t) = \varphi \in G(\mathcal{A}^1) \subset \mathcal{U}(\mathcal{A}^1)$$

while we take the function f_2 to be scalar valued.

One has $\delta_1(s) = e^{sX} \delta_1 e^{-sX} = \sum_{n=0}^{\infty} \delta_{n+1} \frac{s^n}{n!}$, and its left action on $\mathcal{U}(\mathcal{A}^1)$ is given, on group like elements φ by

$$(7) \quad \delta_1(s) \varphi = \langle \delta_1(s), \varphi \rangle \varphi.$$

(Using $\langle \delta_1(s) \varphi, P \rangle = \langle \varphi, P \delta_1(s) \rangle = \langle \Delta \varphi, P \otimes \delta_1(s) \rangle = \langle \varphi, P \rangle \langle \varphi, \delta_1(s) \rangle$.)

Moreover by (50) section III, one has

$$(8) \quad \langle \delta_1(s), \varphi \rangle = - \sum \frac{s^n}{n!} \partial_x^{n+1} (\pi(\varphi) H)_0$$

while, with $\psi = \varphi^{-1}$, one has $(\pi(\varphi) H)(s, x) = s - \log \psi'(x)$, so that (8) gives

$$(9) \quad \langle \delta_1(s), \varphi \rangle = \sum \frac{s^n}{n!} \partial_x^n \left(\frac{\psi''}{\psi'} \right) (x)_{x=0} = \left(\frac{\psi''}{\psi'} \right) (s), \quad \psi = \varphi^{-1}.$$

Thus we can rewrite (5) as ($t > 0$)

$$(10) \quad (\varphi f)(s, t) = \sum_{n=0}^{\infty} \int_{0 \leq s_1 \leq \dots \leq s_n \leq t} \prod ds_i \prod_1^n \left(\frac{\psi''}{\psi'} \right) (s_i) (\partial_s + (t - s_n) \partial_t) \dots (\partial_s + (t - s_1) \partial_t) f(s, t).$$

We first apply this formula to $f(s, t) = f(s)$, independent of t , we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^t \left(\frac{\psi''}{\psi'} \right) (s) ds \right)^n \partial_s^n f(s) = f(s + \log \psi'(t)).$$

We then apply it to $f(s, t) = t$. The term $(\partial_s + (t - s_n) \partial_t) \dots (\partial_s + (t - s_1) \partial_t) f$ gives $(t - s_n)$, thus we get,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{0 \leq s_n \leq t} \frac{\psi''(s_n)}{\psi'(s_n)} \frac{1}{(n-1)!} \left(\int_0^{s_n} \left(\frac{\psi''}{\psi'} \right) (u) du \right)^{n-1} (t - s_n) ds_n + t \\ = t + \int_0^t \frac{\psi''(s)}{\psi'(s)} (\exp \log \psi'(s)) (t - s) ds \\ = t + \int_0^t \psi''(s) (t - s) ds = t + \psi(t) - t \psi'(0) - \psi(0). \end{aligned}$$

Thus in general we get,

$$(11) \quad (\varphi f)(s, t) = f(s + \log \psi'(t), \psi(t)) , \quad \psi = \varphi^{-1} .$$

V. Hopf algebra $\mathcal{H}(G)$ associated to a matched pair of subgroups

In this section we recall a basic construction of Hopf algebras $([K], [B-S], [M])$. We let G be a finite group, G_1, G_2 be subgroups of G such that,

$$(1) \quad G = G_1 G_2 , G_1 \cap G_2 = 1 ,$$

i.e. we assume that any $g \in G$ admits a unique decomposition as

$$(2) \quad g = k a \quad , \quad k \in G_1 , a \in G_2 .$$

Since $G_1 \cong G/G_2$ one has a natural left action of G on G_1 which for $g \in G_1$ coincides with the left action of G_1 on itself, it is given by

$$(3) \quad g(k) = \pi_1(gk) \quad \forall g \in G , k \in G_1 ,$$

where $\pi_j : G \rightarrow G_j$ are the two projections.

For $g \in G_1$ one has $g(k) = gk$ while for $a \in G_2$ one has,

$$(4) \quad a(1) = 1 \quad \forall a \in G_2 .$$

(Since $\pi_1(a) = 1 \quad \forall a \in G_2$.)

Since $G_2 \cong G_1 \setminus G$, one has a right action of G on G_2 which restricted to $G_2 \subset G$ is the right action of G_2 on itself,

$$(5) \quad a \cdot g = \pi_2(ag) \quad \forall a \in G_2 , g \in G .$$

As above one has

$$(6) \quad 1 \cdot k = 1 \quad \forall k \in G_1 .$$

Lemma 1. a) For $a \in G_2, k_1, k_2 \in G_1$ one has $a(k_1 k_2) = a(k_1)(a \cdot k_1)(k_2)$.

b) For $k \in G_1, a_1, a_2 \in G_2$ one has $(a_1 a_2) \cdot k = (a_1 \cdot a_2(k))(a_2 \cdot k)$.

Proof. a) One has $a k_1 = k'_1 a'$ with $k'_1 = a(k_1), a' = a \cdot k_1$. Then $(a k_1) k_2 = k'_1 a' k_2 = k'_1 k'_2 a''$ with $k'_2 = a'(k_2)$. Thus $k'_1 k'_2 = a(k_1 k_2)$ which is the required equality.

b) One has $a_2 k = k' a'_2$ with $k' = a_2(k), a'_2 = a_2 \cdot k$. Then $a_1 a_2 k = a_1(k' a'_2) = (a_1 k') a'_2 = k'' a'_1 a'_2$ where $a'_1 = a_1 \cdot k'$, thus $(a_1 a_2) \cdot k = a'_1 a'_2$ as required. ■

One defines a Hopf algebra \mathcal{H} as follows. As an algebra \mathcal{H} is the crossed product of the algebra of functions h on G_2 by the action of G_1 . Thus elements of \mathcal{H} are of the form $\sum h_k X_k$ with the rule,

$$(7) \quad X_k h X_k^{-1} = k(h), \quad k(h)(a) = h(a \cdot k) \quad \forall a \in G_2, k \in G_1.$$

The coproduct Δ is defined as follows,

$$(8) \quad \Delta \varepsilon_c = \sum_{ba=c} \varepsilon_a \otimes \varepsilon_b, \quad \varepsilon_c(g) = 1 \text{ if } g = c \text{ and } 0 \text{ otherwise.}$$

$$(9) \quad \Delta X_k = \sum_{k'} h_{k'}^k X_k \otimes X_{k'}, \quad h_{k'}^k(a) = 1 \text{ if } k' = a(k) \text{ and } 0 \text{ otherwise.}$$

One first checks that Δ defines a covariant representation. The equality (8) defines a representation of the algebra of functions on G_2 . Let us check that (9) defines a representation of G_1 . First, for $k = 1$ one gets by (4) that $\Delta X_1 = X_1 \otimes X_1$. One has $\Delta X_{k_1} \Delta X_{k_2} = \sum_{k'_1, k'_2} h_{k'_1}^{k_1} X_{k_1} h_{k'_2}^{k_2} X_{k_2} \otimes X_{k'_1 k'_2} = \sum_{k'_1, k'_2} h_{k'_1}^{k_1} k_1(h_{k'_2}^{k_2}) X_{k_1 k_2} \otimes X_{k'_1 k'_2}$.

For $a \in G_2$ one has $(h_{k'_1}^{k_1} k_1(h_{k'_2}^{k_2}))(a) \neq 0$ only if $k'_1 = a(k_1)$, $k'_2 = (a \cdot k_1)(k_2)$. Thus given a there is only one term in the sum to contribute, and by lemma 1, one then has $k'_1 k'_2 = a(k_1 k_2)$, thus,

$$(10) \quad \Delta X_{k_1} \Delta X_{k_2} = \sum h_{k''}^{k_1 k_2} X_{k_1 k_2} \otimes X_{k''} = \Delta X_{k_1 k_2}.$$

Next, one has $\Delta X_k \Delta \varepsilon_c = \sum_{k'} \sum_{ba=c} h_{k'}^k X_k \varepsilon_a \otimes X_{k'} \varepsilon_b$, and $X_k \varepsilon_a = \varepsilon_{a \cdot k^{-1}} X_k$, so that

$$(11) \quad \Delta X_k \Delta \varepsilon_c = \sum_{k'} \sum_{ba=c} h_{k'}^k \varepsilon_{a \cdot k^{-1}} X_k \otimes \varepsilon_{b \cdot k'^{-1}} X_{k'}.$$

One has $\Delta \varepsilon_{c \cdot k^{-1}} \Delta X_k = \sum_{k'} \sum_{b' a' = c \cdot k^{-1}} h_{k'}^k \varepsilon_{a'} X_k \otimes \varepsilon_{b'} X_{k'}$. In (11), given a, b with $ba = c$ the only k' that appears is $k' = (a \cdot k^{-1})(k)$. But $(a \cdot k^{-1})(k) = a(k^{-1})^{-1}$ and $b \cdot k'^{-1} = b \cdot a(k^{-1})$ so that by lemma 1, $(b \cdot a(k^{-1}))(a \cdot k^{-1}) = (ba) \cdot k^{-1} = c \cdot k^{-1}$. Thus one gets

$$(12) \quad \Delta X_k \Delta \varepsilon_c = \Delta \varepsilon_{c \cdot k^{-1}} \Delta X_k$$

which shows that Δ defines an algebra homomorphism.

To show that the coproduct Δ is coassociative let us identify the dual algebra \mathcal{H}^* with the crossed product,

$$(13) \quad (G_1)_{\text{space}} \rtimes G_2.$$

For $f U_a^* \in \mathcal{H}^*$ we define the pairing with \mathcal{H} by

$$(14) \quad \langle h X_k, f U_a^* \rangle = f(k) h(a)$$

while the crossed product rules are

$$(15) \quad \begin{aligned} U_a^* f &= f^a U_a^*, \quad f^a(k) = f(a(k)) \quad \forall k \in G_1 \\ U_{ab}^* &= U_b^* U_a^* \quad \forall a, b \in G_2. \end{aligned}$$

What we need to check is

$$(16) \quad \langle \Delta h X_k, f U_a^* \otimes g U_b^* \rangle = \langle h X_k, f U_a^* g U_b^* \rangle.$$

We can assume that $h = \varepsilon_c$ so that $\Delta h X_k = \sum_{ba=c} \varepsilon_a X_k \otimes \varepsilon_b X_{a(k)}$. The left hand side of (16) is then $f(k) g(a(k))$ or 0 according to $ba = c$ or $ba \neq c$ which is the same as the right hand side.

Let us now describe the antipode S . The counit is given by

$$(17) \quad \varepsilon(h X_k) = h(1) \quad 1 \text{ unit of } G_2.$$

We can consider the Hopf subalgebra \mathcal{H}^1 of \mathcal{H} given by the $h X_k$, for $k = 1$. The antipode S^1 of \mathcal{H}^1 is given by (group case)

$$(18) \quad (S^1 h)(a) = h(a^{-1}) = \tilde{h}(a).$$

Thus it is natural to expect,

$$(19) \quad S(\varepsilon_a X_k) = X_{a(k)}^{-1} \varepsilon_{a^{-1}}.$$

One needs to check that given $c \in G_2$ one has

$$(20) \quad \sum_{ba=c} S(\varepsilon_a X_k) \varepsilon_b X_{a(k)} = \sum \varepsilon_a X_k S(\varepsilon_b X_{a(k)}) = \varepsilon(\varepsilon_c X_k).$$

The first term is $\sum_{ba=c} X_{a(k)}^{-1} \varepsilon_{a^{-1}} \varepsilon_b X_{a(k)}$ which is 0 unless $c = 1$. When $c = 1$ it is equal to 1 since the $(a^{-1}) \cdot a(k) = (a \cdot k)^{-1}$ label G_2 when a varies in G_2 . Similarly the second term gives $\varepsilon_a X_k X_{b(a(k))}^{-1} \varepsilon_{b^{-1}}$ which is non zero only if $a \cdot k c(k)^{-1} = b^{-1}$, i.e. $a \cdot k = b^{-1} \cdot c(k)$, i.e. if $c \cdot k = 1$ (since by lemma 1, one has $(b^{-1} c) \cdot k = ((b^{-1}) \cdot c(k))(c \cdot k)$). Thus $c = 1$ and the sum gives 1.

Let us now compute the \mathcal{H} -bimodule structure of \mathcal{H}^* .

Lemma 2. a) *The left action of $h X_k \in \mathcal{H}$ on $f U_a^* \in \mathcal{H}^*$ is given by $(h X_k) \cdot (f U_a^*) = h_a X_k(f) U_a^*$, where for $k_1 \in G_1$, $h_a(k_1) = h(a \cdot k_1)$ and $X_k(f)(k_1) = f(k_1 k)$.*

b) *The right action of $h X_k \in \mathcal{H}$ on $f U_a^* \in \mathcal{H}^*$ is given by $(f U_a^*) \cdot (h X_k) = h(a) L_{k^{-1}}(f) U_{a \cdot k}^*$, where for $k_1 \in G_1$, $(L_{k^{-1}} f)(k_1) = f(k k_1)$.*

Proof. a) By definition $\langle (h X_k) \cdot f U_a^*, h_0 X_{k_0} \rangle = \langle f U_a^*, h_0 X_{k_0} h X_k \rangle$. Thus one has to check that $f(k_0 k) h_0(a) h(a \cdot k_0) = h_0(a) (h_a X_k(f))(k_0)$ which is clear.

b) One has $\langle (f U_a^*) \cdot h X_k, h_0 X_{k_0} \rangle = \langle f U_a^*, h X_k h_0 X_{k_0} \rangle = f(k k_0) h(a) h_0(a \cdot k)$ while $\langle h(a) L_{k^{-1}}(f) U_{a \cdot k}^*, h_0 X_{k_0} \rangle = h(a) f(k k_0) h_0(a \cdot k)$. ■

VI. Duality between \mathcal{H} and $C_c^\infty(G_1) \rtimes G_2$, $G = G_1 G_2 = \text{Diff } \mathbb{R}$

Let, as above, \mathcal{H} be the Hopf algebra generated by X, Y, δ_n . While there is a formal group $G(\mathcal{A}^1)$ associated to the subalgebra \mathcal{A}^1 of the Lie algebra of formal vector fields, there is no such group associated to \mathcal{A} itself. As a substitute for this let us take,

$$(1) \quad G = \text{Diff } (\mathbb{R}).$$

(We take smooth ones but restrict to real analytic if necessary).

We let $G_1 \subset G$ be the subgroup of affine diffeomorphisms,

$$(2) \quad k(x) = ax + b \quad \forall x \in \mathbb{R}$$

and we let $G_2 \subset G$ be the subgroup,

$$(3) \quad \varphi \in G, \varphi(0) = 0, \varphi'(0) = 1.$$

Given $\varphi \in G$ it has a unique decomposition $\varphi = k \psi$ where $k \in G_1, \psi \in G_2$ and one has,

$$(4) \quad a = \varphi'(0), b = \varphi(0), \psi(x) = \frac{\varphi(x) - \varphi(0)}{\varphi'(0)}.$$

The left action of G_2 on G_1 is given by applying (4) to $x \rightarrow \varphi(ax + b)$, for $\varphi \in G_2$. This gives

$$(5) \quad b' = \varphi(b), a' = a \varphi'(b)$$

which is the natural action of G_2 on the frame bundle $F(\mathbb{R})$. Thus,

Lemma 1. *The left action of G_2 on G_1 coincides with the action of G_2 on $F(\mathbb{R})$.*

Let us then consider the right action of G_1 on G_2 . In fact we consider the right action of $\varphi_1 \in G$ on $\varphi \in G_2$, it is given by

$$(6) \quad (\varphi \cdot \varphi_1)(x) = \frac{\varphi(\varphi_1(x)) - \varphi(\varphi_1(0))}{\varphi'(\varphi_1(0)) \varphi_1'(0)}.$$

Lemma 2. a) *The right action of G on G_2 is affine in the coordinates δ_n on G_2 .*

b) When restricted to G_1 it coincides with the action of the Lie algebra X, Y .

Proof. a) By definition one lets

$$(7) \quad \delta_n(\psi) = (\log \psi')^{(n)}(0).$$

With $\psi = \varphi \cdot \varphi_1$, the first derivative $\psi'(x)$ is $\varphi'(\varphi_1(x)) \varphi_1'(x) / \varphi'(\varphi_1(0)) \varphi_1'(0)$ so that up to a constant one has,

$$(8) \quad \log \psi'(x) = (\log \varphi')(\varphi_1(x)) + \log \varphi_1'(x).$$

Differentiating n times the equality (8) proves a).

To prove b) let $\varphi_1(x) = ax + b$ while, up to a constant,

$$(9) \quad (\log \varphi')(x) = \sum_1^{\infty} \frac{\delta_n^0}{n!} x^n, \quad \delta_n^0 = \delta_n(\varphi).$$

Then the coordinates $\delta_n = \delta_n(\varphi \cdot \varphi_1)$ are obtained by replacing x by $ax + b$ in (9), which gives

$$(10) \quad \delta_n = a^n \delta_n^0 \text{ if } b = 0, \quad \frac{\partial}{\partial b} \delta_n = \delta_{n+1}^0 \text{ at } b = 0, \quad a = 1.$$

■

We now consider the discrete crossed product of $C_c^\infty(G_1)$ by G_2 , i.e. the algebra of finite linear combinations of terms

$$(11) \quad f U_\psi^*, \quad f \in C_c^\infty(G_1), \quad \psi \in G_2$$

where the algebraic rules are

$$(12) \quad U_\psi^* f = (f \circ \psi) U_\psi^*.$$

We want to define a pairing between the (enveloping) algebra \mathcal{H} and the crossed product $C_c^\infty(G_1) \rtimes G_2$, by the equality

$$(13) \quad \langle h X_k, f U_\psi^* \rangle = h(\psi) f(k) \quad \forall k \in G_1, \quad \psi \in G_2.$$

In order to make sense of (13) we need to explain how we write an element of \mathcal{H} in the form $h X_k$.

Given a polynomial $P(\delta_1, \dots, \delta_n)$, we want to view it as a function on G_2 in such a way that the left action of that function h given by lemma 2 of section V coincides with the multiplication of U_ψ^* by

$$(14) \quad P(\gamma_1, \dots, \gamma_n), \quad \gamma_j = \left(\frac{\partial}{\partial x} \right)^j \log \psi'(x) e^{-js}, \quad k = (e^{-s}, x) \in G_1.$$

The formula of lemma 2 of section V gives the multiplication by

$$(15) \quad h(\psi \cdot k)$$

which shows that with δ_n defined by (7) one has,

$$(16) \quad h = P(\delta_1, \dots, \delta_n).$$

We then need to identify the Lie algebra generated by X, Y with the Lie algebra \mathbf{G}_1 of G_1 (generated by Z_{-1}, Z_0) in such a way that the left action of the latter coincides with

$$(17) \quad X f U_\psi^* = \left(e^{-s} \frac{\partial}{\partial x} f \right) U_\psi^*, \quad Y f U_\psi^* = - \left(\frac{\partial}{\partial s} f \right) U_\psi^* \quad (k = (e^{-s}, x)).$$

The formula of lemma 2 of section V gives $X_k f U_\psi^* = (X_k f) U_\psi^*$ with $(X_k f)(k_1) = f(k_1 k)$. One has $f(k) = f(s, x)$ for $k = (e^{-s}, x)$, i.e. $k(t) = e^{-s} t + x$. With $k_1 = (e^{-s_1}, x_1)$ and $k(\varepsilon) = (e^\varepsilon, 0)$ one gets

$$(18) \quad \frac{\partial}{\partial \varepsilon} f(k_1 k)_{\varepsilon=0} = - \frac{\partial}{\partial s_1} f(k_1) = (Y f)(k_1)$$

so that Y corresponds to the one parameter subgroup $(e^\varepsilon, 0)$ of G_1 . With $k(\varepsilon) = (1, \varepsilon)$ one has

$$(19) \quad \frac{\partial}{\partial \varepsilon} f(k_1 k_\varepsilon)_{\varepsilon=0} = (e^{-s} \partial_{x_1} f)(k_1) = (X f)(k_1)$$

so that X corresponds to the one parameter subgroup $(1, \varepsilon)$ of G_1 . Now the element $e^{tX} e^{sY}$ of G_1 considered in section IV is given by

$$(20) \quad k = e^{tX} e^{sY} = (e^s, t)$$

which has the effect of changing s to $-s$ in our formulas and thus explains the equality (11) of section IV.

This gives a good meaning to (13) as a pairing between \mathcal{H} and the crossed product $C_c^\infty(G_1) \rtimes G_2$.

VII. Hopf algebras and cyclic cohomology

Let us first make sense of the right action of \mathcal{H} on $C_c^\infty(G_1) \rtimes G_2$. We use the formula of lemma 2.b) section V

$$(1) \quad (f U_\psi^*) \cdot (h X_k) = h(\psi) L_{k^{-1}}(f) U_{\psi \cdot k}^*$$

where $(L_{k^{-1}}(f))(k_1) = f(k k_1)$ for $k_1 \in G_1$.

For the action of functions $h = P(\delta_1, \dots, \delta_n)$ we see that the difference with the left action is that we multiply U_ψ^* by a *constant*, namely $h(\psi)$. Next, since we took a *discrete* crossed product to get $C_c^\infty(G_1) \rtimes G_2$, we can only act by the same type of elements on the right, i.e. by elements in

$$(2) \quad \tilde{\mathcal{H}} = \text{finite linear combinations of } h X_k.$$

The algebra $\tilde{\mathcal{H}}$ has little in common with \mathcal{H} , but both are multipliers of the smooth crossed product by G_1 .

In fact, $\tilde{\mathcal{H}}$ acts on both sides on $C_c^\infty(G_1) \rtimes G_2$ but only the left action makes sense at the Lie algebra level, i.e. as an action of \mathcal{H} .

The coproduct Δ is not defined for $\tilde{\mathcal{H}}$ since $\Delta(e^{tX})$ cannot be written in $\tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}}$. Thus there is a problem to make sense of the right invariance property of an n -cochain,

$$(3) \quad \varphi(x^0, \dots, x^n) ; x^j \in C_c^\infty(G_1) \rtimes G_2$$

which we would usually write as

$$(4) \quad \sum \varphi(x^0 y_{(0)}, \dots, x^n y_{(n)}) = \varepsilon(y) \varphi(x^0, \dots, x^n)$$

for $\Delta^n y = \sum y_{(0)} \otimes \dots \otimes y_{(n)}$, $y \in \tilde{\mathcal{H}}$.

In fact it is natural to *require* as part of the right invariance property of the cochain, that it possesses the right continuity property in the variables $\psi_j \in G_2$ so that the integration required in the coproduct formula (3) section IV, does make sense.

This problem does not arise for $n = 0$, in which case we define the functional,

$$(5) \quad \tau_0(f U_\psi^*) = 0 \text{ if } \psi \neq 1, \quad \tau_0(f) = \int f(s, x) e^s ds dx,$$

where we used $k = (e^{-s}, x) \in G_1$.

One has $(f \circ \psi)(s, x) = f(s - \log \psi'(x), \psi(x))$ by (5) section VI, so that τ_0 is a *trace* on the algebra $C_c^\infty(G_1) \rtimes G_2 = \mathcal{H}_*$.

Let us compute $\tau_0((f U_\psi^*)(h X_k))$ and compare it with $\varepsilon(h X_k) \tau_0(f U_\psi^*)$. First $f U_\psi^* h X_k = h(\psi)(L_{k^{-1}} f) U_{\psi \cdot k}^*$ so that τ_0 vanishes unless $\psi \cdot k = 1$ i.e. unless $\psi = 1$. We can thus assume that $\psi = 1$. Then we just need to compare $\tau_0(L_{k^{-1}} f)$ with $\tau_0(f)$. For $k = (e^{-s_1}, x_1)$ one has $k \circ (e^{-s}, x)(t) = e^{-s_1}(e^{-s} t + x) + x_1 = e^{-(s+s_1)} t + (e^{-s_1} x + x_1)$. This corresponds to $\psi(x) = e^{-s_1} x + x_1$ and preserves τ_0 . Thus

Lemma 1. τ_0 is a right invariant trace on $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$.

Let us now introduce a bilinear pairing between $\mathcal{H}^{\otimes(n+1)}$ and $\mathcal{H}_*^{\otimes(n+1)}$ by the formula,

$$(6) \quad \langle y_0 \otimes \dots \otimes y_n, x^0 \otimes x^1 \otimes \dots \otimes x^n \rangle = \tau_0(y_0(x^0) \dots y_n(x^n))$$

$\forall y_j \in \mathcal{H}, x^k \in \mathcal{H}_*$.

This pairing defines a corresponding weak topology and we let

Definition 2. An n -cochain $\varphi \in C^n$ on the algebra \mathcal{H}_* is right invariant iff it is in the range of the above pairing.

We have a natural linear map θ from $\mathcal{H}^{\otimes(n+1)}$ to right invariant cochains on \mathcal{H}_* , given by

$$(7) \quad \theta(y_0 \otimes \dots \otimes y_n)(x^0, \dots, x^n) = \langle y_0 \otimes \dots \otimes y_n, x^0 \otimes \dots \otimes x^n \rangle$$

and we investigate the subcomplex of the cyclic complex of \mathcal{H}_* given by the range of θ .

It is worthwhile to lift the cyclic operations at the level of

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes(n+1)}$$

and consider θ as a morphism of Λ -modules.

Thus let us recall that the basic operations in the cyclic complex of an algebra are given on cochains $\varphi(x^0, \dots, x^n)$ by,

$$(8) \quad \begin{aligned} (\delta_i \varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^i x^{i+1}, \dots, x^n) & i = 0, 1, \dots, n-1 \\ (\delta_n \varphi)(x^0, \dots, x^n) &= \varphi(x^n x^0, x^1, \dots, x^{n-1}) \\ (\sigma_j \varphi)(x^0, \dots, x^n) &= \varphi(x^0, \dots, x^j, 1, x^{j+1}, \dots, x^n) & j = 0, 1, \dots, n \\ (\tau_n \varphi)(x^0, \dots, x^n) &= \varphi(x^n, x^0, \dots, x^{n-1}). \end{aligned}$$

These operations satisfy the following relations

$$(9) \quad \begin{aligned} \tau_n \delta_i &= \delta_{i-1} \tau_{n-1} & 1 \leq i \leq n, & \quad \tau_n \delta_0 = \delta_n \\ \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} & 1 \leq i \leq n, & \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \\ \tau_n^{n+1} &= 1_n. \end{aligned}$$

In the first line $\delta_i : C^{n-1} \rightarrow C^n$. In the second line $\sigma_i : C^{n+1} \rightarrow C^n$. Note that $(\sigma_n \varphi)(x^0, \dots, x^n) = \varphi(x^0, \dots, x^n, 1)$, $(\sigma_0 \varphi)(x^0, \dots, x^n) = \varphi(x^0, 1, x^1, \dots, x^n)$. The map θ maps $\mathcal{H}^{\otimes(n+1)}$ to C^n thus there is a shift by 1 in the natural index n . We let

$$(10) \quad \delta_i(h^0 \otimes h^1 \otimes \dots \otimes h^i \otimes \dots \otimes h^{n-1}) = h^0 \otimes \dots \otimes h^{i-1} \otimes \Delta h^i \otimes h^{i+1} \otimes \dots \otimes h^{n-1}$$

and this makes sense for $i = 0, 1, \dots, n-1$.

One has $\tau_0(h^0(x^0) \dots h^i(x^i x^{i+1}) h^{i+1}(x^{i+2}) \dots h^{n-1}(x^n)) = \sum \tau_0(h^0(x^0) \dots h_{(0)}^i(x^i) h_{(1)}^i(x^{i+1}) \dots h^{n-1}(x^n))$

$$(11) \quad \delta_n(h^0 \otimes h^1 \otimes \dots \otimes h^{n-1}) = \sum h_{(1)}^0 \otimes h^1 \otimes \dots \otimes h^{n-1} \otimes h_{(0)}^0$$

which is compatible with $h^0(x^n x^0) = \sum h_{(0)}^0(x^n) h_{(1)}^0(x^0)$, together with the trace property of τ_0 .

With $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$ the counit, we let

$$(12) \quad \sigma_j(h^0 \otimes \dots \otimes h^{n+1}) = h^0 \otimes \dots \otimes \varepsilon(h^{j+1}) h^{j+2} \dots \otimes h^{n+1} \quad j = 0, 1, \dots, n$$

which corresponds to $h^j(1) = \varepsilon(h^j) 1$.

Finally we let τ_n act on $\mathcal{H}^{\otimes(n+1)}$ by

$$(13) \quad \tau_n(h^0 \otimes \dots \otimes h^n) = h^1 \otimes h^2 \otimes \dots \otimes h^{n-1} \otimes h^n \otimes h^0$$

which corresponds to $\tau_0(h^0(x^n) h^1(x^0) \dots h^n(x^{n-1})) = \tau_0(h^1(x^0) \dots h^0(x^n))$.

One checks that with these operations \mathcal{H}_{\natural} is a Λ -module where Λ is the cyclic category. To the relations (9) one has to add the relations of the simplicial Δ , namely,

$$(14) \quad \delta_j \delta_i = \delta_i \delta_{j-1} \text{ for } i < j, \quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad i \leq j$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1 & \text{if } i = j \text{ or } i = j + 1 \\ \delta_{i-1} \sigma_j & i > j + 1. \end{cases}$$

The small category Λ is best defined as a quotient of the following category $E\Lambda$. The latter has one object (\mathbb{Z}, n) for each n and the morphisms $f : (\mathbb{Z}, n) \rightarrow (\mathbb{Z}, m)$ are non decreasing maps, $(n, m \geq 1)$

$$(15) \quad f : \mathbb{Z} \rightarrow \mathbb{Z}, \quad f(x+n) = f(x) + m \quad \forall x \in \mathbb{Z}.$$

In defining Λ (cf. [Co]) one uses homotopy classes of non decreasing maps from S^1 to S^1 of degree 1, mapping \mathbb{Z}/n to \mathbb{Z}/m . Given such a map we can lift it to a map satisfying (15). Such an f defines uniquely a homotopy class downstairs and, if we replace f by $f + km$, $k \in \mathbb{Z}$ the result downstairs is the same. When $f(x) = a(m) \quad \forall x$, one can restrict f to $\{0, 1, \dots, n-1\}$

Figure

then $f(j)$ is either a or $a + m$ which labels the various choices. One has $\Lambda = (E\Lambda)/\mathbb{Z}$.

We recall that δ_i is the injection that misses i , while σ_j is the surjection which identifies j with $j + 1$.

Proposition 3. \mathcal{H}_{\natural} is a Λ -module and θ is a Λ -module morphism to the Λ -module $C^*(\mathcal{H}_*)$ of cochains on $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$.

This is clear by construction.

Now the definition of \mathcal{H}_{\natural} only involves ((10) ... (13)) the coalgebra structure of \mathcal{H} , it is thus natural to compare it with the more obvious duality which pairs $\mathcal{H}^{\otimes(n+1)}$ with $\mathcal{H}_*^{\otimes(n+1)}$ namely,

$$(16) \quad \langle h^0 \otimes h^1 \otimes \dots \otimes h^n, x^0 \otimes \dots \otimes x^n \rangle = \prod_0^n \langle h^j, x^j \rangle.$$

One has $\langle h^i, x^i x^{i+1} \rangle = \langle \Delta h^i, x^i \otimes x^{i+1} \rangle$, so that the rules (10) and (11) are the correct ones. One has $\langle h^{j+1}, 1 \rangle = \varepsilon(h^{j+1})$ so that (12) is right. Finally (13) is also right.

This means that $C^*(\mathcal{H}_*) = \mathcal{H}_\natural^{**}$ as Λ -modules. Thus,

$$(17) \quad \theta : \mathcal{H}_\natural \rightarrow \mathcal{H}_\natural^{**}$$

is a cyclic morphism.

To understand the algebraic nature of θ , let us compute it in the simplest cases first. We first take $\mathcal{H} = \mathbb{C}G_1$ where G_1 is a finite group, and use the Hopf algebra $\mathcal{H}(G)$ for $G = G_1, G_2 = \{e\}$. Thus as an algebra it is the group ring $\sum \lambda_k X_k, k \in G_1$. As a right invariant trace on \mathcal{H}^* we take

$$(18) \quad \tau_0(f) = \sum_{G_1} f(k), \quad \forall f \in \mathcal{H}^*.$$

The pairing $\langle h, f \rangle$, for $h = \sum \lambda_k X_k, f \in \mathcal{H}^*$ is given by $\sum \lambda_k f(k)$. The left action $h(f)$ is given by lemma 2 section V, i.e.

$$(19) \quad h(f)(x) = \sum \lambda_k f(xk).$$

Thus the two pairings are, for (16): $\sum_{k_i} h^0(k_0) f_0(k_0) \dots h^n(k_n) f_n(k_n)$ and for (6), $\sum_{k, k_i} h^0(k_0) f_0(k k_0) \dots h^n(k_n) f_n(k k_n)$.

Thus at the level of the $h^0 \otimes \dots \otimes h^n$ the map θ is just the sum of the left translates,

$$(20) \quad \sum_{G_1} (L_g \otimes L_g \otimes \dots \otimes L_g).$$

Next, we take the dual case, $G_1 = \{e\}, G_2 = G$, with G finite as above. Then \mathcal{H} is the algebra of functions h on G_2 , and the dual \mathcal{H}^* is the group ring of $G_2^{\text{op}} \simeq G_2$, with generators $U_g^*, g \in G_2$. For a trace τ_0 on this group ring, the right invariance under \mathcal{H} means that τ_0 is the regular trace,

$$(21) \quad \tau_0 \left(\sum f_g U_g^* \right) = f_e.$$

This has a natural normalization, $\tau_0(1) = 1$, for which we should expect θ to be an idempotent. The pairing between h and f is,

$$(22) \quad \langle h, f \rangle = \sum h(g) f_g.$$

Thus the two pairings (16) and (6) give respectively, for (16) $\sum h_0(g_0) f_0(g_0) h_1(g_1) f_1(g_1) \dots h_n(g_n) f_n(g_n)$ and for (6), knowing that $h(U_g^*) = h(g) U_g^*$, i.e. $h \sum f_g U_g^* = \sum h(g) f_g U_g^*$ one gets,

$$(23) \quad \sum_{g_n \dots g_1 g_0 = 1} h_0(g_0) f_0(g_0) h_1(g_1) f_1(g_1) \dots h_n(g_n) f_n(g_n).$$

Thus, at the level of $\mathcal{H}^{\otimes n+1}$ the map θ is exactly the localisation on the conjugacy class of e .

These examples clearly show that in general $\text{Ker } \theta \neq \{0\}$. Let us compute in our case how τ_0 is modified by the *left* action of \mathcal{H} on \mathcal{H}_* . By lemma 2 section V one has $(h X_k)(f U_\psi^*) = h_\psi X_k(f) U_\psi^*$ and τ_0 vanishes unless $\psi = 1$. In this case h_ψ is the constant $h(1) = \varepsilon(h)$, while

$$(24) \quad X_k(f)(k_1) = f(k_1 k).$$

Thus we need to compare $\int f((e^{-s}, x)(a, b)) e^s ds dx$ with its value for $a = 1$, $b = 0$. With $k^{-1} = (a^{-1}, -b/a)$ the right multiplication by k^{-1} transforms (y, x) to (y', x') with $y' = y a^{-1}$, $x' = x - y b/a$, so that $\frac{dy'}{y'^2} \wedge dx' = a \frac{dy}{y^2} \wedge dx$,

$$(25) \quad \tau_0((h X_k) \cdot f U_\psi^*) = \varepsilon(h) \delta(k) \tau_0(f U_\psi^*)$$

where the module δ of the group G_1 is,

$$(26) \quad \delta(a, b) = a.$$

In fact we view δ as a character of \mathcal{H} , with

$$(27) \quad \delta(h X_k) = \varepsilon(h) \delta(k).$$

(Note that $1 \cdot k = 1$ for all $k \in G_1$ so that (27) defines a character of \mathcal{H} .) Thus in our case we have a (non trivial) character of \mathcal{H} such that

$$(28) \quad \tau_0(y(x)) = \delta(y) \tau_0(x) \quad \forall x \in \mathcal{A}, y \in \mathcal{H}.$$

In fact we need to write the invariance property of τ_0 as a formula for integrating by parts. To do this we introduce the twisted antipode,

$$(29) \quad \tilde{S}(y) = \sum \delta(y_{(0)}) S(y_{(1)}), \quad y \in \mathcal{H}, \quad \Delta y = \sum y_{(0)} \otimes y_{(1)}.$$

One has $\tilde{S}(y) = S(\sigma(y))$ where σ is the automorphism obtained by composing $(\delta \otimes 1) \circ \Delta : \mathcal{H} \rightarrow \mathcal{H}$. One can view \tilde{S} as $\delta * S$ in the natural product (cf.(38) sectionV) on the algebra of linear maps from the coalgebra \mathcal{H} to the algebra \mathcal{H} . Since S is the inverse of the identity map, i.e. $I * S = S * I = \varepsilon$, one has $(\delta * S) * I = \delta$, i.e.

$$(30) \quad \sum \tilde{S}(y_{(0)}) y_{(1)} = \delta(y) \quad \forall y \in \mathcal{H}.$$

The formula that we need as a working hypothesis on τ_0 is,

$$(31) \quad \tau_0(y(a) b) = \tau_0(a \tilde{S}(y)(b)) \quad \forall a, b \in \mathcal{A}, y \in \mathcal{H}.$$

Using this formula we shall now determine $\text{Ker } \theta$ purely algebraically. We let $h = \sum h_0^i \otimes \dots \otimes h_n^i \in \mathcal{H}^{\otimes(n+1)}$, we associate to h the following element of $\mathcal{H}^{\otimes(n)}$:

$$(32) \quad t(h) = \sum_i (\Delta^{n-1} \tilde{S}(h_0^i)) h_1^i \otimes \dots \otimes h_n^i$$

where we used both the coproduct of \mathcal{H} and the product of $\mathcal{H}^{\otimes(n)}$ to perform the operations.

Lemma 3. $h \in \text{Ker } \theta$ iff $t(h) = 0$.

Proof. Let us first show that $h - 1 \otimes t(h) \in \text{Ker } \theta$ for any h . One can assume that $h = h^0 \otimes \dots \otimes h^n$. Using (31) one has $\tau_0(h^0(x^0) h^1(x^1) \dots h^n(x^n)) = \tau_0(x^0(\tilde{S} h^0)(h^1(x^1) \dots h^n(x^n))) = \tau_0(x^0(\tilde{S} h^0)_{(0)} h^1(x^1)(\tilde{S} h^0)_{(1)} h^2(x^2) \dots (\tilde{S} h^0)_{(n-1)} h^n(x^n))$.

It follows that if $t(h) = 0$ then $h \in \text{Ker } \theta$. Conversely, let us show that if $1 \otimes t(h) \in \text{Ker } \theta$ then $t(h) = 0$. We assume that the Haar measure τ_0 is faithful i.e. that

$$(33) \quad \tau_0(ab) = 0 \quad \forall a \in \mathcal{A} \quad \text{implies } b = 0.$$

Thus, with $\tilde{h} = t(h)$, $\tilde{h} = \sum \tilde{h}_1^i \otimes \dots \otimes \tilde{h}_n^i$, one has

$$(34) \quad \sum \tilde{h}_1^i(x^1) \dots \tilde{h}_n^i(x^n) = 0 \quad \forall x^j \in \mathcal{A}.$$

Applying the unit $1 \in \mathcal{H}$ to both sides we get,

$$(35) \quad \sum \langle \tilde{h}_1^i, x^1 \rangle \dots \langle \tilde{h}_n^i, x^n \rangle = 0 \quad \forall x^j \in \mathcal{A}$$

which implies that $\tilde{h} = 0$ in $\mathcal{H}^{\otimes n}$. ■

Definition 4. The cyclic module $C^*(\mathcal{H})$ of a Hopf algebra \mathcal{H} is the quotient of \mathcal{H}_\natural by the kernel of t .

Note that to define t we needed the module $\delta : \mathcal{H} \rightarrow \mathbb{C}$, but that any reference to analysis has now disappeared in the definition of $C^*(\mathcal{H})$.

Note also that the construction of $C^*(\mathcal{H})$ uses in an essential way both the coproduct and the product of \mathcal{H} . As we shall see it provides a working definition of the analogue of Lie algebra cohomology in general. (We did assume however that τ_0 was a trace. This is an unwanted restriction which should be removed by making use of the modular theory.)

When $\mathcal{H} = \mathcal{U}(\mathbf{G})$ is the (complex) envelopping algebra of a (real) Lie algebra \mathbf{G} , there is a natural interpretation of the Lie algebra cohomology,

$$(36) \quad H^*(\mathbf{G}, \mathbb{C}) = H^*(\mathcal{U}(\mathbf{G}), \mathbb{C})$$

where the right hand side is the Hochschild cohomology with coefficients in the $\mathcal{U}(\mathbf{G})$ -bimodule \mathbb{C} obtained using the augmentation. In general, given a Hopf algebra \mathcal{H} we can dualise the construction of the Hochschild complex $C^n(\mathcal{H}^*, \mathbb{C})$ where \mathbb{C} is viewed as a bimodule on \mathcal{H}^* using the augmentation, i.e. the counit of \mathcal{H}^* . This gives the following operations: $\mathcal{H}^{\otimes(n-1)} \rightarrow \mathcal{H}^{\otimes n}$, defining a cosimplicial space

$$(37) \quad \begin{aligned} \delta_0(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1}, \\ \delta_j(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes \Delta h^j \otimes \dots \otimes h^{n-1}, \\ \delta_n(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes h^{n-1} \otimes 1, \\ \sigma_i(h^1 \otimes \dots \otimes h^{n+1}) &= h^1 \otimes \dots \varepsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}, \quad 0 \leq i \leq n. \end{aligned}$$

Proposition 5. *The map t is an isomorphism of cosimplicial spaces.*

Proof. Modulo $\text{Ker } t = \text{Ker } \theta$ any element of $\mathcal{H}^{\otimes(n+1)}$ is equivalent to an element of the form $\sum 1 \otimes h_i^1 \otimes \dots \otimes h_i^n = \xi$. One has $t(\xi) = \sum h_i^1 \otimes \dots \otimes h_i^n$. It is enough to show that the subspace $1 \otimes \mathcal{H}^{\otimes*}$ is a cosimplicial subspace isomorphic to (37) through t . Thus we let $h^0 = 1$ in the definition (10) of δ_1 and (11) of δ_n and check that they give (37). Similarly for σ_i . ■

This shows that the underlying cosimplicial space of the cyclic module $C^*(\mathcal{H})$ is a standard object of homological algebra attached to the coalgebra \mathcal{H} together with 1 , $\Delta 1 = 1 \otimes 1$. The essential new feature, due to the Hopf algebra structure is that this cosimplicial space carries a cyclic structure. The latter is determined by giving the action of τ_n which is,

$$(38) \quad \tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h_1)) h^2 \otimes \dots \otimes h^n \otimes 1$$

where one uses the product in $\mathcal{H}^{\otimes n}$ and the twisted antipode \tilde{S} . It is nontrivial to check directly that $(\tau_n)^{n+1} = 1$, for instance for $n = 1$ this means that \tilde{S} is an involution, i.e. $\tilde{S}^2 = 1$. Note that the antipode S of the Hopf algebra of section III is *not* an involution, while \tilde{S} is one. The first two cases in which we shall compute the cyclic cohomology of \mathcal{H} are the following.

Proposition 6. 1) *The periodic cyclic cohomology $H^*(\mathcal{H})$, for $\mathcal{H} = \mathcal{U}(\mathbf{G})$ the envelopping algebra of a Lie algebra \mathbf{G} is isomorphic to the Lie algebra homology $H_*(\mathbf{G}, \mathbb{C}_\delta)$ where $\mathbb{C}_\delta = \mathbb{C}$ is viewed a \mathbf{G} -module by using the modular function δ of G .*

2) *The periodic cyclic cohomology $H^*(\mathcal{H})$, for $\mathcal{H} = \mathcal{U}(\mathbf{G})_*$, is isomorphic to the Lie algebra cohomology of \mathbf{G} with trivial coefficients, provided G is an affine space in the coordinates of \mathcal{H} . This holds in the nilpotent case.*

Proof. 1) One has a natural inclusion $\mathbf{G} \subset \mathcal{U}(\mathbf{G})$. Let us consider the corresponding inclusion of $\Lambda^n \mathbf{G}$ in $\mathcal{H}^{\otimes n}$, given by

$$(41) \quad X_1 \wedge \dots \wedge X_n \rightarrow \sum (-1)^\sigma X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}.$$

Let $b : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes(n+1)}$ be the Hochschild coboundary, one has

$$(42) \quad \text{Im } b \oplus \Lambda^n \mathbf{G} = \text{Ker } b \quad \forall n.$$

For $n = 1$ one has $b(h) = h \otimes 1 - \Delta h + 1 \otimes h$ so that $b(h) = 0$ iff $h \in \mathbf{G}$. For $n = 0$, $b(\lambda) = \lambda - \lambda = 0$ so that $b = 0$. In general the statement (42) only uses the cosimplicial structure, i.e. only the coalgebra structure of \mathcal{H} together with the element $1 \in \mathcal{H}$. This structure is unaffected if we replace the Lie algebra structure of \mathbf{G} by the trivial commutative one. More precisely let us define the linear isomorphism,

$$(43) \quad \pi : S(\mathbf{G}) \rightarrow \mathcal{U}(\mathbf{G}), \quad \pi(X^n) = X^n \quad \forall X \in \mathbf{G}.$$

Then $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta_S$ where Δ_S is the coproduct of $S(\mathbf{G})$. Indeed it is enough to check this equality on X^n , $X \in \mathbf{G}$ and both sides give $\sum C_n^k X^k \otimes X^{n-k}$.

The result then follows by dualising the homotopy between the standard resolution and the Koszul resolution $S(E) \otimes \Lambda(E)$ of the module \mathbb{C} over $S(E)$ for a vector space E .

Let us then compute $B(X_1 \wedge \dots \wedge X_n)$. Note that $B_0(X_1 \wedge \dots \wedge X_n)$ corresponds to the functional $(-1)^\sigma \tau_0(X_{\sigma(1)}(x^0) \dots X_{\sigma(n)}(x^{n-1}))$ which is already cyclic. Thus it is enough to compute B_0 . One has

$$(44) \quad \tilde{S}(X) = -X + \delta(X) \quad \forall X \in \mathbf{G},$$

thus $\Delta^{n-2} \tilde{S}X = -\sum 1 \otimes \dots \otimes X \otimes \dots \otimes 1 + \delta(X) 1 \otimes 1 \dots \otimes 1$.

We get $B_0(X_1 \wedge \dots \wedge X_n) = \sum_{\sigma} (-1)^\sigma \delta(X_{\sigma(1)}) X_{\sigma(2)} \otimes \dots \otimes X_{\sigma(n)} - \sum_{\sigma} (-1)^\sigma X_{\sigma(2)} \otimes \dots \otimes X_{\sigma(1)} X_{\sigma(j)} \otimes \dots \otimes X_{\sigma(n)} = \sum (-1)^{k+1} \delta(X_k) X_1 \wedge \dots \wedge \overset{\vee}{X}_k \wedge \dots \wedge X_n + \sum_{i < j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \dots \wedge \overset{\vee}{X}_i \wedge \dots \wedge \overset{\vee}{X}_j \wedge \dots \wedge X_n$. This shows that B leaves $\Lambda^* \mathbf{G}$ invariant and coincides there with the boundary map of Lie algebra homology. The situation is identical to what happens in computing cyclic cohomology of the algebra of smooth functions on a manifold.

2) The Hochschild complex of \mathcal{H} is by construction the dual of the standard chain complex which computes the Hochschild homology of $\mathcal{U}(\mathbf{G})$ with coefficients in \mathbb{C} (viewed as a bimodule using ε). Recall that in the latter complex the boundary d_n is

$$(45) \quad d_n(\lambda_1 \otimes \dots \otimes \lambda_n) = \varepsilon(\lambda_1) \lambda_2 \otimes \dots \otimes \lambda_n - \lambda_1 \lambda_2 \otimes \dots \otimes \lambda_n + \dots + (-1)^i \lambda_1 \otimes \dots \otimes \lambda_i \lambda_{i+1} \otimes \dots \otimes \lambda_n + \dots + (-1)^n \lambda_1 \otimes \dots \otimes \lambda_{n-1} \varepsilon(\lambda_n).$$

One has a homotopy between the complex (45) and the subcomplex $V(\mathbf{G})$ obtained by the following map from $\Lambda^* \mathbf{G}$,

$$(46) \quad X_1 \wedge \dots \wedge X_n \rightarrow \sum (-1)^\sigma X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)} \in \mathcal{U}(\mathbf{G})^{\otimes n}.$$

This gives a subcomplex on which d_n coincides with the boundary in Lie algebra homology with trivial coefficients,

$$(47) \quad d^t(X_1 \wedge \dots \wedge X_n) = \sum_{i < j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \dots \wedge \overset{\vee}{X}_i \wedge \dots \wedge \overset{\vee}{X}_j \wedge \dots \wedge X_n.$$

It is thus natural to try and dualise the above homotopy to the Hochschild complex of \mathcal{H} . Now a Hochschild cocycle

$$(48) \quad h = \sum h_1^i \otimes \dots \otimes h_n^i \in \mathcal{H}^{\otimes n}$$

gives an n -dimensional group cocycle on $G = \{g \in \mathcal{U}(\mathbf{G})_{\text{completed}}; \Delta g = g \otimes g\}$ where,

$$(49) \quad c(g_1, \dots, g_n) = \sum \langle h_1^i, g_1 \rangle \dots \langle h_n^i, g_n \rangle.$$

These cocycles are quite special in that they depend polynomially on the g_i 's. Thus we need to construct a map of cochain complexes from the complex of Lie algebra cohomology to the complex of polynomial cocycles and prove that it gives an isomorphism in cohomology,

$$(50) \quad s : \Lambda^n(\mathbf{G})^* \rightarrow \mathcal{H}^{\otimes n}.$$

If we let j be the restriction of a polynomial cochain to $\Lambda(\mathbf{G})^*$ we expect to have $j \circ s = \text{id}$ and to have a homotopy,

$$(51) \quad s \circ j - 1 = dk + kd.$$

Using the affine coordinates on G we get for $g^0, \dots, g^n \in G$, an affine simplex

$$(52) \quad \Delta(g^0, \dots, g^n) = \left\{ \sum_0^n \lambda_i g^i ; \lambda_i \in [0, 1], \sum \lambda_i = 1 \right\},$$

moreover the right multiplication by $g \in G$ being affine, we have,

$$(53) \quad \Delta(g^0 g, \dots, g^n g) = \Delta(g^0, \dots, g^n) g.$$

The map s is then obtained by the following formula,

$$(54) \quad (s\omega)(g^1, \dots, g^n) = \gamma(1, g^1, g^1 g^2, \dots, g^1 \dots g^n)$$

where

$$\gamma(g^0, \dots, g^n) = \int_{\Delta(g^0, \dots, g^n)} \mathbf{w}$$

where \mathbf{w} is the right invariant form on G associated to $\omega \in \Lambda^n(\mathbf{G})^*$. To prove the existence of the homotopy (51) we introduce the bicomplex of the proof of the Van Est theorem but we restrict to forms with polynomial coefficients: $A^n(G)$ and to group cochains which are polynomial. Thus an element of $C^{n,m}$ is an n -group cochain

$$(55) \quad c(g^1, \dots, g^n) \in A^m$$

and one uses the right action of G on itself to act on forms,

$$(56) \quad (g\omega)(x) = \omega(xg).$$

The first coboundary d_1 is given by

$$(57) \quad \begin{aligned} (d_1 c)(g^1, \dots, g^{n+1}) &= g^1 c(g^2, \dots, g^{n+1}) - c(g^1 g^2, \dots, g^{n+1}) + \dots \\ &+ (-1)^n c(g^1, \dots, g^n g^{n+1}) + (-1)^{n+1} c(g^1, \dots, g^n). \end{aligned}$$

The second coboundary is simply

$$(58) \quad (d_2 c)(g^1, \dots, g^n) = dc(g^1, \dots, g^n).$$

(One should put a sign so that $d_1 d_2 = -d_2 d_1$.)

We need to write down explicitly the homotopies of lines and columns in order to check that they preserve the polynomial property of the cochains. In the affine coordinates δ_μ on G we let

$$(59) \quad X = \sum \delta_\mu \frac{\partial}{\partial \delta_\mu}$$

be the vector field which contracts G to a point.

Then the homotopy k_2 for d_2 comes from,

$$(60) \quad k\omega = \int_0^1 (i_X \omega)(t\delta) \frac{dt}{t}$$

which preserves the space of forms with polynomial coefficients^(*). The homotopy k_1 for d_1 comes from the structure of induced module, i.e. from viewing a cochain $c(g_1, \dots, g_n)$ as a function of $x \in G$ with values in $\Lambda^m \mathbf{G}^*$,

$$(61) \quad (k_1 c)(g_1, \dots, g_{n-1})(x) = c(x, g_1, \dots, g_{n-1})(e).$$

One has $(d_1 k_1 c)(g_1, \dots, g_n)(x) = (k_1 c)(g_2, \dots, g_n)(x g_1) - (k_1 c)(g_1 g_2, \dots, g_n)(x) + \dots + (-1)^{n-1} (k_1 c)(g_1, \dots, g_{n-1} g_n)(x) + (-1)^n (k_1 c)(g_1, \dots, g_{n-1})(x) = \{c(x g_1, g_2, \dots, g_n) - c(x, g_1 g_2, \dots, g_n) + \dots + (-1)^{n-1} c(x, g_1, \dots, g_{n-1} g_n) + (-1)^n c(x, g_1, \dots, g_{n-1})\}(e) = c(g_1, \dots, g_n)(x) - (d_1 c)(x, g_1, \dots, g_n)(e) = c(g_1, \dots, g_n)(x) - (k_1 d_1 c)(g_1, \dots, g_n)(x)$.

This homotopy k_1 clearly preserves the polynomial cochains. This is enough to show that the Hochschild cohomology of \mathcal{H} is isomorphic to the Lie algebra cohomology $H^*(\mathbf{G}, \mathbb{C})$. But it follows from the construction of the cocycle $s\omega$ (54) that

$$(62) \quad (s\omega)(g_1, \dots, g_n) = 0 \quad \text{if} \quad g_1 \dots g_n = 1,$$

and this implies (as in the case of discrete groups) that the corresponding cocycle is also cyclic. ■

Our goal now is to compute the cyclic cohomology of our original Hopf algebra \mathcal{H} . This should combine the two parts of Proposition 6. In the first part the Hochschild cohomology was easy to compute and the operator B was non

^(*) For instance for $G = \mathbb{R}$, $\omega = f(x) dx$ one gets $k\omega = \alpha$, $\alpha(x) = x \int_0^1 f(tx) dt$. In general take $\omega = P(\delta) d\delta_1 \wedge \dots \wedge d\delta_k$, then $i_X \omega$ is of the same form and one just needs to know that $\int_0^1 P(t\delta) t^a dt$ is still a polynomial in δ , which is clear.

trivial. In the second part b was non trivial. At the level of G_1 , one needs to transform the Lie algebra cohomology into the Lie algebra homology with coefficients in \mathbb{C}_δ . The latter corresponds to invariant currents on G and the natural isomorphism is a Poincaré duality. For G_1 with Lie algebra $\{X, Y\}$, $[Y, X] = X$, $\delta(X) = 0$, $\delta(Y) = 1$, one gets that $X \wedge Y$ is a 2-dimensional cycle, while since $bY = 1$, there is no zero dimensional cycle. For the Lie algebra cohomology one checks that there is no 2-dimensional cocycle.

We shall start by constructing an explicit map from the Lie algebra cohomology of $\mathbf{G} = \mathcal{A}$, the Lie algebra of formal vector fields, to the cyclic cohomology of \mathcal{H} . As an intermediate step in the construction of this map, we shall use the following double complex $(C^{n,m}, d_1, d_2)$. For $0 \leq k \leq \dim G_1$ and let $\Omega_k(G_1) = \Omega_k$ be the space of de Rham currents on G_1 , and we let $\Omega_k = \{0\}$ for $k \notin \{0, \dots, \dim G_1\}$. We let $C^{n,m} = \{0\}$ unless $n \geq 0$ and $-\dim G_1 \leq m \leq 0$, and let $C^{n,m}$ be the space of totally antisymmetric polynomial maps $\gamma : G_2^{n+1} \rightarrow \Omega_{-m}$ such that,

$$(63) \quad \gamma(g_0 g, \dots, g_n g) = g^{-1} \gamma(g_0, \dots, g_n) \quad \forall g_i \in G_2, g \in G$$

where we use the right action of G on $G_2 = G_1 \setminus G$ to make sense of $g_i g$ and the left action of G on $G_1 = G/G_2$ to make sense of $g^{-1} \gamma$. The coboundary $d_1 : C^{n,m} \rightarrow C^{n+1,m}$ is given by

$$(64) \quad (d_1 \gamma)(g_0, \dots, g_{n+1}) = (-1)^m \sum_{j=0}^{n+1} (-1)^j \gamma(g_0, \dots, \overset{\vee}{g}_j, \dots, g_{n+1}).$$

The coboundary $d_2 : C^{n,m} \rightarrow C^{n,m+1}$ is the de Rham boundary,

$$(65) \quad (d_2 \gamma)(g_0, \dots, g_n) = d^t \gamma(g_0, \dots, g_n).$$

For $g_0, \dots, g_n \in G_2$, we let $\Delta(g_0, \dots, g_n)$ be the affine simplex with vertices the g_i in the affine coordinates δ_k on G_2 . Since the right action of G on G_2 is affine in these coordinates, we have

$$(66) \quad \Delta(g_0 g, \dots, g_n g) = \Delta(g_0, \dots, g_n) g \quad \forall g_i \in G_2, g \in G.$$

Let ω be a left invariant differential form on G associated to a cochain of degree k in the complex defining the Lie algebra cohomology of the Lie algebra \mathcal{A} . For each pair of integers $n \geq 0$, $-\dim G_1 \leq m \leq 0$ such that $n + m = k - \dim G_1$, let

$$(67) \quad \langle C_{n,m}(g^0, \dots, g^n), \alpha \rangle = (-1)^{\frac{m(m+1)}{2}} \int_{(G_1 \times \Delta(g^0, \dots, g^n))^{-1}} \pi_1^*(\alpha) \wedge \omega$$

for any smooth differential form α , with compact support on G_1 and of degree $-m$.

In this formula we use $G_1 \times \Delta(g_0, \dots, g_n)$ as a cycle in G and we need to show that if $K_i \subset G_i$ are compact subsets, the subset of G

$$(68) \quad K = \{g \in G; \pi_1(g) \in K_1, \pi_2(g^{-1}) \in K_2\}$$

is compact.

For $g \in K$ one has $g = ka$ with $k \in K_1$ and $\pi_2(g^{-1}) \in K_2$ so that $\pi_2(a^{-1}k^{-1}) \in K_2$. But $\pi_2(a^{-1}k^{-1}) = a^{-1} \cdot k^{-1}$ and one has $a^{-1} \in K_2 \cdot K_1^{-1}$ and $a \in (K_2 \cdot K_1^{-1})^{-1}$, thus, the required compactness follows from

$$(69) \quad K \subset K_1(K_2 \cdot K_1^{-1})^{-1}.$$

We let $C^*(\mathcal{A})$ be the cochain complex defining the Lie algebra cohomology of \mathcal{A} and let C be the map defined by (67).

Lemma 7. *The map C is a morphism to the total complex of $(C^{n,m}, d_1, d_2)$.*

Proof. Let us first check the invariance condition (63). One has (66) so that for $C_{n,m}(g^0 g, \dots, g^n g)$ the integration takes place on $\{h \in G; \pi_2(h^{-1}) \in \Delta(g^0, \dots, g^n)g\} = \sum'$. Since $\pi_2(h^{-1})g^{-1} = \pi_2(h^{-1}g^{-1})$ one has $\sum' = g^{-1}\sum$ in G , with $\sum = \{h \in G; \pi_2(h^{-1}) \in \Delta(g^0, \dots, g^n)\}$. One has, with $\beta = \pi_1^*(\alpha) \wedge \omega$, the equality

$$(70) \quad \int_{g^{-1}\sum} \beta = \int_{\sum} L_g \beta$$

where $g \rightarrow L_g$ is the natural action of G on forms on G by left translation. One has $\pi_1(gk) = g\pi_1(k)$ so that $L_g \pi_1^*(\alpha) = \pi_1^*(L_g \alpha)$. Moreover ω is left invariant by hypothesis, so one gets,

$$(71) \quad \langle C_{n,m}(g^0 g, \dots, g^n g), \alpha \rangle = \langle C_{n,m}(g^0, \dots, g^n), L_g \alpha \rangle$$

which since $L_g^\dagger = L_{g^{-1}}$, is the invariance condition (63).

Before we check that $C_{n,m}$ is polynomial in the g^i 's, let us check that

$$(72) \quad C(d\omega) = (d_1 + d_2)C(\omega).$$

One has $d(\pi_1^*(\alpha) \wedge \omega) = \pi_1^*(d\alpha) \wedge \omega + (-1)^m \pi_1^*(\alpha) \wedge d\omega$ and since

$$(73) \quad \int_{\pi_2(g^{-1}) \in \Delta(g^0, \dots, g^{n+1})} d\beta = \sum (-1)^i \int_{\pi_2(g^{-1}) \in \Delta(g^0, \dots, \overset{\vee}{g^i}, \dots, g^n, g^{n+1})} \beta.$$

With $\beta = \pi_1^*(\alpha) \wedge \omega$, the r.h.s. gives $(-1)^m (d_1 C)(g^0, \dots, g^{n+1})$, while the l.h.s. gives $\langle C(g^0, \dots, g^{n+1}), d\alpha \rangle + (-1)^m \langle C'(g^0, \dots, g^{n+1}), \alpha \rangle$, with $C' = C(d\omega)$. Thus we get,

$$(74) \quad d_1 C_{n,m} + d_2 C_{n+1,m-1} = C'_{n+1,m}$$

provided we use the sign: $(-1)^{\frac{m(m+1)}{2}}$ in the definition (67) of C .

We shall now be more specific on the polynomial expression of $C_{n,m}(g^0, \dots, g^n)$ and write this de Rham current on G_1 in the form,

$$(75) \quad \sum c_j \rho_j$$

where $\rho_0 = 1$, $\rho_1 = ds$, $\rho_2 = e^s dx$, $\rho = \rho_1 \wedge \rho_2$ form a basis of left G_1 invariant forms on G_1 , while the c_j are functions on G_1 which are finite linear combinations of finite products of the following functions,

$$(76) \quad k \in G_1 \rightarrow \delta_p(g_j \cdot k), \quad j \in \{0, \dots, n\}.$$

The equality (67) defines $C_{n,m}$ as the integration over the fibers for the map $\pi_1 : G \rightarrow G_1$ of the product of the smooth form ω by the current $\tilde{\Delta}$ of integration on $(G_1 \times \Delta)^{-1} = \{g \in G; \pi_2(g^{-1}) \in \Delta\}$,

$$(77) \quad C_{n,m} = (\pi_1)_*(\omega \wedge \tilde{\Delta}).$$

Thus c_0 , which is a function, is obtained as the integral of $\omega \wedge \tilde{\Delta}$ along the fibers, and its value at $1 \in G_1$ is

$$(78) \quad c_0(1) = \int_{g^{-1} \in \Delta} \omega|_{G_2}.$$

To obtain the value of $c_j(1)$ by a similar formula, one can contract by a vector field Z on G_1 given by left translation, $k \in G_1 \rightarrow \partial_\varepsilon(k(\varepsilon)k)_{\varepsilon=0}$, $k(\varepsilon) \in G_1$. Let \tilde{Z} be the vector field on G given by the same left translation, $g \in G_1 \rightarrow \partial_\varepsilon(k(\varepsilon)g)_{\varepsilon=0}$. The equality $\pi_1(k(\varepsilon)g) = k(\varepsilon)\pi_1(g)$ shows that \tilde{Z} is a left of Z for the fibration π_1 . It follows that for any current β on G one has

$$(79) \quad i_Z \pi_{1*}(\beta) = \pi_{1*}(i_{\tilde{Z}}\beta).$$

(One has $\langle i_Z \pi_{1*}(\beta), \alpha \rangle = \langle \pi_{1*}(\beta), i_Z \alpha \rangle = \langle \beta, \pi_1^*(i_Z \alpha) \rangle = \int \pi_1^*(i_Z \alpha) \wedge \beta = \int i_{\tilde{Z}} \pi_1^*(\alpha) \wedge \beta = \int \pi_1^*(\alpha) \wedge i_{\tilde{Z}} \beta$.)

Next one has $i_{\tilde{Z}}(\omega \wedge \tilde{\Delta}) = (i_{\tilde{Z}}\omega) \times \tilde{\Delta} + (-1)^{\partial\omega} \omega \wedge i_{\tilde{Z}}\tilde{\Delta}$. The contribution of the first term is simple,

$$(80) \quad \int_{g^{-1} \in \Delta} (i_{\tilde{Z}}\omega)|_{G_2}.$$

To compute the contribution of the second term, one needs to understand the current $i_{\tilde{Z}}\tilde{\Delta}$ on G . In general, if β is the current of integration on a manifold $M \hookrightarrow G$ (possibly with boundary), the contraction $i_{\tilde{Z}}\beta$ is obtained as a limit for $\varepsilon \rightarrow 0$ from the manifold $M \times [-\varepsilon, \varepsilon]$ which maps to G by $(x, t) \rightarrow (\exp t \tilde{Z})(x)$. Applying this to $\tilde{\Delta}$ one gets the map from $G_1 \times \Delta \times [-\varepsilon, \varepsilon]$ to G given by

$$(81) \quad (k, a, t) \rightarrow k(t) a^{-1} k^{-1}.$$

One has $k(t) a^{-1} k^{-1} = (ka k(t)^{-1})^{-1} = k^1(a \cdot k(t)^{-1})^{-1}$ where the $a \cdot k^{-1}(t)$ belong to the simplex $\Delta \cdot k(t)^{-1}$ while $k^1 \in G_1$ is arbitrary.

Thus, if we let Z' be the vector field on G_2 given by,

$$(82) \quad a \rightarrow \partial_\varepsilon (a \cdot k^{-1}(\varepsilon))_{\varepsilon=0}$$

we see that the contribution of the second term is

$$(83) \quad \int_{g \in \Delta} i_{Z'} \tilde{\omega}|_{G_2}, \quad \tilde{\omega}(g) = \omega(g^{-1}).$$

Thus there exists a differential form μ_j on G_2 obtained from $\tilde{\omega}$ by contraction by suitable vector fields and restriction to G_2 , such that

$$(84) \quad c_j(1) = \int_{\Delta(g^0, \dots, g^n)} \mu_j.$$

The value of c_j at $k \in G_1$ can now be computed using (71) for $g = k \in G_1$. The forms ρ_j are left invariant under the action of G_1 and by (71) one has $L_{g^{-1}} C(g_0, \dots, g_n) = C(g_0 g, \dots, g_n g)$, $g = k$. Thus $c_j(k)$ corresponds to the current $L_{k^{-1}} C(g_0, \dots, g_n)$ evaluated at 1 and one has,

$$(85) \quad c_j(k) = \int_{\Delta(g^0, \dots, g^n) \cdot k} \mu_j = \int_{\Delta(g^0 \cdot k, \dots, g^n \cdot k)} \mu_j.$$

Let us now apply (71) for $g \in G_2$. The point $1 \in G_1$, is fixed by the action of G_2 while the forms ρ_j vary as follows under the action of G_2 ,

$$(86) \quad L_{g^{-1}} \rho_j = \rho_j, \quad j \neq 1, \quad L_{g^{-1}} \rho_1 = \rho_1 - \delta_1(g \cdot k) \rho_2 \quad (\text{at } k \in G_1).$$

Thus we see that while μ_j , $j \neq 1$ are right invariant forms on G_2 , the form μ_1 satisfies,

$$(87) \quad \mu_1(ag) = \mu_1(a) - \delta_1(g) \mu_2(a).$$

Now in G_2 the product rule as well as $g \rightarrow g^{-1}$ are polynomial in the coordinates δ_n . It follows that the forms μ_j are polynomial forms in these coordinates and that the formula (84) is a polynomial function of the $\delta_j(g^k)$. Using (85) we obtain the desired form (76) for c_j . ■

We shall now use the canonical map Φ of [Co] Theorem 14 p. 220, from the bicomplex $(C^{n,m}, d_1, d_2)$ to the (b, B) bicomplex of the algebra $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$. What we need to prove is that the obtained cochains on \mathcal{H}_* are right invariant in the sense of definition 2 above.

Let us first rewrite the construction of Φ using the notation fU_ψ^* for the generators of \mathcal{H}_* . As in [Co], we let \mathcal{B} be the tensor product,

$$(88) \quad \mathcal{B} = A^*(G_1) \otimes \Lambda \mathbb{C}(G'_2)$$

where $A^*(G_1)$ is the algebra of smooth forms with compact support on G_1 , while we label the generators of the exterior algebra $\Lambda \mathbb{C}(G_2')$ as $\delta_\psi, \psi \in G_2$, with $\delta_0 = 0$. We take the crossed product,

$$(89) \quad C = \mathcal{B} \rtimes G_2$$

of \mathcal{B} by the product action of G_2 , so that

$$(90) \quad \begin{aligned} U_\psi^* \omega U_\psi &= \psi^* \omega = \omega \circ \psi & \forall \omega \in A^*(G_1), \\ U_{\psi_1}^* \delta_{\psi_2} U_{\psi_1} &= \delta_{\psi_2 \circ \psi_1} - \delta_{\psi_1} & \forall \psi_j \in G_2. \end{aligned}$$

The differential d in C is given by

$$(91) \quad d(b U_\psi^*) = db U_\psi^* - (-1)^{\partial b} b \delta_\psi U_\psi^*$$

where the first term comes from the exterior differential in $A^*(G_1)$. Thus the δ_ψ play the role of

$$(92) \quad -\delta_\psi = (d U_\psi^*) U_\psi = -U_\psi^* d U_\psi.$$

A cochain $\gamma \in C^{n,m}$ in the above bicomplex determines a linear form $\tilde{\gamma}$ on C , by,

$$(93) \quad \tilde{\gamma}(\omega \otimes \delta_{g_1} \dots \delta_{g_n}) = \langle \omega, \gamma(1, g_1, \dots, g_n) \rangle, \quad \tilde{\gamma}(b U_\psi^*) = 0 \quad \text{if } \psi \neq 1, \dots$$

What we shall show is that the following cochains on \mathcal{H}_* satisfy definition 2,

$$(94) \quad \varphi(x^0, \dots, x^\ell) = \tilde{\gamma}(dx^{j+1} \dots dx^\ell x^0 dx^1 \dots dx^j), \quad x^j \in \mathcal{H}_*.$$

We can assume that $\gamma(1, g_1, \dots, g_n) = \prod_{j=1}^n P_j(\delta(g_j \cdot k)) \rho_i(k)$ where each P_j is a polynomial (in fact monomial) in the δ_n .

We take the $\rho_j, j = 0, 1, 2, 3$ as a basis of $A^*(G_1)$ viewed as a module over $C_c^\infty(G_1)$ and for $f \in C_c^\infty(G_1)$ we write df as

$$(95) \quad df = -(Yf) \rho_1 + (Xf) \rho_2, \quad Y = -\frac{\partial}{\partial s}, \quad X = e^{-s} \frac{\partial}{\partial x}$$

which is thus expressed in terms of the left action of \mathcal{H} on $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$. Moreover, using (86), one has $U_\psi^* \rho_j U_\psi = \rho_j, j \neq 1$ and

$$(96) \quad U_\psi^* \rho_1 U_\psi = \rho_1 - \delta_1(U_\psi^*) U_\psi \rho_2$$

or in other terms $\rho_1 U_\psi^* = \delta_1(U_\psi^*) \rho_2 + U_\psi^* \rho_1$.

This shows that provided we replace some of the x_j 's in (94) by the $h_j(x_j), h_j \in \mathcal{H}$, we can get rid of all the exterior differentials df and move all the ρ_j 's to the end of the expression which becomes,

$$(97) \quad \tilde{\gamma}(f^0 \delta_{\psi_0} U_{\psi_0}^* f^1 \delta_{\psi_1} U_{\psi_1}^* \dots f^\ell \delta_{\psi_\ell} U_{\psi_\ell}^* \rho_i)$$

provided we relabel the x_j 's in a cyclic way (which is allowed by definition 2) and we omit several δ_{ψ_j} .

To write (97) in the form (6) we can assume that $\psi_\ell \dots \psi_1 \psi_0 = 1$ since otherwise one gets 0. We first simplify the parenthesis using the crossed product rule in C and get,

$$(98) \quad \begin{aligned} & f^0(f^1 \circ \psi^0)(f^2 \circ \psi^1 \circ \psi^0) \dots (f^\ell \circ \psi^{\ell-1} \dots \psi^0) \\ & \delta_{\psi_0}(\delta_{\psi_1 \psi_0} - \delta_{\psi_0}) \dots (\delta_{\psi_\ell \dots \psi_0} - \delta_{\psi_{\ell-1} \dots \psi_0}) \rho_i. \end{aligned}$$

Let $f = f^0(f^1 \circ \psi^0) \dots (f^\ell \circ \psi^{\ell-1} \dots \psi^0)$. When we apply $\tilde{\gamma}$ to (98) we get

$$(99) \quad \int_{G_1} f(k) \prod_{j=1}^{\ell} P_j(\delta(\psi^{j-1} \dots \psi^0 \cdot k)) \rho(k)$$

where we used the equality $\delta_g^2 = 0$ in $\Lambda \mathbb{C} G'_2$. The same result holds if we omit several δ_{ψ_j} , one just takes $P_j = 1$ in the expression (99).

We now rewrite (99) in the form,

$$(100) \quad \tau_0(f(P_0(\delta)U_{\psi_0}^*)U_{\psi_0}(P_1(\delta)U_{\psi_1 \psi_0}^*)U_{\psi_1 \psi_0} \dots (P_\ell(\delta)U_{\psi_{\ell-1} \dots \psi_0}^*)U_{\psi_{\ell-1} \dots \psi_0}).$$

Let us replace f by $f^0(f^1 \circ \psi^0) \dots (f^\ell \circ \psi^{\ell-1} \dots \psi^0)$ and move the f^j so that they appear without composition, we thus get

$$(101) \quad \tau_0(f^0(P_0(\delta)U_{\psi_0}^*)f^1(U_{\psi_0}P_1(\delta)U_{\psi_1 \psi_0}^*)f^2(U_{\psi_1 \psi_0}P_2(\delta)U_{\psi_2 \psi_1 \psi_0}^*) \dots).$$

We now use the coproduct rule to rearrange the terms, thus

$$(102) \quad P_1(\delta)U_{\psi_1 \psi_0}^* = \sum P_1^{(1)}(\delta)U_{\psi_0}^* P_1^{(2)}(\delta)U_{\psi_1}^*$$

and we can permute f^1 with $U_{\psi_0}P_1^{(1)}(\delta)U_{\psi_0}^*$ and use the equality

$$(103) \quad (P(\delta)U_{\psi_0}^*)(U_{\psi_0}Q(\delta)U_{\psi_0}^*) = (PQ)(\delta)U_{\psi_0}^*.$$

Proceeding like this we can rewrite (101) in the form,

$$(104) \quad \tau_0(f^0Q_0(\delta)U_{\psi_0}^*f^1Q_1(\delta)U_{\psi_1}^* \dots f^\ell Q_\ell(\delta)U_{\psi_\ell}^*)$$

which shows that the functional (94) satisfies definition 2.

Thus the map Φ of [Co] p. 220 together with Lemma 7 gives us a morphism θ of complexes from the complex $C^*(\mathcal{A})$ of the Lie algebra cohomology of the Lie algebra \mathcal{A} of formal vector fields, to the (b, B) bicomplex of the Hopf algebra \mathcal{H} .

Since the current $c(g^0, \dots, g^n)$ is determined by its value at $1 \in G_1$, i.e. by

$$(105) \quad \sum c_j(1) \rho_j$$

we can view the map C as a map from $C^*(\mathcal{A})$ to the cochains of the group cohomology of G_2 with coefficients in the module E ,

$$(106) \quad E = \Lambda \mathbf{G}_1^*$$

which is the exterior algebra on the cotangent space $T_1^*(G_1)$. Since the action of G_2 on G_1 fixes 1 it acts on $T_1^*(G_1)$ and in the basis ρ_i the action is given by (86).

Since \mathcal{A} is the direct sum $\mathcal{A} = \mathbf{G}_1 \oplus \mathbf{G}_2$ of the Lie algebras of G_1 and G_2 viewed as Lie subalgebras of \mathcal{A} (it is a direct sum as vector spaces, not as Lie algebras), one has a natural isomorphism

$$(107) \quad \Lambda \mathcal{A}^* \simeq \Lambda \mathbf{G}_1^* \otimes \Lambda \mathbf{G}_2^*$$

of the cochains in $C^*(\mathcal{A})$ with cochains in $C^*(\mathbf{G}_2, E)$, the Lie algebra cohomology of \mathbf{G}_2 with coefficients in E .

Lemma 8. *Under the above identifications, the map C coincides with the cochain implementation $C^*(\mathbf{G}_2, E) \rightarrow C^*(G_2, E)$ of the van Est isomorphism, which associates to a right invariant form μ on G_2 , with values in E , the totally antisymmetric homogeneous cochain*

$$C(\mu)(g^0, \dots, g^n) = \int_{\Delta(g^0, \dots, g^n)} \mu.$$

Proof. By (84) we know that there exists a right invariant form, $\mu = \sum \mu_j \rho_j$ on G_2 with values in E , such that

$$(108) \quad c_j(1) = \int_{\Delta(g^0, \dots, g^n)} \mu_j.$$

The value of μ_j at $1 \in G_2$ is obtained by contraction of ω evaluated at $1 \in G$, by a suitable element of $\Lambda \mathbf{G}_1$. Indeed this follows from (80) and the vanishing of the vector field Z' of (82) at $a = 1 \in G_2$. Thus the map $\omega \rightarrow \mu$ is the isomorphism (107). ■

Of course the coboundary d_1 in the cochain complex $C^*(\mathbf{G}_2, E)$ is not equal to the coboundary d of $C^*(\mathcal{A})$, but it corresponds by the map C to the coboundary d_1 of the bicomplex $(C^{n,m}, d_1, d_2)$. We should thus check directly that d_1 and d anticommute in $C^*(\mathcal{A})$. To see this, we introduce a bigrading in $C^*(\mathcal{A})$ associated to the decomposition $\mathcal{A} = \mathbf{G}_1 \oplus \mathbf{G}_2$. What we need to check is that the Lie algebra cohomology coboundary d transforms an element of bidegree (n, m) into a sum of two elements of bidegree $(n+1, m)$ and $(n, m+1)$ respectively. It is enough to do that for 1-forms. Let ω be of bidegree $(1, 0)$, then

$$(109) \quad d\omega(X_1, X_2) = -\omega([X_1, X_2]) \quad \forall X_1, X_2 \in \mathcal{A}.$$

This vanishes if $X_1, X_2 \in \mathbf{G}_2$ thus showing that $d\omega$ has no component of bidegree $(0, 2)$.

We can then decompose d as $d = d_1 + d_2$ where d_1 is of bidegree $(1, 0)$ and d_2 of bidegree $(0, 1)$.

Let us check that d_1 is the same as the coboundary of Lie algebra cohomology of \mathbf{G}_2 with coefficients in $\Lambda \mathbf{G}_1^*$. Let $\alpha \in \Lambda^m \mathbf{G}_1^*$ and $\omega \in \Lambda^n \mathbf{G}_2^*$. The component of bidegree $(n+1, m)$ of $d(\alpha \wedge \omega) = (d\alpha) \wedge \omega + (-1)^m \alpha \wedge d\omega$ is

$$(110) \quad d_1 \alpha \wedge \omega + (-1)^m \alpha \wedge d_1 \omega$$

where $d_1 \omega$ takes care of the second term in the formula for the coboundary in Lie algebra cohomology,

$$(111) \quad \sum (-1)^{i+1} X_i \omega(X_1, \dots, \overset{\vee}{X}_i, \dots, X_{n+1}) + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \overset{\vee}{X}_i, \dots, \overset{\vee}{X}_j, \dots, X_{n+1}).$$

Thus, it remains to check that $d_1 \alpha \wedge \omega$ corresponds to the first term in (111) for the natural action of \mathbf{G}_2 on $E = \Lambda^* \mathbf{G}_1$, which can be done directly for α a 1-form, since $d_1 \alpha$ is the transpose of the action of \mathbf{G}_2 on \mathbf{G}_1 .

We can now state the main lemma allowing to prove the surjectivity of the map θ .

Lemma 9. *The map θ from $(C^*(\mathcal{A}), d_1)$ to the Hochschild complex of \mathcal{H} gives an isomorphism in cohomology.*

Proof. Let us first observe that by Lemma 8 and the above proof of the van Est theorem, the map C gives an isomorphism in cohomology from $(C^*(\mathcal{A}), d_1)$ to the complex (C^*, d_1) of polynomial G_2 -group cochains with coefficients in E . Thus it is enough to show that the map Φ gives an isomorphism at the level of Hochschild cohomology. This of course requires to understand the Hochschild cohomology of the algebra \mathcal{H}_* with coefficients in the module \mathbb{C} given by the augmentation on \mathcal{H}_* .

In order to do this we shall use the abstract version ([C-E] Theorem 6.1 p. 349) of the Hochschild-Serre spectral sequence [Ho-Se].

A subalgebra $\mathcal{A}_1 \subset \mathcal{A}_0$ of a augmented algebra \mathcal{A}_0 is called *normal* iff the right ideal J generated in \mathcal{A}_0 by the $\text{Ker } \varepsilon$ (of the augmentation ε of \mathcal{A}_1) is also a left ideal.

When this is so, one lets

$$(112) \quad \mathcal{A}_2 = \mathcal{A}_0/J$$

be the quotient of \mathcal{A}_0 by the ideal J . One has then a spectral sequence converging to the Hochschild cohomology $H_{\mathcal{A}_0}^*(\mathbb{C})$ of \mathcal{A}_0 with coefficients in the module \mathbb{C} (using the augmentation e) and with E_2 term given by

$$(113) \quad H_{\mathcal{A}_2}^p(H_{\mathcal{A}_1}^q(\mathbb{C})).$$

To prove it one uses the equivalence, for any right \mathcal{A}_2 -module A and right \mathcal{A}_0 -module B ,

$$(114) \quad \text{Hom}_{\mathcal{A}_2}(A, \text{Hom}_{\mathcal{A}_0}(\mathcal{A}_2, B)) \simeq \text{Hom}_{\mathcal{A}_0}(A, B)$$

where in the left term one views \mathcal{A}_2 as an $\mathcal{A}_2 - \mathcal{A}_0$ bimodule using the quotient map $\varphi : \mathcal{A}_0 \rightarrow \mathcal{A}_2$ to get the right action of \mathcal{A}_0 . Also in the right term, one uses φ to turn A into a right \mathcal{A}_0 -module.

Replacing A by a projective resolution of \mathcal{A}_2 -right modules and B by an injective resolution of \mathcal{A}_0 -right modules, and using (since $\mathcal{A}_2 = \mathbb{C} \otimes_{\mathcal{A}_1} \mathcal{A}_0$) the equivalence,

$$(115) \quad \text{Hom}_{\mathcal{A}_0}(\mathcal{A}_2, B) = \text{Hom}_{\mathcal{A}_1}(\mathbb{C}, B) ,$$

one obtains the desired spectral sequence.

In our case we let $\mathcal{A}_0 = \mathcal{H}_*$ and $\mathcal{A}_1 = C_c^\infty(G_1)$. These algebras are non unital but the results still apply. We first need to prove that \mathcal{A}_1 is a *normal* subalgebra of \mathcal{A}_0 . The augmentation ε on \mathcal{H}_* is given by,

$$(116) \quad \varepsilon(f U_\psi^*) = f(1) \quad \forall f \in C_c^\infty(G_1), \psi \in G_2 .$$

Its restriction to \mathcal{A}_1 is thus $f \rightarrow f(1)$. Thus the ideal J is linearly generated by elements $g U_\psi^*$ where

$$(117) \quad g \in C_c^\infty(G_1) , g(1) = 0 .$$

We need to show that it is a left ideal in $\mathcal{A}_0 = \mathcal{H}_*$. For this it is enough to show that $U_{\psi_1}^* g U_{\psi_2}^*$ is of the same form, but this follows because,

$$(118) \quad \psi(1) = 1 \quad \forall \psi \in G_2 .$$

Moreover, with the above notations, the algebra \mathcal{A}_2 is the group ring of G_2 . We thus obtain by [C-E] loc. cit., a spectral sequence which converges to the Hochschild cohomology of \mathcal{H} and whose E_2 term is given by the polynomial group cohomology of G_2 with coefficients in the Hochschild cohomology of the coalgebra $\mathcal{U}(\mathbf{G}_1)$, which according to proposition 6.1 is given by $\Lambda \mathbf{G}_1^*$.

It thus follows that, combining the van Est theorem with the above spectral sequence, the map θ gives an isomorphism in Hochschild cohomology. ■

We can summarize this section by the following result.

Theorem 10. *The map θ defines an isomorphism from the Lie algebra cohomology of \mathcal{A} to the periodic cyclic cohomology of \mathcal{H} .*

This theorem extends to the higher dimensional case, i.e. where the Lie algebra \mathcal{A} is replaced by the Lie algebra of formal vector fields in n -dimensions, while $G = \text{Diff } \mathbb{R}^n$, with G_1 the subgroup of affine diffeomorphisms, and $G_2 = \{\psi; \psi(0) = 0, \psi'(0) = \text{id}\}$. It also admits a relative version in which one considers the Lie algebra cohomology of \mathcal{A} relative to $SO(n) \subset G_0 = GL(n) \subset G_1$.

VIII. Characteristic classes for actions of Hopf algebras

In the above section we have defined and computed the cyclic cohomology of the Hopf algebra \mathcal{H} as the Lie algebra cohomology of \mathcal{A} , the Lie algebra of formal vector fields.

The theory of characteristic classes for actions of \mathcal{H} extends the construction (cf. [Co2]) of cyclic cocycles from a Lie algebra of derivations of a C^* algebra A , together with an *invariant trace* τ on A . At the purely algebraic level, given an algebra A and an action of the Hopf algebra \mathcal{H} on A ,

$$(1) \quad \mathcal{H} \otimes A \rightarrow A, \quad h \otimes a \rightarrow h(a)$$

satisfying $h_1(h_2 a) = (h_1 h_2)(a) \quad \forall h_i \in \mathcal{H}$ and

$$h(ab) = \sum h_{(0)}(a) h_{(1)}(b) \quad \forall a, b \in A$$

we shall say that a trace τ on A is *invariant* iff the following holds,

$$(2) \quad \tau(h(a)b) = \tau(a \tilde{S}(h)(b)) \quad \forall a, b \in A, \quad h \in \mathcal{H}.$$

One has the following straightforward,

Proposition 1. *Let τ be an \mathcal{H} -invariant trace on A , then the following defines a canonical map $\gamma : HC^*(\mathcal{H}) \rightarrow HC^*(A)$,*

$$\begin{aligned} \gamma(h^1 \otimes \dots \otimes h^n) &\in C^n(A), \quad \gamma(h^1 \otimes \dots \otimes h^n)(x^0, \dots, x^n) = \\ &\tau(x^0 h^1(x^1) \dots h^n(x^n)). \end{aligned}$$

In the interesting examples the algebra A is a C^* algebra and the action of \mathcal{H} on A is only densely defined. It is then crucial to know that the common domain,

$$(3) \quad A^\infty = \{a \in A; \quad h(a) \in A, \quad \forall h \in \mathcal{H}\}$$

is a subalgebra stable under the holomorphic functional calculus.

It is clear from the coproduct rule that A^∞ is a subalgebra of A , the question is to show the stability under holomorphic functional calculus.

Our aim is to show that for any action of our Hopf algebra \mathcal{H} (of section 2) on a C^* algebra A the common domain (3) is stable under holomorphic functional calculus. For short we shall say that a Hopf algebra \mathcal{H} is *differential* iff this holds.

Lemma 2. *Let \mathcal{H} be a Hopf algebra, $\mathcal{H}_1 \subset \mathcal{H}$ a Hopf subalgebra. We assume that \mathcal{H}_1 is differential and that as an algebra \mathcal{H} is generated by \mathcal{H}_1 and an element $\delta \in \mathcal{H}$, ($\varepsilon(\delta) = 0$) such that the following holds,*

a) For any $h \in \mathcal{H}_1$, there exist $h_1, h_2, h'_1, h'_2 \in \mathcal{H}_1$ such that

$$\delta h = h_1 \delta + h_2, \quad h \delta = \delta h'_1 + h'_2.$$

b) There exists $R \in \mathcal{H}_1 \otimes \mathcal{H}_1$ such that $\Delta \delta = \delta \otimes 1 + 1 \otimes \delta + R$.

Then \mathcal{H} is a differential Hopf algebra.

Proof. Let $A^{1,\infty} = \{a \in A; h(a) \in A \quad \forall h \in \mathcal{H}_1\}$. By hypothesis if u is invertible in A and $u \in A^{1,\infty}$ one has $u^{-1} \in A^{1,\infty}$. It is enough, modulo some simple properties of the resolvent ([Co]) to show that the same holds in A^∞ . One has using a) that $A^\infty = \bigcap_k \{a \in A^{1,\infty}; \delta^j(a) \in A^{1,\infty} \quad 1 \leq j \leq k\} = \bigcap_k A^{\infty,k}$. Let $u \in A^\infty$ be invertible in A , then $u^{-1} \in A^{1,\infty}$. Let us show by induction on k that $u^{-1} \in A^{\infty,k}$. With $R = \sum h_i \otimes k_i$ it follows from b) that

$$(4) \quad \delta(u u^{-1}) = \delta(u) u^{-1} + u \delta(u^{-1}) + \sum h_i(u) k_i(u^{-1})$$

so that,

$$(5) \quad \delta(u^{-1}) = u^{-1} \delta(1) - u^{-1} \delta(u) u^{-1} - \sum u^{-1} h_i(u) k_i(u^{-1}).$$

Since $\varepsilon(\delta) = 0$ one has $\delta(1) = 0$ and the r.h.s. of (5) belongs to $A^{1,\infty}$. This shows that $u^{-1} \in A^{\infty,1}$. Now by a), $A^{\infty,1}$ is stable by the action of \mathcal{H}_1 and by b) it is a subalgebra of $A^{1,\infty}$. Thus (5) shows that $\delta(u^{-1}) \in A^{\infty,1}$, i.e. that $u^{-1} \in A^{\infty,2}$. Now similarly since,

$$(6) \quad A^{\infty,2} = \{a \in A^{\infty,1}; \delta(a) \in A^{\infty,1}\}$$

we see that $A^{\infty,2}$ is a subalgebra of $A^{\infty,1}$ stable under the action of \mathcal{H}_1 , so that by (5) we get $\delta(u^{-1}) \in A^{\infty,2}$ and $u^{-1} \in A^{\infty,3}$. The conclusion follows by induction. ■

Proposition 3. *The Hopf algebra \mathcal{H} of Section 2 is differential.*

Proof. Let $\mathcal{H}^1 \subset \mathcal{H}$ be the inductive limit of the Hopf subalgebras \mathcal{H}_n . For each n the inclusion of \mathcal{H}_n in \mathcal{H}_{n+1} fulfills the hypothesis of Lemma 2 so that $\mathcal{H}^1 = \bigcup_n \mathcal{H}_n$ is differential.

Let then $\mathcal{H}^2 \subset \mathcal{H}$ be generated by \mathcal{H}^1 and Y , again the inclusion $\mathcal{H}^1 \subset \mathcal{H}^2$ fulfills the hypothesis of the Lemma 2 since $\Delta Y = Y \otimes 1 + 1 \otimes Y$ while $h \rightarrow Yh - hY$ is a derivation of \mathcal{H}^1 . Finally \mathcal{H} is generated over \mathcal{H}^2 by X and one checks again the hypothesis of Lemma 2, since in particular $XY = (Y+1)X$. ■

It is not difficult to give examples of Hopf algebras which are not differential such as the Hopf algebra generated by θ with

$$(7) \quad \Delta \theta = \theta \otimes \theta.$$

IX. The index formula

We use the notations of section II, so that $\mathcal{A} = C_c^\infty(F^+) \rtimes \Gamma$ is the crossed product of the positive frame bundle of a flat manifold M by a pseudogroup Γ . We let v be the canonical Diff^+ invariant volume form on F^+ and,

$$(1) \quad L^2 = L^2(F^+, v),$$

be the corresponding Hilbert space.

The canonical representation of \mathcal{A} in L^2 is given by

$$(2) \quad (\pi(f U_\psi^*) \xi)(j) = f(j) \xi(\tilde{\psi}(j)) \quad \forall j \in F^+, \xi \in L^2, f U_\psi^* \in \mathcal{A}.$$

The invariance of v shows that π is a unitary representation for the natural involution of \mathcal{A} . When no confusion can arise we shall write simply $a\xi$ instead of $\pi(a)\xi$ for $a \in \mathcal{A}$, $\xi \in L^2$.

The flat connection on M allows to extend the canonical action of the group $G_0 = GL^+(n, \mathbb{R})$ on F^+ to an action of the affine group,

$$(3) \quad G_1 = \mathbb{R}^n \rtimes G_0$$

generated by the vector fields X_i and Y_ℓ^k of section I.

This representation of G_1 admits the following compatibility with the representation π of \mathcal{A} ,

Lemma 1. *Let $a \in \mathcal{A}$ one has, 1) $Y_\ell^k \pi(a) = \pi(a) Y_\ell^k + \pi(Y_\ell^k(a))$ 2) $X_i \pi(a) = \pi(a) X_i + \pi(X_i(a)) + \pi(\delta_{ij}^k(a)) Y_k^j$.*

Proof. With $a = f U_\psi^* \in \mathcal{A}$ and $\xi \in C_c^\infty(F^+)$ one has $Y_\ell^k \pi(a) \xi = Y_\ell^k(f \xi \circ \psi) = Y_\ell^k(f) \xi \circ \psi + f Y_\ell^k(\xi \circ \psi)$. Since Y_ℓ^k commutes with diffeomorphisms the last term is $(Y_\ell^k \xi) \circ \psi$ which gives 1).

The operators X_i acting on \mathcal{A} satisfy (11) of section II which one can specialize to $b \in C_c^\infty(F^+)$, to get

$$(4) \quad X_i(f U_\psi^* b) = X_i(f U_\psi^*) b + f U_\psi^* X_i(b) + \delta_{ij}^k(f U_\psi^*) Y_k^j(b).$$

One has $X_i(f U_\psi^* b) = X_i(f(b \circ \psi)) U_\psi^* = X_i(\pi(a)b) U_\psi^*$, $X_i(f U_\psi^*) b = \pi(X_i(a)) b U_\psi^*$, $f U_\psi^* X_i(b) = \pi(a) X_i(b) U_\psi^*$ and $\delta_{ij}^k(f U_\psi^*) Y_k^j(b) = \pi(\delta_{ij}^k(a)) Y_k^j(b) U_\psi^*$, thus one gets 2). ■

Corollary 2. *For any element Q of $\mathcal{U}(\mathbf{G}_1)$ there exists finitely many elements $Q_i \in \mathcal{U}(\mathbf{G}_1)$ and $h_i \in \mathcal{H}$ such that*

$$Q \pi(a) = \sum \pi(h_i(a)) Q_i \quad \forall a \in \mathcal{A}.$$

Proof. This condition defines a subalgebra of $\mathcal{U}(\mathbf{G}_1)$ and we just checked it for the generators. ■

Let us now be more specific and take for Q the hypoelliptic signature operator on F^+ . It is not a scalar operator but it acts in the tensor product

$$(5) \quad \mathcal{H}_0 = L^2(F^+, v) \otimes E$$

where E is a finite dimensional representation of $SO(n)$ specifically given by

$$(6) \quad E = \Lambda \mathbf{P}_n \otimes \Lambda \mathbb{R}^n, \quad \mathbf{P}_n = S^2 \mathbb{R}^n.$$

The operator Q is the graded sum,

$$(7) \quad Q = (d_V^* d_V - d_V d_V^*) \oplus (d_H + d_H^*)$$

where the horizontal (resp. vertical) differentiation d_H (resp. d_V) is a matrix in the X_i resp. Y_ℓ^k . When n is equal to 1 or 2 modulo 4 one has to replace F^+ by its product by S^1 so that the dimension of the vertical fiber is even (it is then $1 + \frac{n(n+1)}{2}$) and the vertical signature operator makes sense. The longitudinal part is not elliptic but only transversally elliptic with respect to the action of $SO(n)$. Thus, to get a hypoelliptic operator, one restricts Q to the Hilbert space

$$(8) \quad \mathcal{H}_1 = (L^2(F^+, v) \otimes E)^{SO(n)}$$

and one takes the following subalgebra of \mathcal{A} ,

$$(9) \quad \mathcal{A}_1 = (\mathcal{A})^{SO(n)} = C_c^\infty(P) \rtimes \Gamma, \quad P = F^+ / SO(n).$$

Let us note that the operator Q is in fact the image under the right regular representation of the affine group G_1 of a (matrix-valued) hypoelliptic symmetric element in $\mathcal{U}(\mathbf{G}_1)$. By an easy adaptation of a theorem of Nelson and Stinespring, it then follows that Q is essentially selfadjoint (with core any dense, G_1 -invariant subspace of the space of C^∞ -vectors of the right regular representation of G_1).

We let τ be the trace on \mathcal{A}_1 which is dual to the invariant volume form v_1 on P , that is,

$$(10) \quad \tau(f U_\psi^*) = 0 \quad \text{if } \psi \neq 1, \quad \tau(f) = \int_P f dv_1.$$

Also we adapt the result of the previous sections to the relative case, and use the action of \mathcal{H} on \mathcal{A} to get a characteristic map,

$$(11) \quad HC^*(\mathcal{H}, SO(n)) \rightarrow HC^*(\mathcal{A}_1)$$

associated to the trace τ .

Proposition 3. *Let us assume that the action of Γ on M has no degenerate fixed point. Then any cochain on \mathcal{A}_1 of the form,*

$$\varphi(a^0, \dots, a^n) = \int a^0 [Q, a^1]^{(k_1)} \dots [Q, a^n]^{(k_n)} (Q^2)^{-\frac{1}{2}(n+2|k|)} \quad \forall a^j \in \mathcal{A}_1$$

(with $T^{(k)} = [Q^2, \dots [Q^2, T] \dots]$), is in the range of the characteristic map.

Proof. By Corollary 2 one can write each operator $a^0 [Q, a^1]^{(k_1)} \dots [Q, a_n]^{(k_n)}$ in the form

$$(12) \quad \sum_{\alpha} a^0 h_1^{\alpha}(a^1) \dots h_n^{\alpha}(a^n) Q_{\alpha} \quad h_j \in \mathcal{H}, Q_{\alpha} \in \mathcal{U}(G_1),$$

thus we just need to understand the cochains of the form,

$$(13) \quad \int a^0 h_1(a^1) \dots h_n(a^n) R = \int a R$$

where R is pseudodifferential in the hypoelliptic calculus and commutes with the affine subgroup of Diff,

$$(14) \quad R U_{\psi} = U_{\psi} R \quad \forall \psi \in G'_1 \subset \text{Diff}.$$

Since R is given by a smoothing kernel outside the diagonal and the action of Γ on $F^+ = F^+(M)$ is free by hypothesis, one gets that

$$(15) \quad \int f U_{\psi}^* R = 0 \quad \text{if } \psi \neq 1.$$

Also by (14) the functional $\int f R$ is G'_1 invariant and is hence proportional to

$$(16) \quad \int f dv_1.$$

We have thus proved that φ can be written as a finite linear combination,

$$(17) \quad \varphi(a^0, \dots, a^n) = \sum_{\alpha} \tau(a^0 h_1^{\alpha}(a^1) \dots h_n^{\alpha}(a^n)).$$

Since we restrict to the subalgebra $\mathcal{A}_1 \subset \mathcal{A}$, the cochain

$$(18) \quad c = \sum h_1^{\alpha} \otimes \dots \otimes h_n^{\alpha}$$

should be viewed as a *basic* cochain in the cyclic complex of \mathcal{H} , relative to the subalgebra $\mathcal{H}_0 = \mathcal{U}(\mathbf{O}(n))$. ■

Now by theorem 10 of section VII one has an isomorphism,

$$(19) \quad H^*(\mathcal{A}_n, SO(n)) \xrightarrow{\theta} HC^*(\mathcal{H}, SO(n))$$

where the left hand side is the relative Lie algebra cohomology of the Lie algebra of formal vector fields.

Let us recall the result of Gelfand-Fuchs (cf. [G-F], [G]) which allows to compute the left hand side of (19).

One lets $G_0 = GL^+(n, \mathbb{R})$ and \mathbf{G}_0 its Lie algebra, viewed as a subalgebra of $\mathbf{G}_1 \subset \mathcal{A}_n$. One then views (cf. [G]) the natural projection,

$$(20) \quad \pi : \mathcal{A}_n \rightarrow \mathbf{G}_0$$

as a connection 1-form. Its restriction to $\mathbf{G}_0 \subset \mathcal{A}_n$ is the identity map and,

$$(21) \quad \pi([X_0, X]) = [X_0, \pi(X)] \quad \forall X_0 \in \mathbf{G}_0, X \in \mathcal{A}_n.$$

The curvature of this connection is

$$(22) \quad \Omega = d\pi + \frac{1}{2}[\pi, \pi]$$

is easy to compute and is given by

$$(23) \quad \Omega(X, Y) = [Y_1, X_{-1}] - [X_1, Y_{-1}], \quad \forall X, Y \in \mathcal{A}_n,$$

in terms of the projections $X \rightarrow (X)_j$ associated to the grading of the Lie algebra \mathcal{A}_n .

It follows from the Chern-Weil theory that one has a canonical map from the Weil complex, $S^*(\mathbf{G}_0) \wedge \Lambda^*(\mathbf{G}_0) = W(\mathbf{G}_0)$,

$$(24) \quad W(\mathbf{G}_0) \xrightarrow{\varphi} C^*(\mathcal{A}_n).$$

For $\xi \in \mathbf{G}_0^*$ viewed as an odd element $\xi^- \in \Lambda^* \mathbf{G}_0$, one has

$$(25) \quad \varphi(\xi^-) \in \mathcal{A}_n^*, \quad \varphi(\xi^-) = \xi \circ \pi.$$

For $\xi \in \mathbf{G}_0^*$ viewed as an even element $\xi^+ \in S^* \mathbf{G}_0$, one has

$$(26) \quad \varphi(\xi^+) \in \Lambda^2 \mathcal{A}_n^*, \quad \varphi(\xi^+) = \xi \circ \Omega,$$

moreover the map φ is an algebra morphism, which fixes it uniquely. By construction φ vanishes on the ideal J of $W = W(\mathbf{G}_0)$ generated by $\sum_{r>n} S^r(\mathbf{G}_0^*)$.

By definition one lets

$$(27) \quad W_n = W/J$$

be the corresponding differential algebra and $\tilde{\varphi}$ the quotient map,

$$(28) \quad \tilde{\varphi} : H^*(W_n) \rightarrow H^*(\mathcal{A}_n).$$

It is an isomorphism by [G-F].

The discussion extends to the relative situation and yields a subcomplex $WSO(n)$ of the elements of W_n which are basic relative to the action of $SO(n)$. Again by [G-F], the morphism $\tilde{\varphi}$ gives an isomorphism,

$$(29) \quad \tilde{\varphi} : H^*(WSO(n)) \rightarrow H^*(\mathcal{A}_n, SO(n)).$$

A concrete description of $H^*(WSO(n))$ is obtained (cf. [G]) as a small variant of $H^*(WO(n))$, i.e. the orthogonal case. The latter is the cohomology of the complex

$$(30) \quad E(h_1, h_3, \dots, h_m) \otimes P(c_1, \dots, c_n), \quad d$$

where $E(h_1, h_3, \dots, h_m)$ is the exterior algebra in the generators h_i of dimension $2i - 1$, (m is the largest odd integer less than n) and i odd $\leq n$, while $P(c_1, \dots, c_n)$ is the polynomial algebra in the generators c_i of degree $2i$ truncated by the ideal of elements of weight $> 2n$. The coboundary d is defined by,

$$(31) \quad dh_i = c_i, \quad i \text{ odd}, \quad dc_i = 0 \text{ for all } i.$$

One lets $p_i = c_{2i}$ be the Pontrjagin classes, they are non trivial cohomology classes for $2i \leq n$. One has $H^*(WSO(n)) = H^*(WO(n))$ for n odd, while for n even one has

$$(32) \quad H^*(WSO(n)) = H^*(WO(n)) [\chi] / (\chi^2 - c_n).$$

Let us now recall the index theorem of [C-M] for spectral triples $(\mathcal{A}, \mathcal{H}, D)$ whose dimension spectrum is discrete and simple, which is the case (cf. [C-M]) for the transverse fundamental class, (we treat the *odd* case)

Theorem 4. a) The equality $\int P = \text{Res}_{z=0} \text{Trace}(P|D|^{-z})$ defines a trace on the algebra generated by \mathcal{A} , $[D, \mathcal{A}]$ and $|D|^z$, $z \in \mathbb{C}$. b) The following formula only has a finite number of non zero terms and defines the components $(\varphi_n)_{n=1,3,\dots}$ of a cocycle in the (b, B) bicomplex of \mathcal{A} ,

$$\varphi_n(a^0, \dots, a^n) = \sum_k c_{n,k} \int a^0 [D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} |D|^{-n-2|k|}$$

$\forall a^j \in \mathcal{A}$ where one lets $T^{(k)} = \nabla^k(T)$, $\nabla(T) = D^2T - TD^2$ and where k is a multiindex, $c_{n,k} = (-1)^{|k|} \sqrt{2i} (k_1! \dots k_n!)^{-1} (k_1 + 1)^{-1} \dots (k_1 + k_2 + \dots + k_n + n)^{-1} \Gamma(|k| + \frac{n}{2})$, $|k| = k_1 + \dots + k_n$. c) The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.

Let us remark that the theorem is unchanged if we replace everywhere the operator D by

$$(33) \quad Q = D|D|.$$

This follows directly from the proof in [C-M].

In our case the operator Q is differential, given by (7), so that by proposition 3 we know that the components φ_n belong to the range of the characteristic map. Since the computation is local we thus get, collecting together the results of this paper,

Theorem 5. *There exists for each n a universal polynomial $\tilde{L}_n \in H^*(WSO_n)$ such that,*

$$Ch_*(Q) = \underline{\theta}(\tilde{L}_n).$$

Here θ is the isomorphism of theorem 10 section VII, in its relative version (19) and $\underline{\theta}$ denotes the composition of θ with the relative characteristic map (11) associated to the action of the Hopf algebra \mathcal{H} on \mathcal{A} .

One can end the computation of \tilde{L}_n by evaluating the index on the range of the assembly map,

$$(34) \quad \mu : K_{*,\tau}(P \rtimes_{\Gamma} E\Gamma) \rightarrow K(\mathcal{A}_1),$$

provided one makes use of the conjectured (but so far only partially verified, cf. [He], [K-T]) injectivity of the natural map,

$$(35) \quad H_d^*(\Gamma_n, \mathbb{R}) \rightarrow H^*(B\Gamma_n, \mathbb{R})$$

from the smooth cohomology of the Haefliger groupoid Γ_n to its real cohomology.

One then obtains that \tilde{L}_n is the product of the usual L -class by another universal expression in the Pontrjagin classes p_i , accounting for the cohomological analogue of the K -theory Thom isomorphism

$$(36) \quad \beta : K_*(C_0(M) \rtimes \Gamma) \rightarrow K_*(C_0(P) \rtimes \Gamma)$$

of [Co1, §V]. This can be checked directly in small dimension. It is noteworthy also that the first Pontrjagin class p_1 already appears with a non zero coefficient for $n = 2$.

Appendix: the one-dimensional case

In the one dimensional case the operator Q is readily reduced to the following operator on the product \tilde{F} of the frame bundle F^+ by an auxiliary S^1 whose corresponding periodic coordinate is called α (and whose role is as mentioned above to make the vertical fiber even dimensional).

$$(1) \quad Q = Q_V + Q_H.$$

We work with 2 copies of $L^2(\tilde{F}, e^s d\alpha ds dx)$ and the following gives the vertical operator Q_V ,

$$(2) \quad Q_V = \begin{bmatrix} -\partial_\alpha^2 + \partial_s(\partial_s + 1) & -2\partial_\alpha \partial_s - \partial_\alpha \\ -2\partial_\alpha \partial_s - \partial_\alpha & \partial_\alpha^2 - \partial_s(\partial_s + 1) \end{bmatrix}.$$

The horizontal operator Q_H is given by

$$(3) \quad Q_H = \frac{1}{i} e^{-s} \partial_x \gamma_2$$

where $\gamma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ anticommutes with Q_V .

We use the following notation for 2×2 matrices

$$(4) \quad \gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We can thus write the full operator Q acting in 2 copies of $L^2(\tilde{F}, e^s d\alpha ds dx)$ as

$$(5) \quad Q = (-2\partial_\alpha \partial_s - \partial_\alpha) \gamma_1 + \frac{1}{i} e^{-s} \partial_x \gamma_2 + (-\partial_\alpha^2 + \partial_s(\partial_s + 1)) \gamma_3.$$

Theorem A. *Up to a coboundary $ch_* Q$ is equal to twice the transverse fundamental class $[\tilde{F}]$.*

The factor 2 is easy to understand since it is the local index of the signature operator along the fibers of \tilde{F} .

The formulas of theorem 4 for the components φ_1 and φ_3 of the character are,

$$(6) \quad \begin{aligned} \varphi_1(a^0, a^1) &= \sqrt{2i} \Gamma\left(\frac{1}{2}\right) \int (a^0 [Q, a^1] (Q^2)^{-1/2}) \\ &\quad - \sqrt{2i} \frac{1}{2} \Gamma\left(\frac{3}{2}\right) \int (a^0 \nabla [Q, a^1] (Q^2)^{-3/2}) \\ &\quad + \sqrt{2i} \frac{1}{2} \frac{1}{3} \Gamma\left(\frac{5}{2}\right) \int (a^0 \nabla^2 [Q, a^1] (Q^2)^{-5/2}) \\ &\quad - \sqrt{2i} \frac{1}{2 \cdot 3 \cdot 4} \Gamma\left(\frac{7}{2}\right) \int (a^0 \nabla^3 [Q, a^1] (Q^2)^{-7/2}) \end{aligned}$$

$$(7) \quad \begin{aligned} \varphi_3(a^0, a^1, a^2, a^3) &= \sqrt{2i} \frac{1}{3i} \Gamma\left(\frac{3}{2}\right) \int (a^0 [Q, a^1] [Q, a^2] [Q, a^3] (Q^2)^{-3/2}) \\ &\quad - \sqrt{2i} \frac{1}{2 \cdot 3 \cdot 4} \Gamma\left(\frac{5}{2}\right) \int (a^0 \nabla [Q, a^1] [Q, a^2] [Q, a^3] (Q^2)^{-5/2}) \\ &\quad - \sqrt{2i} \frac{1}{3 \cdot 4} \Gamma\left(\frac{5}{2}\right) \int (a^0 [Q, a^1] \nabla ([Q, a^2] [Q, a^3] (Q^2)^{-5/2}) \\ &\quad - \sqrt{2i} \frac{1}{2 \cdot 4} \Gamma\left(\frac{5}{2}\right) \int (a^0 [Q, a^1] [Q, a^2] \nabla [Q, a^3] (Q^2)^{-5/2}). \end{aligned}$$

The computation gives the following result,

$$(8) \quad \varphi_1(a^0, a^1) = 0 \quad \forall a^0, a^1 \in \mathcal{A}$$

in fact, each of the 4 terms of (6) turns out to be 0.

$$(9) \quad \varphi_3 = 2([\widetilde{F}] + b\psi),$$

where $[\widetilde{F}]$ is the tranverse fundamental class ($[Co]$), i.e. the extension to the crossed product of the following invariant cyclic 3-cocycle on the algebra $C_c^\infty(\widetilde{F})$,

$$(10) \quad \mu(f^0, f^1, f^2, f^3) = \int_{\widetilde{F}} f^0 df^1 \wedge df^2 \wedge df^3.$$

We shall now give the explicit form of both $b\psi$ and ψ with $B\psi = 0$.

We let τ be the trace on \mathcal{A} given by the measure

$$(11) \quad f \in C_c^\infty(\widetilde{F}) \rightarrow \int_{\widetilde{F}} f e^s d\alpha ds dx.$$

This measure is invariant under the action of Diff^+ and thus gives a dual trace on the crossed product.

The two derivations ∂_α and ∂_s of $C_c^\infty(\widetilde{F})$ are invariant under the action of Diff^+ and we denote by the same letter their canonical extension to \mathcal{A} ,

$$(12) \quad \partial_\alpha(fU_\psi^*) = (\partial_\alpha f)U_\psi^*, \quad \partial_s(fU_\psi^*) = (\partial_s f)U_\psi^*.$$

We let δ_1 be the derivation of \mathcal{A} defined in section II.

By construction both δ_1 and τ are invariant under ∂_α but neither of them is invariant under ∂_s .

But the following derivation $\partial_u : \mathcal{A} \rightarrow \mathcal{A}^*$ commutes with both ∂_α and ∂_s ,

$$(13) \quad \langle \partial_u(a), b \rangle = \tau(\delta_1(a)b).$$

We then view ∂_α , ∂_s and ∂_u as three commuting derivations, where ∂_u cannot be iterated and use the notation (us, α, s) for the cochain

$$a^0, a^1, a^2, a^3 \rightarrow \langle a^0, (\partial_u \partial_s a^1)(\partial_\alpha a^2)(\partial_s a^3) \rangle.$$

The formula for $b\psi$ is then the following,

$$\begin{aligned} & \frac{1}{8} (- (u, \alpha, s) + (u, s, \alpha) + (\alpha, s, u) - (s, \alpha, u)) \\ & \frac{1}{2} (- (us, \alpha, s) + (\alpha, us, s) + (\alpha, s, us) \\ & \quad + (us, s, \alpha) + (s, us, \alpha) - (s, \alpha, us) \quad + (\alpha, u\alpha, \alpha)) \\ & \quad - (u, \alpha s, s) - (s, u\alpha, s) - (s, \alpha s, u) \\ & \frac{1}{4} (- (u, \alpha, ss) + (ss, u, \alpha) - (ss, \alpha, u) \\ & \quad + (u, ss, \alpha) + (\alpha, u, ss) + (\alpha, ss, u) \\ & \quad - (u, \alpha\alpha\alpha) + (\alpha\alpha, u, \alpha) - (\alpha\alpha, \alpha, u) \\ & \quad - (u, \alpha\alpha, \alpha) + (\alpha, u, \alpha\alpha) - (\alpha, \alpha\alpha, u)). \end{aligned}$$

This formula is canonical and a possible choice of ψ is given by,

$$\begin{aligned} & \frac{1}{8}((\alpha, su) - (\alpha u, s) - (s, \alpha u) + (su, \alpha)) \\ & \frac{1}{4}(-(\alpha\alpha u, \alpha) - \frac{1}{3}(\alpha\alpha\alpha, u) + \alpha u, \alpha\alpha)) \\ & \frac{1}{4}((u\alpha s, s) - (\alpha, ssu) + \frac{1}{2}(\alpha ss, u) + (\alpha s, us) + \frac{1}{2}(\alpha u, ss) - \frac{1}{2}(ssu, \alpha)). \end{aligned}$$

The natural domain of ψ is the algebra C^3 of 3 times differentiable elements of \mathcal{A} where the derivation ∂_u is only used once.

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